ACCELERATOR PHYSICS BASICS

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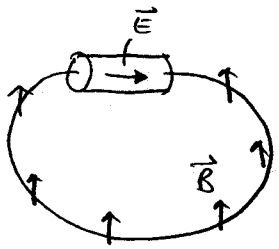
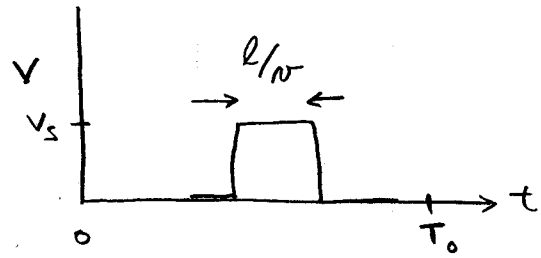
energy gain $\Delta E = q\,|\vec{E}|\cdot l$

$= qV = eV$ $\quad e = 1.6\times10^{-19}$ coul for proton

Synchrotron:



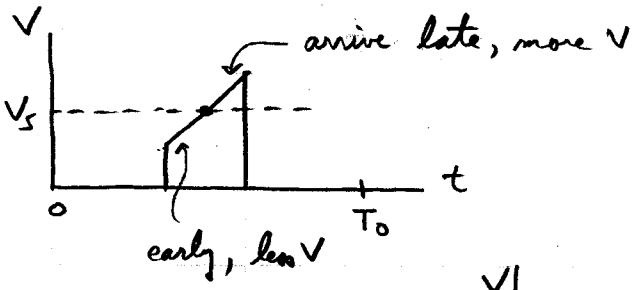
for one particle, would want:



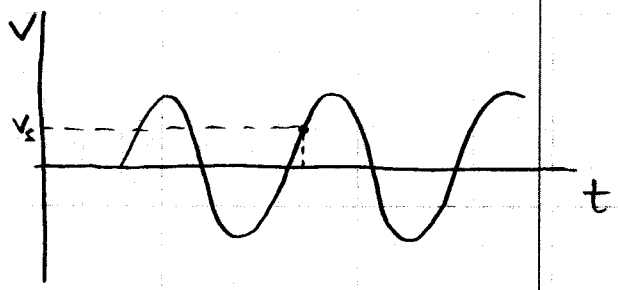
but, more than 1 particle...

$\therefore$ want longitudinal focusing ("phase stability")

a) try



b) resonant cavities: more efficient



what frequency? $\quad f_0$ = revolution frequency $= \dfrac{1}{T_0}$

$$f_{RF} = h f_0 \qquad h = \text{harmonic \#}$$

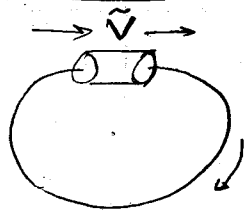
$= $ # of possible "phase stable" points

ex: TEVATRON: $\quad h = 1113$

$$f_{RF} = h f_0 = 1113 \cdot \frac{3\cdot10^8\ \text{m/s}}{2\pi\cdot10^3\ \text{m}} \approx 53\ \text{MHz} \qquad (\text{spacing} \sim 19\ \text{nsec})$$

Longitudinal Motion

circular accelerator

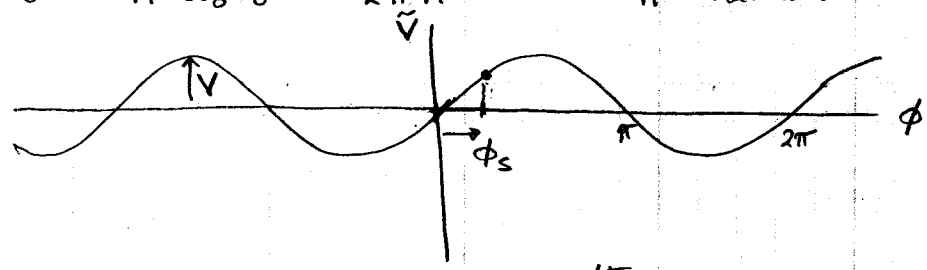


$$\tilde{V} = V \sin \omega_{RF} t$$

assumes resonant cavity

$$C = 2\pi R = v T_0 = v/f_0$$

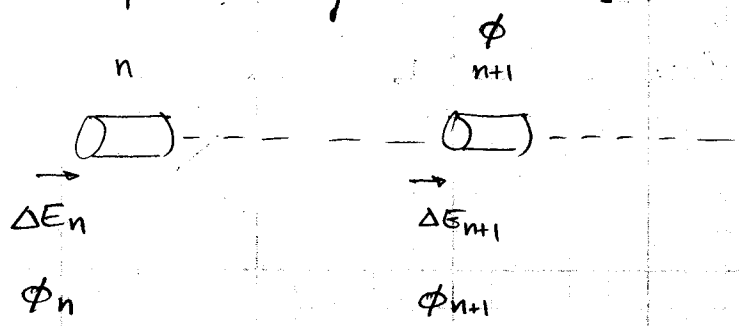
let $\omega_{RF} T_0 = h\, \omega_0 T_0 = 2\pi h$ $\qquad$ h = harmonic #



synchronous particle: $\quad E_s \qquad \dfrac{dE_s}{dn} = eV \sin \phi_s$

[note: must adjust Bend field so that $\int B\, ds = \frac{p_s}{e} \cdot 2\pi$]

look at particle nearby: $\quad E = E_s + \Delta E$



easy to see $\quad \Delta E_{n+1} = \Delta E_n + eV(\sin \phi_n - \sin \phi_s)$

what about ϕ ?

particle will take more/less time to go around, depending upon E (or, p)

$$T = \frac{C}{v} \qquad \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} \qquad \frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

$$\Rightarrow \frac{\Delta T}{T} = \underbrace{\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)}_{\eta}\frac{\Delta p}{p} \qquad (\eta = \text{slip factor})$$

$$\frac{\Delta C}{C} \equiv \frac{1}{\gamma_t^2}\frac{\Delta p}{p}$$ (more later)

so, going once around, the phase will slip by an amount

$$\frac{\Delta\phi}{2\pi h} = \frac{\Delta T}{T} = \eta\frac{\Delta p}{p} = \frac{\eta}{\beta^2}\frac{\Delta E}{E}$$

$$\Rightarrow \qquad \phi_{n+1} = \phi_n + 2\pi h\eta\left(\frac{\Delta p}{p_s}\right)_{n+1}$$

or, we have

$$\boxed{\begin{aligned}\Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_n - \sin\phi_s)\\ \phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E_s}\Delta E_{n+1}\end{aligned}}$$

(BASIC program)

TRACK, or approximate w/ diff eq's ...

$$\frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s)$$

$$\frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E_s}\Delta E$$

$$\Rightarrow \quad \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta eV}{\beta^2 E_s}(\sin\phi - \sin\phi_s)$$

suppose $\phi = \phi_s + \Delta\phi$, $\Delta\phi \ll 1$, then

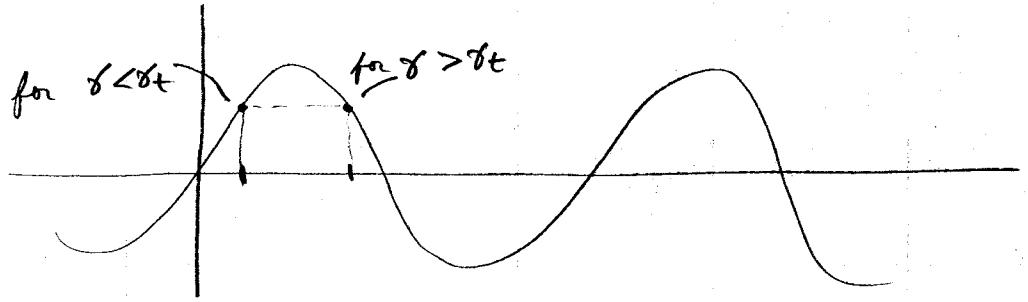
$$\begin{aligned}\sin\phi - \sin\phi_s &= \sin\phi_s \cancel{\cos\Delta\phi} + \sin\Delta\phi\cos\phi_s - \cancel{\sin\phi_s}\\ &\approx \Delta\phi\cos\phi_s\end{aligned}$$

$$\Rightarrow \quad \boxed{\frac{d^2\Delta\phi}{dn^2} - \frac{2\pi h\eta eV\cos\phi_s}{\beta^2 E_s}\Delta\phi = 0}$$

then motion is stable if $\boxed{\eta\cos\phi_s < 0}$

So, when $\eta < 0$ $[\gamma < \gamma_t]$, we want $\cos\phi_s > 0$

when $\eta > 0$ $[\gamma > \gamma_t]$, we want $\cos\phi_s < 0$



$\therefore$ if γ_t exists, need "phase jump @ transition"

γmc^2 = transition energy

BOO:	$\gamma_t \approx 4.5$	$\therefore$ cross
MI:	$\gamma_t \approx 20$	$\therefore$ cross
TEV:	$\gamma_t \approx 18$	$\therefore$ do NOT cross

small oscillation period...

$$\omega_s = \sqrt{-\frac{2\pi h \eta e V \cos\phi_s}{\beta^2 E_s}}$$

$$\nu_s \equiv \sqrt{-\frac{h \eta e V \cos\phi_s}{2\pi \beta^2 E_s}} = \text{synchrotron tune}$$

$$\frac{1}{\nu_s} = \text{synchrotron period (\# turns)} \gg 1$$

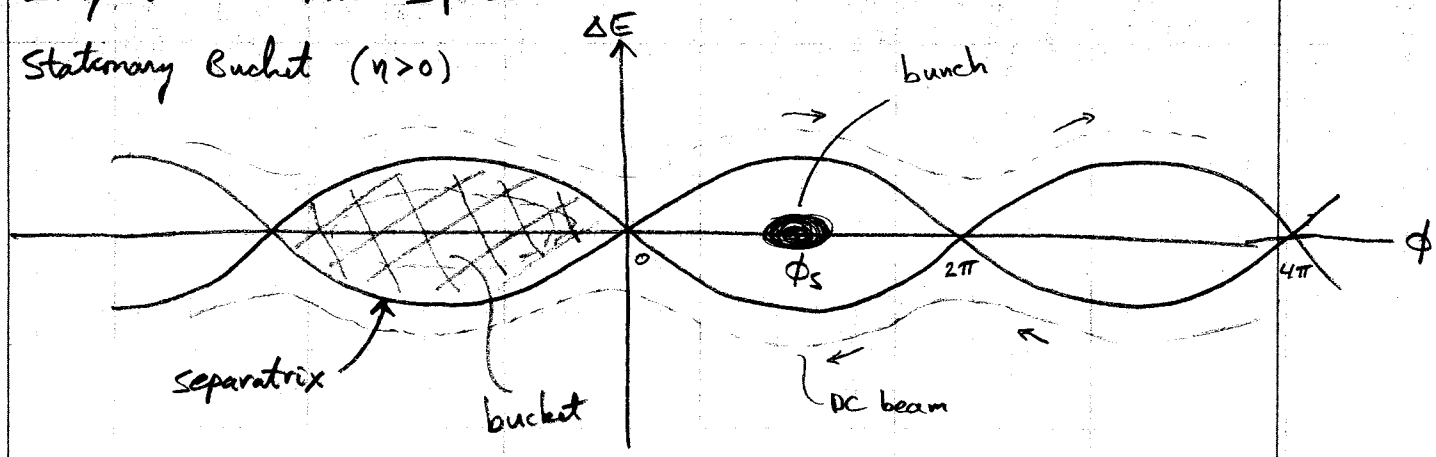
ex: TEV @ 150 GeV

$$\nu_s = \sqrt{-\frac{(1113)(\frac{1}{18^2})(10^6\,eV)(-1)}{2\pi \quad 150\cdot 10^9\,eV}} \approx \frac{1}{500}$$

synchrotron "frequency" $= \nu_s f_0 = \dfrac{\nu_s c\beta}{C} \approx \dfrac{3\cdot 10^8\,m/s}{500\cdot 2\pi\cdot 10^3\,m} \approx 100\,Hz$

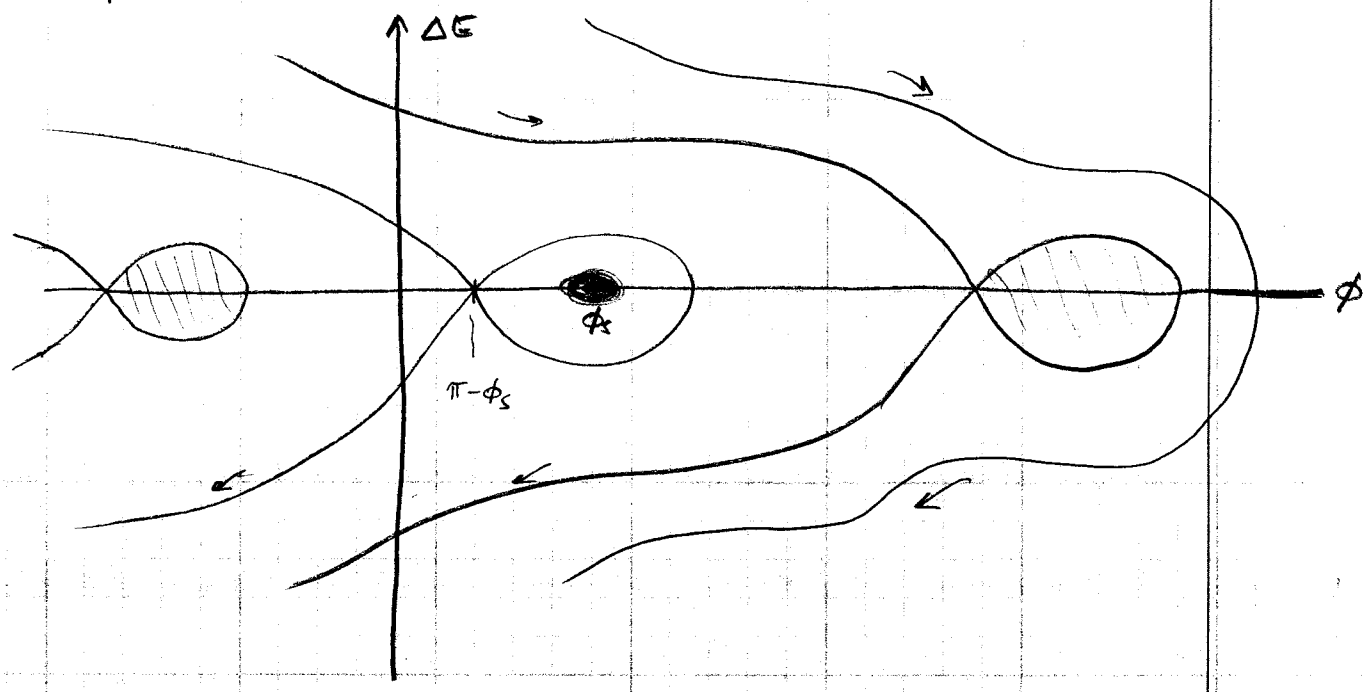
Longitudinal Phase Space

Stationary Bucket ($\eta > 0$)

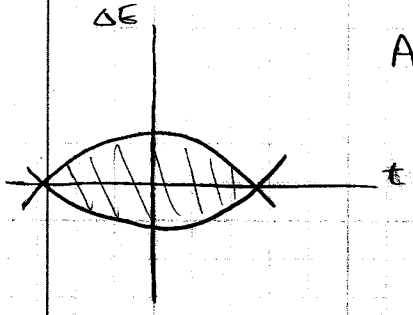


ν_s depends on amplitude of motion; near separatrix, $\nu_s \to 0$

Accelerating Bucket ($\eta > 0$)



Change to canonical variables: $\oint \Delta E \cdot dt$ = adiabatic invariant



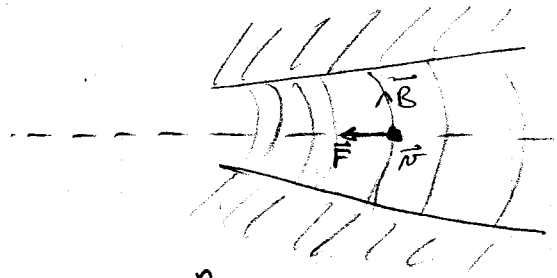
Area of Stationary bucket:

$$\boxed{A = \frac{8C}{\pi h c}\sqrt{\frac{eV \cdot E_s}{2\pi h |\eta|}}}$$

Transverse Motion

Need to force particles back toward the ideal trajectory
∴ would like a force ∝ displacement

early accelerators used "weak focusing" gradient magnets



$$\vec{B}_y = B_0 \left(\frac{R_0}{r}\right)^n \qquad r = R_0 + x$$

$$= B_0 \left(1 + \frac{x}{R_0}\right)^{-n} \approx B_0 \left(1 - \frac{n}{R_0} x\right)$$

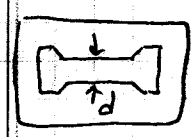
$$\therefore B_x \approx -n \frac{B_0}{R_0} y \qquad \left[\text{since } \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}\right]$$

system is stable (x + y) if $\underline{\underline{0 < n < 1}}$.

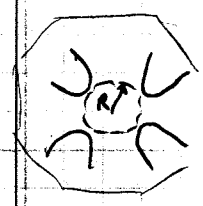
Modern accelerators (HEP) use "strong focusing" (Courant, et al.)...

Can separate Bending + Focusing functions using different magnets: dipoles, quadrupoles

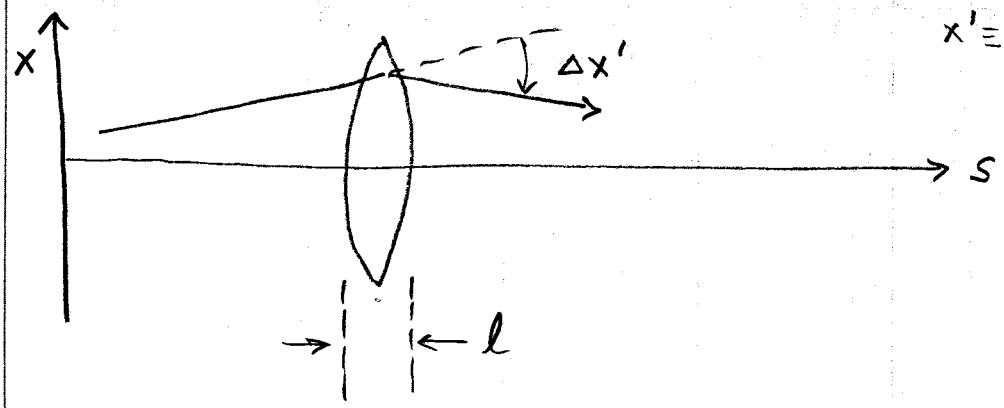
dipole: $B_y = B$, $B_x = 0$ $\qquad \left[B = \frac{2\mu_0 N I}{d}\right]$



quadrupole: $B_y = B' x$, $B_x = B' y$ $\qquad \left[B' = \frac{2\mu_0 N I}{R^2}\right]$



quadrupoles act like <u>lenses</u> ...



$$x' \equiv \frac{dx}{ds} = \text{slope}$$

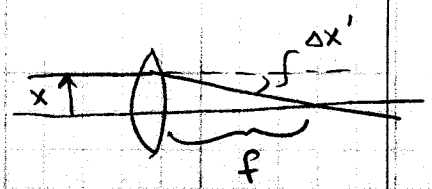
$$\Delta x' = \frac{\Delta p_x}{p} = \frac{\int F_x\,dt}{p} = \frac{-\int e v B_y\,dt}{p} \approx \frac{-e\int B_y\,ds}{p}$$

$$\approx -\frac{e B' l}{p} \cdot x$$

ratio $\frac{p}{e} \equiv$ "magnetic rigidity" $\equiv (B\rho)$

$$\boxed{(B\rho) \approx \frac{10}{3} \cdot p_{[GeV/c]} \quad \text{Tesla-m}}$$

so, $$\Delta x' = -\underbrace{\frac{B' l}{B\rho}}_{\frac{1}{f}} \cdot x$$



"thin lens" approximation : valid if $l \ll f$

ex: Tevatron @ 980 GeV :

$$(B\rho) = \frac{10}{3} \cdot 980 \text{ T·m} = 3270 \text{ T·m}$$

$$B' = 78 \text{ T/m} \qquad l = 1.68\text{m}$$

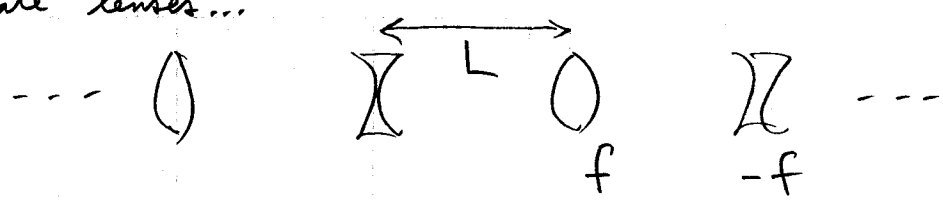
and $$f = \frac{(B\rho)}{B' l} = \frac{3270 \text{ T-m}}{(78 \text{T/m})\, 1.68\text{m}} = 25\text{m} \gg l$$

note:
$$\left.\begin{aligned} B_y &= B' x \\ B_x &= B' y \end{aligned}\right\} \text{ quadrupole field}$$

$$\vec{F} = e\vec{v} \times \vec{B} \Rightarrow \quad F_x = -evB_y = -evB'x$$
$$F_y = +evB_x = +evB'y$$

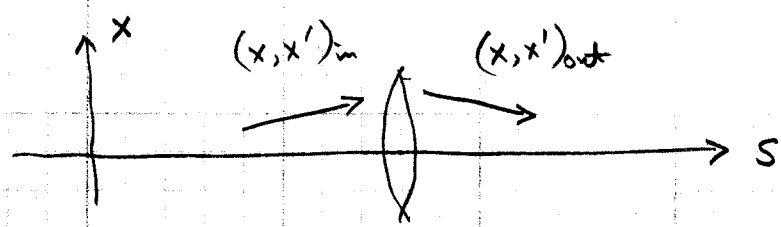
$\therefore B' > 0 \Rightarrow$ focuses in x-direction
defocuses in y-direction

alternate lenses...



system is stable (x + y)
if $\boxed{f > L/2}$

2x2 matrices...



quad (thin lens): $\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$

drift (or bend region): $\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$

∴ build up accelerator out of pieces, each piece described by 2X2 matrix (actually, 2 (x, y) matrices)

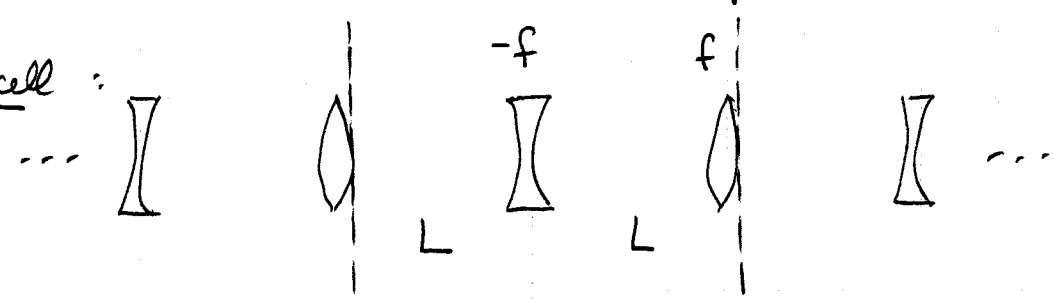
then, total ring described by

$$M = M_n M_{n-1} \cdots M_2 M_1$$

motion is stable if $\boxed{|\text{Trace } M| < 2}$

stability criterion

<u>FODO cell</u>:



$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L \\ -\frac{1}{f} & 1-\frac{L}{f} \end{pmatrix}\begin{pmatrix} 1 & L \\ \frac{1}{f} & 1+\frac{L}{f} \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{f} & 2L+\frac{L^2}{f} \\ -\frac{L}{f^2} & 1-\frac{L}{f}-\frac{L^2}{f^2} \end{pmatrix}$$

$$\text{trace } M = 2 - \frac{L^2}{f^2}$$

∴ criterion: $-2 < 2 - \frac{L^2}{f^2} < 2$

$$-4 < -\frac{L^2}{f^2} < 0$$

or, $\frac{L^2}{f^2} < 4 \Rightarrow f > \frac{L}{2}$

can always solve specific problem with a "piece-wise" solution; next, we look at a more general, analytical approach...

System is analogous to SHM, but the "spring constant" depends on time (or, here, on longitudinal position)...

$$\ddot{x} + kx = 0 \quad \Rightarrow \quad x = A\sin(\omega t + \delta)$$

we have

$$x'' + K(s)x = 0 \quad \Rightarrow \quad x(s) = A(s)\sin\left[\int \omega(s)\,dt + \delta\right]$$

our "spring constant":

$$\Delta x' = -\frac{B'\Delta s}{B\rho}\cdot x \quad \Rightarrow \quad x'' = -\frac{B'}{B\rho}x \qquad \text{(quad. field)}$$

in fact, we also have a "rotating" reference system (bending magnets), so we have

$$K_x(s) = \frac{\frac{\partial B_y}{\partial x}(s)}{(B\rho)} + \frac{1}{\rho(s)^2}$$

ρ = local bending radius (typically very large!)

$$K_y(s) = \frac{\frac{\partial B_y}{\partial x}(s)}{B\rho}$$

but, in each transverse degree of freedom we have

$$\boxed{x'' + K(s)x = 0}$$

Hill's Equation

Trial Solution: $x(s) = A\sqrt{\beta(s)}\,\sin[\,\psi(s) + \delta\,]$

from here, take $x'(s) = \frac{1}{2}A\beta^{-1/2}\beta'\sin[\,] + A\beta^{1/2}\cos[\,]\,\psi'$

and take $x''(s) = \ldots$

+ plug into Hill's Equation

we find that we must have

$$\underbrace{\psi' = \frac{1}{\beta}}, \quad \text{and} \quad \underbrace{K\beta = -\frac{1}{2}\beta'' + \frac{1+\frac{1}{4}(\beta')^2}{\beta}}_{\text{diff. eq. that } \beta \text{ must obey}}$$

$\Downarrow$

β is related to local amplitude of motion

AND

β is a local wavelength of the motion

So, β is called the Amplitude Function. The oscillation phase advances according to

$$\Delta\psi = \int \frac{1}{\beta(s)}\,ds$$

Total # of oscillations around the accelerator:

$$\nu = \frac{1}{2\pi}\oint \frac{1}{\beta}\,ds \quad \equiv \quad \text{the "tune"}$$

Transverse Motion -- 2

Betatron Oscillation: first analyzed for the "Betatron" accelerator

Solution to Hill's Equation: $x(s) = A\sqrt{\beta(s)}\,\sin[\psi(s) + \delta]$

Suppose $x = 0$ and $x' = \theta_0$ @ $s = s_0$

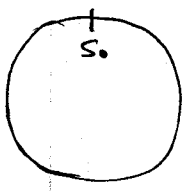
then $\sin[\psi_0 + \delta] = 0 \quad \Rightarrow \quad \delta = -\psi_0$

$$\tfrac{1}{2} A \beta_0^{-1/2} \beta_0' \underbrace{\sin[\psi_0 + \delta]}_{=0} + A\sqrt{\beta_0}\,\underbrace{\cos[\psi_0 + \delta]}_{1}\,\underset{1/\beta_0}{\underset{\shortparallel}{\psi_0'}} = \theta_0$$

$$\Rightarrow A = \sqrt{\beta_0}\,\theta_0$$

$$\therefore \boxed{x(s) = \theta_0 \sqrt{\beta_0 \beta(s)}\,\sin(\psi(s) - \psi_0)}$$

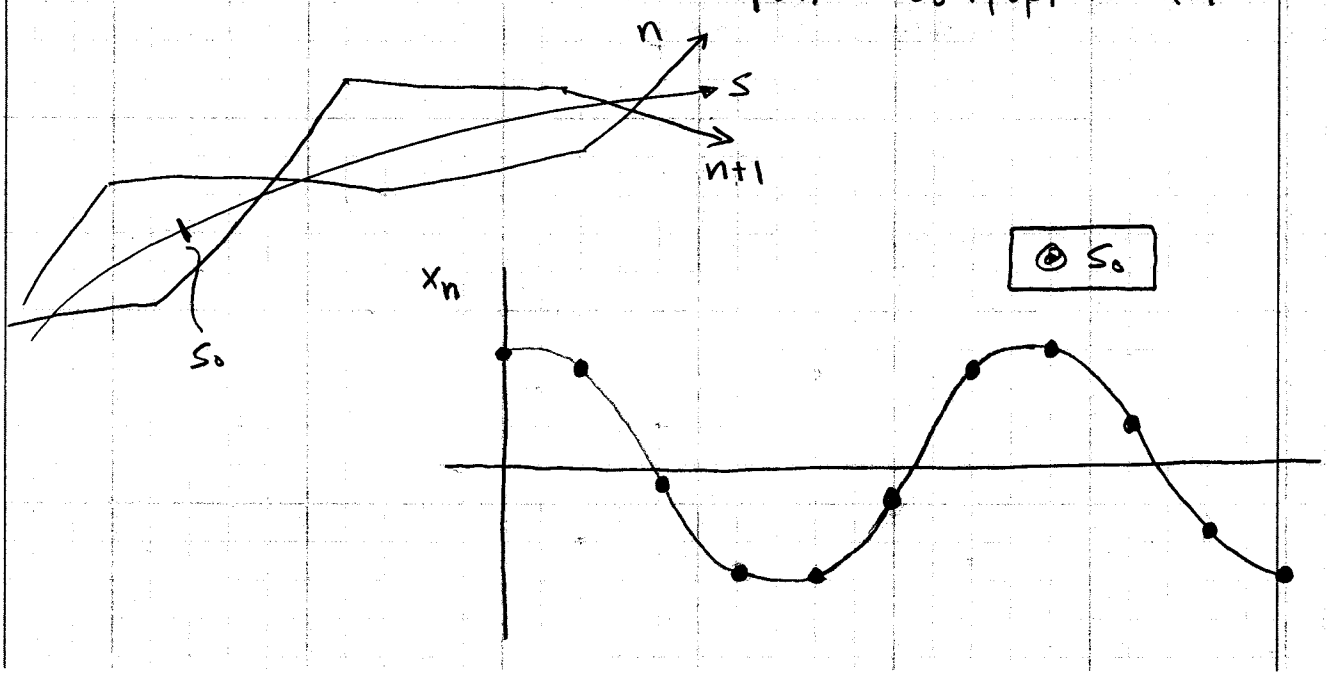
Suppose sit @ location s_0 in the ring, and watch...



$$x(n) = \theta_0 \beta_0 \sin(2\pi\nu n) \qquad (n = \text{turn \#})$$

look somewhere else (s_1, say):

$$x_1(n) = \theta_0 \sqrt{\beta_0 \beta_1}\,\sin(\psi_1 - \psi_0 + 2\pi\nu n)$$



<u>Phase Space</u>

$$x = A\beta^{1/2}\sin(\psi+\delta)$$

$$x' = \tfrac{1}{2}A\beta^{-1/2}\beta'\sin(\psi+\delta) + A\beta^{1/2}\cos(\psi+\delta)\psi'$$

$$= -A\alpha\beta^{-1/2}\sin(\psi+\delta) + A\beta^{-1/2}\cos(\psi+\delta)$$

$$\beta x' = -A\alpha\beta^{1/2}\sin(\psi+\delta) + A\beta^{1/2}\cos(\psi+\delta)$$

$$= -\alpha x + A\beta^{1/2}\cos(\psi+\delta)$$

$$\Rightarrow \quad \beta x' + \alpha x = A\beta^{1/2}\cos(\psi+\delta)$$

$$\therefore \quad \boxed{x^2 + (\beta x' + \alpha x)^2 = A^2\beta} \qquad \text{(circle)}$$

or,

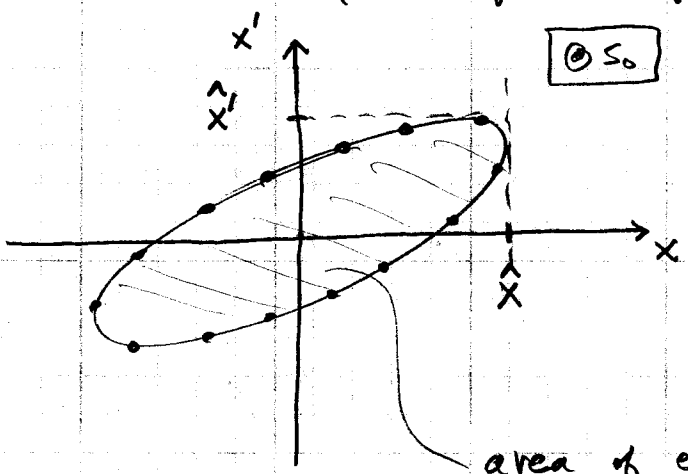
$$x^2 + \beta^2 x'^2 + 2\alpha\beta x x' + \alpha^2 x^2 = A^2\beta$$

$$x^2(1+\alpha^2) + 2\alpha\beta x x' + \beta^2 x'^2 = A^2\beta$$

$$\frac{1+\alpha^2}{\beta}x^2 + 2\alpha x x' + \beta x'^2 = A^2$$

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2} \qquad \text{(ellipse)}$$

note: γ, α, β are functions of s; A^2 is not!



So, @ s_0, we have

$$\hat{x} = A\sqrt{\beta_0}$$

$$\hat{x}' = A\sqrt{\gamma_0}$$

(Computer Demo)

area of ellipse $= \pi A^2$

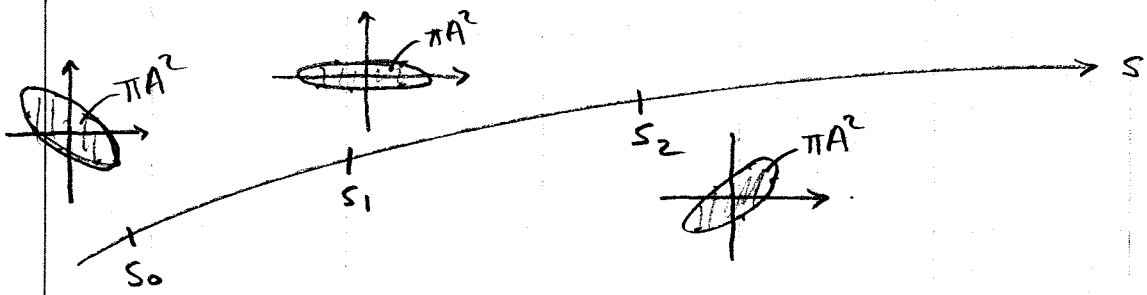
independent of s

<u>Emittance</u> $\equiv$ phase space area

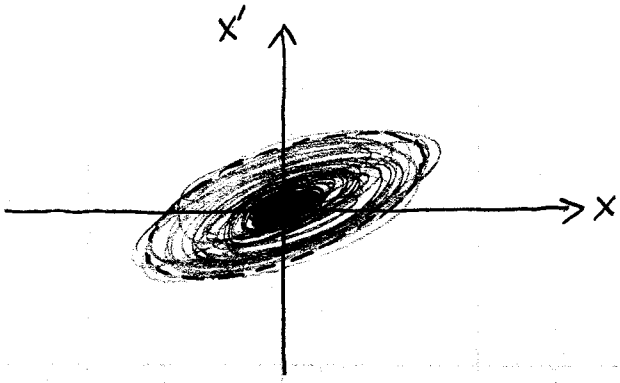
Single particle emittance... $\pi A^2 = \epsilon$, or $A = \sqrt{\epsilon/\pi}$

$$\Rightarrow \quad x = \sqrt{\frac{\epsilon\beta(s)}{\pi}}\,\sin(\psi(s) + \delta)$$

$$\hat{x}_0 = \sqrt{\epsilon\beta_0/\pi} \quad \text{etc.}$$



How to describe a <u>beam</u> of particles?

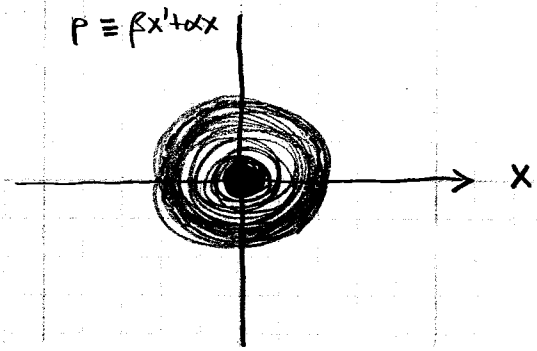


suppose distribution is time independent (i.e., reached an equilibrium condition)

Find area of an ellipse that contains some fraction, f, of the particles.

a) Let's assume a Gaussian distribution, w/ $\langle x^2 \rangle = \sigma^2$

b) Sometimes easier to look at <u>circles</u> rather than ellipses...



Linear Errors

So far, our particles have not been "ideal," but the accelerator has been. Now, we introduce imperfections to the magnets...

Steering errors:

dipole magnet field error: $B_0 \rightarrow B_0 + \Delta B$

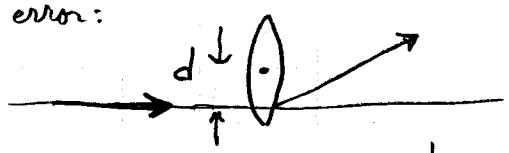
$$\Rightarrow \Delta\theta_x = -\frac{\Delta B \cdot l}{B\rho}$$

dipole magnet roll error: $B_y = B_0$, $B_x = 0$

$$\Rightarrow \begin{cases} B_y = B_0 \cos\phi \approx B_0 \\ B_x = B_0 \sin\phi \approx \phi B_0 \end{cases}$$

$$\Rightarrow \Delta\theta_y = +\phi \frac{B_0 l}{B\rho}$$

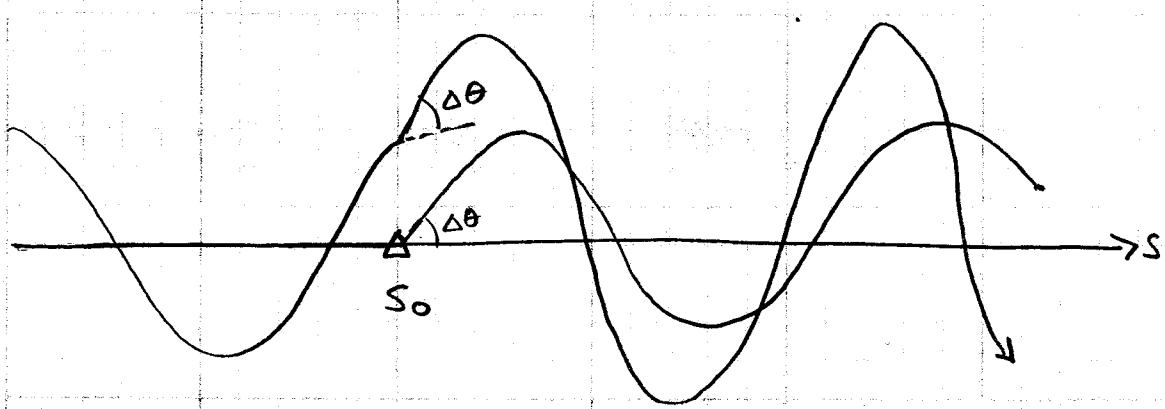
quadrupole alignment error:



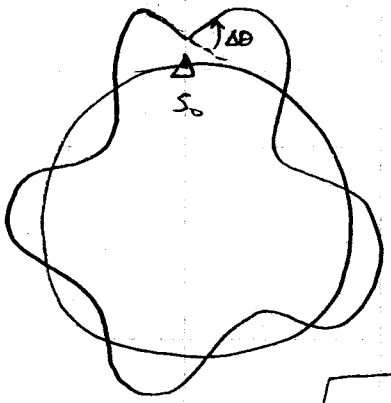
$$\Delta\theta_{x,y} = \frac{d_{x,y}}{f}$$

these are not one-time kicks; they affect the particle motion each revolution...

(Computer Demo)



However, a closed path, or "closed orbit" still exists...



Want to find initial conditions @ s_0, (x_0, x_0') so that after one revolution we come back to s_0, pass through the dipole error, and end up with (x_0, x_0') again...

$$\boxed{M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \qquad M_0 = e^{J_0 \mu} = e^{J_0 2\pi\nu}$$

$$= \left(I - e^{J_0\mu}\right)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \left[e^{J_0\mu/2} \left(e^{-J_0\mu/2} - e^{+J_0\mu/2} \right) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \left[e^{J_0\mu/2} \left(-2 J_0 \sin\tfrac{\mu}{2} \right) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= -\frac{1}{2\sin\pi\nu} J_0^{-1} e^{-J_0\mu/2} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \frac{1}{2\sin\pi\nu} J_0 \left(I\cos\pi\nu - J_0 \sin\pi\nu \right) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

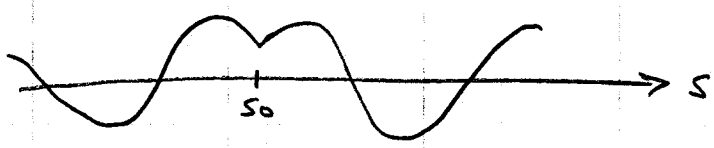
$$= \frac{1}{2\sin\pi\nu} \left(I\sin\pi\nu + J_0 \cos\pi\nu \right) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0 \cos\pi\nu \end{pmatrix}$$

given these initial conditions, can propagate around the ring to find the <u>orbit distortion</u> everywhere...

$$\boxed{\Delta X(s) = \frac{\Delta\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos\left[\left|\psi(s)-\psi_0\right| - \pi\nu\right]}$$

a) symmetric about $s = s_0$, ∴ will see a "cusp" in the orbit @ s_0



b) denominator has $\sin\pi\nu$ ⇒ avoid $\underline{\underline{\nu = \text{integer}}}$!

Of course, will be many errors in the accelerator, + closed orbit will be due to their superposition...

Should see (if unconnected)

$$\Delta X_{rms} \approx \frac{\Delta\theta_{rms}}{2|\sin\pi\nu|}\sqrt{\beta_0\langle\beta\rangle}\sqrt{\frac{N}{2}}$$

@ β_0 due to N errors

TEVATRON, if <u>unconnected</u>, + if $\Delta\theta_{rms} = \frac{d_{rms}}{f} = \frac{0.5\,mm}{25\,m} = 2\cdot10^{-5}$

$$\Rightarrow \Delta X_{rms} \approx \frac{2\cdot10^{-5}}{2}\sqrt{(100\,m)(55\,m)}\sqrt{\frac{200}{2}}$$

$$\approx 7\ mm$$

note: $2.5\cdot\Delta X_{rms} = 18mm$ = edge of pipe!

note: $\Delta\beta, \phi$ ⇒ even larger

Off-momentum particles (again) -- Transverse effects

Let p_0 = ideal momentum, but consider a particle with $p = p_0 + \Delta p$, $\frac{\Delta p}{p_0} \ll 1$.

To first order,

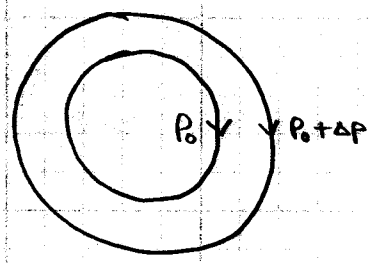
a) dipole magnets will not "bend" as much ($p > p_0$), and so will introduce steering "errors"
$\Rightarrow$ new closed orbit, which is $\propto \frac{\Delta p}{p_0}$

b) quadrupoles will not "focus" as much, and so will introduce gradient "errors"

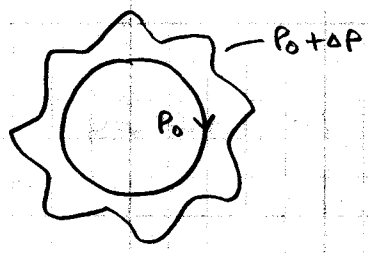
$\Rightarrow$ new tune, which is $\propto \frac{\Delta p}{p_0}$

Dispersion

uniform bend field:



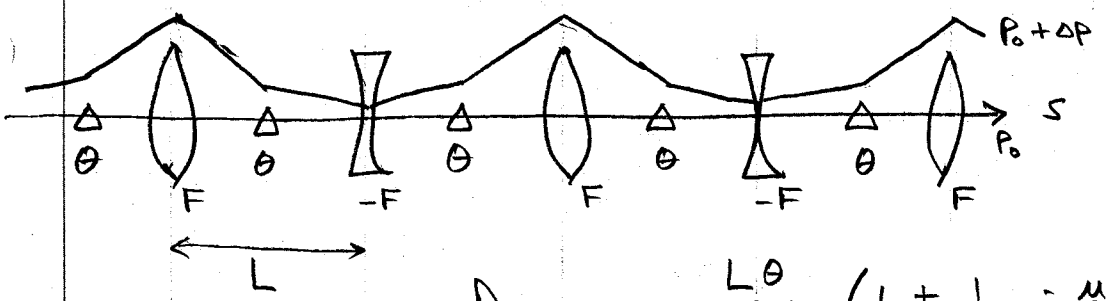
add gradients...



the orbit of an off-momentum particle which closes on itself is given by the Dispersion Function,

$$D(s, p) = \frac{\Delta x(s, p)}{\Delta p / p_0}$$

ex: FODO cells w/ bending ...



$$D_{max,min} = \frac{L\theta}{2\sin^2\frac{\mu}{2}}\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)$$

$$\left(\sin\frac{\mu}{2} = \frac{L}{2F}\right)$$

~ few meters

effect on transverse particle distribution...

$A\sqrt{\beta}\sin\psi$

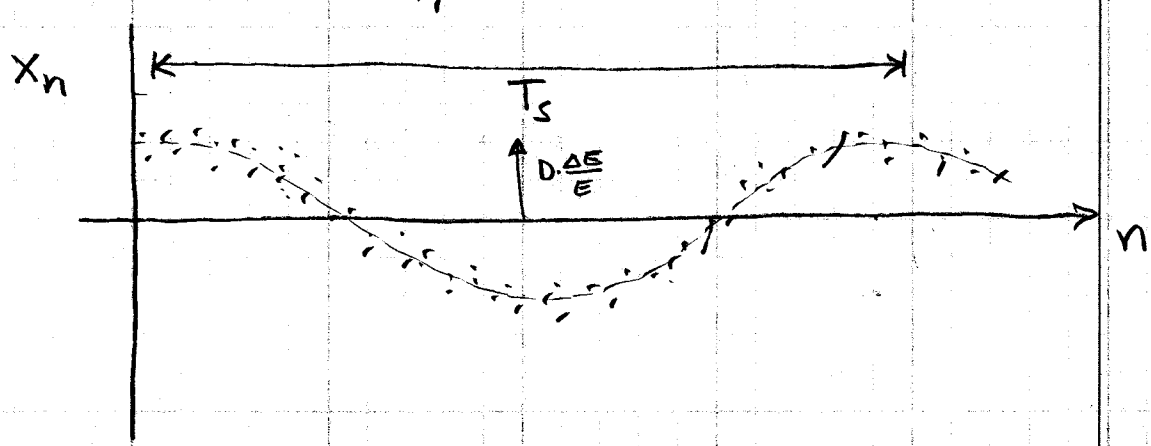
$$x = x_p + x_\beta = D\frac{\Delta p}{p_0} + x_\beta$$

so, $$x^2 = D^2\left(\frac{\Delta p}{p_0}\right)^2 + 2D\,x_\beta\frac{\Delta p}{p_0} + x_\beta^2$$

$$\Rightarrow \langle x^2\rangle = D^2\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle + \langle x_\beta^2\rangle \qquad \left(x_\beta, \frac{\Delta p}{p} \text{ uncorrelated}\right)$$

or, $$\boxed{\sigma_{TOT} = \sqrt{D^2\left(\frac{\sigma_p}{p}\right)^2 + \frac{\epsilon_N\beta}{6\pi(\gamma\beta)}}}$$

ex: injection oscillation w/ energy offset, ΔE: T-B-T display...



CHROMATICITY :

we saw earlier $\Delta x' = -\frac{eB'l}{p} \cdot x$

focusing depends on momentum

$\therefore \quad \nu = \nu(p)$

chromaticity $= \xi \equiv \Delta\nu \Big/ \left(\frac{\Delta p}{p}\right)$

can adjust chromaticity of the synchrotron (h+v) using sextupoles

SO, what BPM's see...

1-particle :

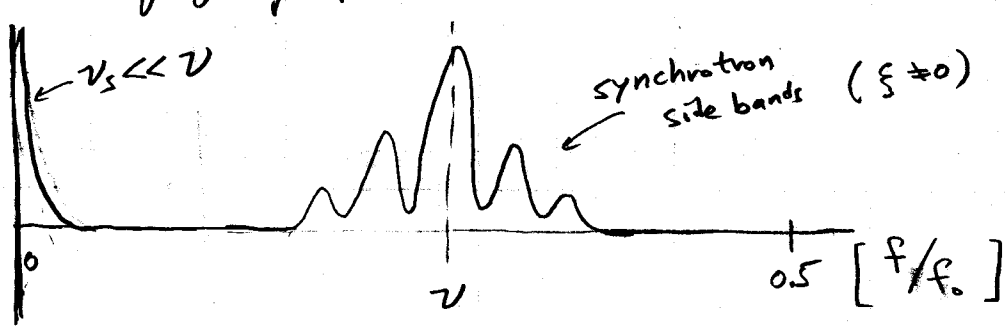
$$\Delta x_n = \Delta x_{c.o.} + D \cdot \frac{\Delta p_n}{p} + \underbrace{a}_{\propto \sqrt{\beta}} \cos\left(2\pi \nu\left(\frac{\Delta p_n}{p}\right) n + \delta\right)$$

average over the beam..

$$\Delta x_n = \Delta x_{c.o.} + D\left\langle \frac{\Delta p}{p} \right\rangle + \left\langle a \cos\left(2\pi \cdot \underbrace{\nu\left(\frac{\Delta p}{p}\right) \cdot n}\right. + \delta)\right\rangle$$

$$\nu\left(\frac{\Delta p}{p}\right) = \xi \cdot \frac{\Delta p}{p} = \xi \cdot \left(\widehat{\frac{\Delta p}{p}}\right) \cos(2\pi \nu_s n + \phi)$$

synch. tune

$\therefore$ frequency modulation

$\therefore$ BPM frequency spectrum