Transverse Motion -- 2

Betatron Oscillation: first analyzed for the "Betatron" accelerator

Solution to Hill's Equation: $x(s) = A\sqrt{\beta(s)}\,\sin[\psi(s) + \delta]$

Suppose $x = 0$ and $x' = \theta_0$ @ $s = s_0$

then $\sin[\psi_0 + \delta] = 0 \Rightarrow \delta = -\psi_0$

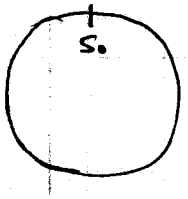
$$\tfrac{1}{2} A \beta_0^{-1/2} \beta_0' \underbrace{\sin[\psi_0+\delta]}_{=0} + A\sqrt{\beta_0}\underbrace{\cos[\psi_0+\delta]}_{1}\psi_0' = \theta_0$$

$\psi_0' = 1/\beta_0$

$$\Rightarrow A = \sqrt{\beta_0}\,\theta_0$$

$$\therefore \boxed{x(s) = \theta_0\sqrt{\beta_0\beta(s)}\,\sin(\psi(s) - \psi_0)}$$

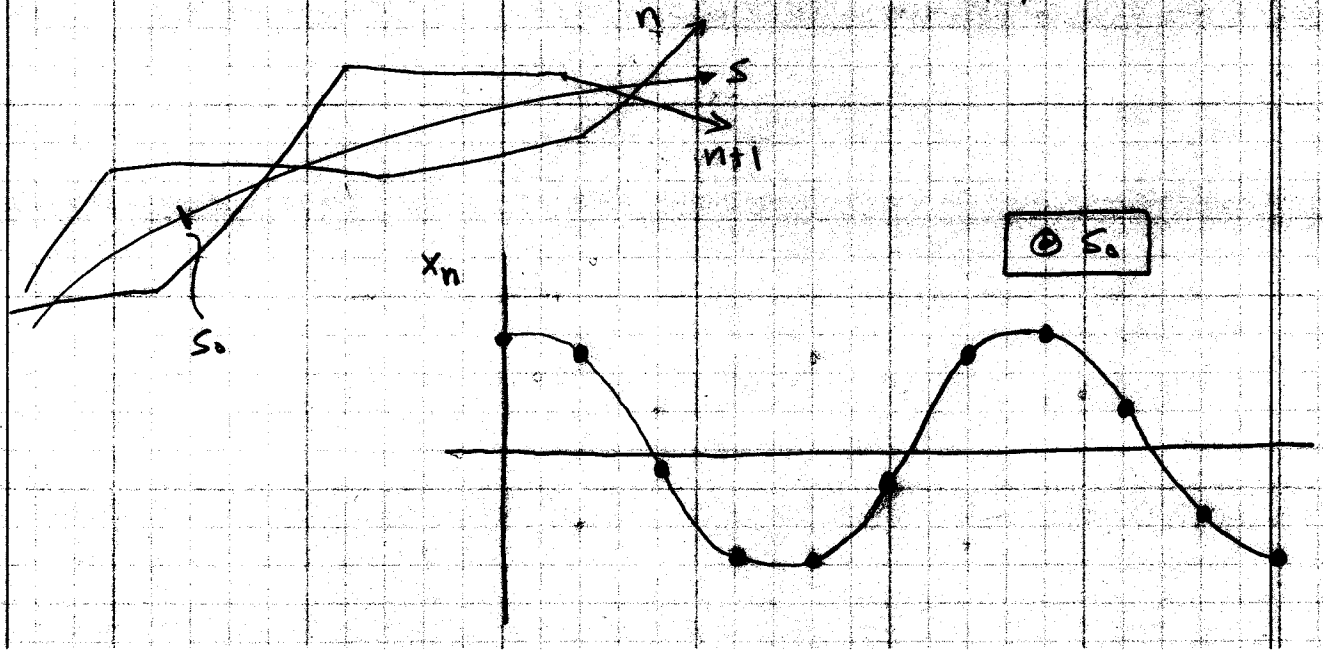
Suppose sit @ location s_0 in the ring, and watch...



$$x(n) = \theta_0\beta_0\sin(2\pi\nu n) \qquad (n = \text{turn \#})$$

look somewhere else (s_1, say):

$$x_1(n) = \theta_0\sqrt{\beta_0\beta_1}\,\sin(\psi_1 - \psi_0 + 2\pi\nu n)$$



Phase Space

$$X = A\beta^{1/2}\sin(\psi+\delta)$$

$$x' = \tfrac{1}{2}A\beta^{-1/2}\beta'\sin(\psi+\delta) + A\beta^{1/2}\cos(\psi+\delta)\psi'$$

$$= -A\alpha\beta^{-1/2}\sin(\psi+\delta) + A\beta^{-1/2}\cos(\psi+\delta)$$

$$\beta x' = -A\alpha\beta^{1/2}\sin(\psi+\delta) + A\beta^{1/2}\cos(\psi+\delta)$$

$$= -\alpha X + A\beta^{1/2}\cos(\psi+\delta)$$

$$\Rightarrow \quad \beta x' + \alpha X = A\beta^{1/2}\cos(\psi+\delta)$$

$$\therefore \quad \boxed{X^2 + (\beta x' + \alpha X)^2 = A^2\beta} \qquad \text{(circle)}$$

or)

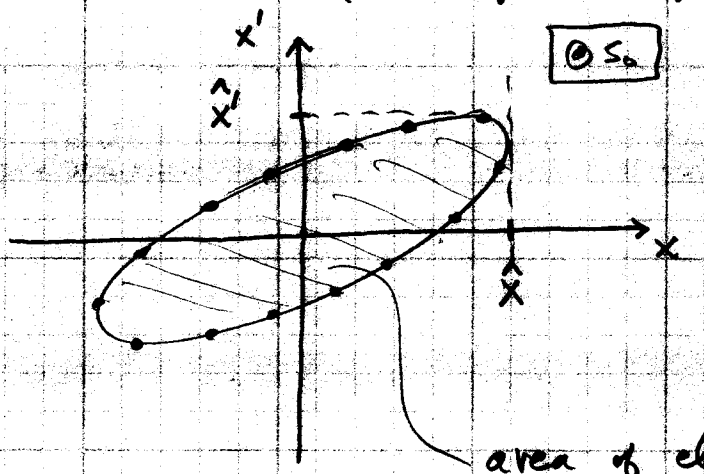
$$X^2 + \beta^2 x'^2 + 2\alpha\beta X x' + \alpha^2 X^2 = A^2\beta$$

$$X^2(1+\alpha^2) + 2\alpha\beta X x' + \beta^2 x'^2 = A^2\beta$$

$$\frac{1+\alpha^2}{\beta}X^2 + 2\alpha X x' + \beta x'^2 = A^2$$

$$\boxed{\gamma X^2 + 2\alpha X x' + \beta x'^2 = A^2} \qquad \text{(ellipse)}$$

note: γ, α, β are functions of s; A^2 is not!



@ s_0, we have

$$\hat{x} = A\sqrt{\beta_0}$$

$$\hat{x}' = A\sqrt{\gamma_0}$$

area of ellipse $= \pi A^2$

independent of s

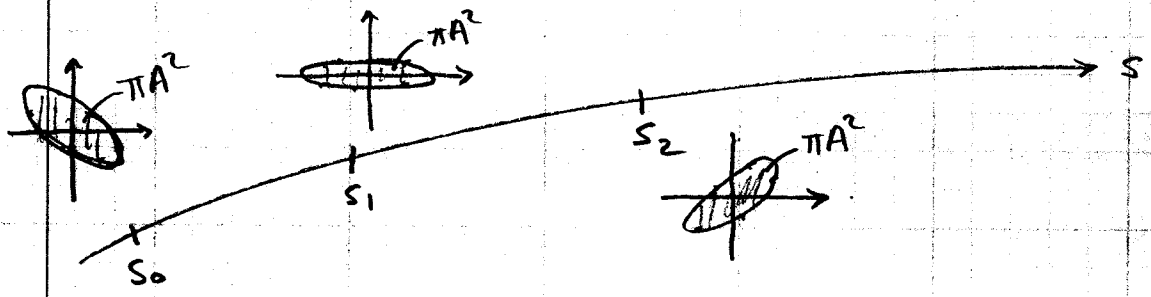
(Computer Demo)

Emittance ≡ phase space area

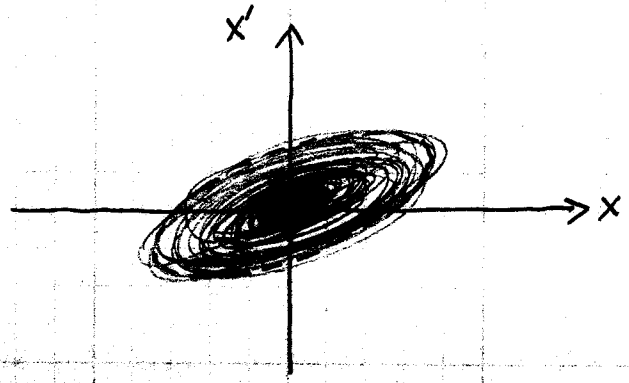
Single particle emittance... $\pi A^2 = \epsilon$, or $A = \sqrt{\epsilon/\pi}$

$$\rightarrow \quad x = \sqrt{\frac{\epsilon\beta(s)}{\pi}}\sin(\psi(s) + \delta)$$

$$\hat{x}_0 = \sqrt{\epsilon\beta_0/\pi} \quad \text{etc.}$$



How to describe a beam of particles?

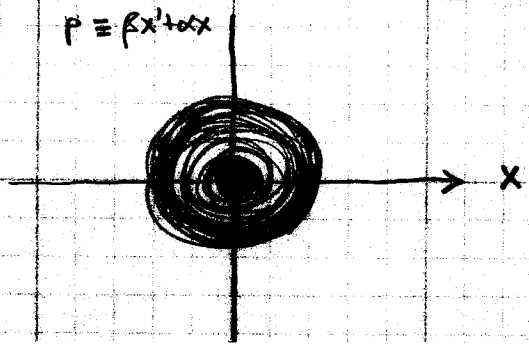


Suppose distribution is time independent (i.e., reached an equilibrium condition)

Find area of an ellipse that contains some fraction, f, of the particles.

a) Let's assume a Gaussian distribution, w/ $\langle x^2 \rangle = \sigma^2$

b) Sometimes easier to look at circles rather than ellipses...



So, distribution would be of the form

$$dn = N \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-p^2/2\sigma^2} dx\,dp$$

$$= \frac{N}{2\pi\sigma^2} e^{-(x^2+p^2)/2\sigma^2} dx\,dp = \frac{N}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta$$

Suppose radius a contains fraction f of the N particles...

$$\int_0^a \int_0^{2\pi} dn = \frac{N}{2\pi\sigma^2} \int_0^a \int_0^{2\pi} e^{-r^2/2\sigma^2} r\,d\theta\,dr$$

$$= N \int_0^a e^{-r^2/2\sigma^2} d\left(\frac{r^2}{2\sigma^2}\right) = N\left[1 - e^{-a^2/2\sigma^2}\right] = fN$$

$$\Rightarrow \quad 1-f = e^{-a^2/2\sigma^2} \quad \text{or} \quad a^2 = -2\sigma^2 \ln(1-f)$$

from page ②, area of ellipse is πA^2, and area of circle is $\pi A^2\beta = \pi a^2$

∴ for our calculation here, the x-x' phase space ellipse that contains fraction f of the particles has area

$$\boxed{\epsilon = -\frac{2\pi\sigma^2(s)}{\beta(s)} \ln(1-f)}$$

example: $f = 95\% \Rightarrow \ln(1-0.95) \approx -3$

$$\therefore \text{ 95\% emittance} \equiv \frac{6\pi\sigma^2}{\beta} \quad \text{(used @ FNAL)}$$

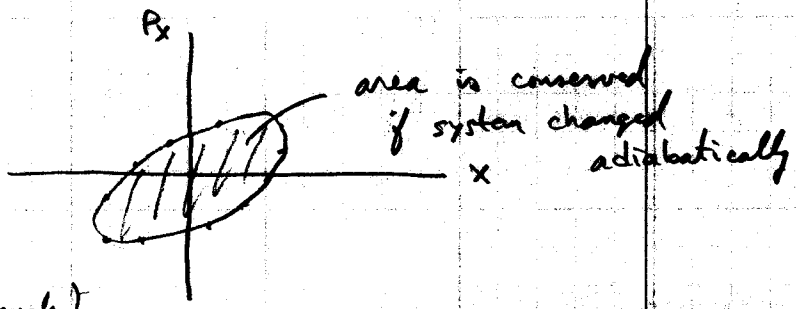
(typically...)

Adiabatic Damping / Normalized Emittance

As discussed earlier for longitudinal dynamics, convenient to talk about canonical variables

$x - x'$ are not

but $x - p_x$ are $\Rightarrow$



$p_x = p \cdot x'$ (x' is small angle)

so, we scale the emittance with momentum and define a Normalized emittance:

$$\epsilon_N \equiv \frac{6\pi\sigma^2}{\beta}(\gamma\beta)$$

($\gamma\beta$: Lorentz; β: Lattice)

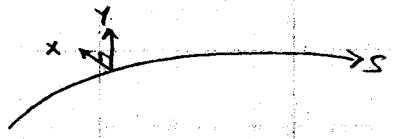
Note: if accelerate relatively slowly (w.r.t. β-tron motion), which we do, the ϵ_N should be preserved

$$\Rightarrow \quad \sigma(s) = \sqrt{\frac{\epsilon_N \beta(s)}{6\pi(\gamma\beta)}} \propto \frac{1}{\sqrt{\text{momentum}}}$$

i.e., betatron oscillations, and hence beam size, shrink as accelerate to higher energy

Linear Errors

So far, our particles have not been "ideal," but the accelerator has been. Now, we introduce imperfections to the magnets...



Steering errors:

dipole magnet field error: $B_0 \rightarrow B_0 + \Delta B$

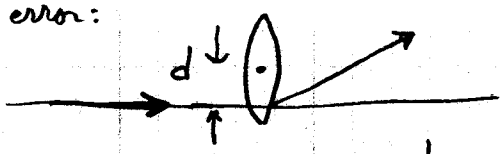
$$\Rightarrow \Delta\theta_x = -\frac{\Delta B \cdot l}{B\rho}$$

dipole magnet roll error: $B_y = B_0$, $B_x = 0$

$$\Rightarrow \begin{cases} B_y = B_0 \cos\phi \approx B_0 \\ B_x = B_0 \sin\phi \approx \phi B_0 \end{cases}$$

$$\Rightarrow \Delta\theta_y = +\phi \frac{B_0 l}{B\rho}$$

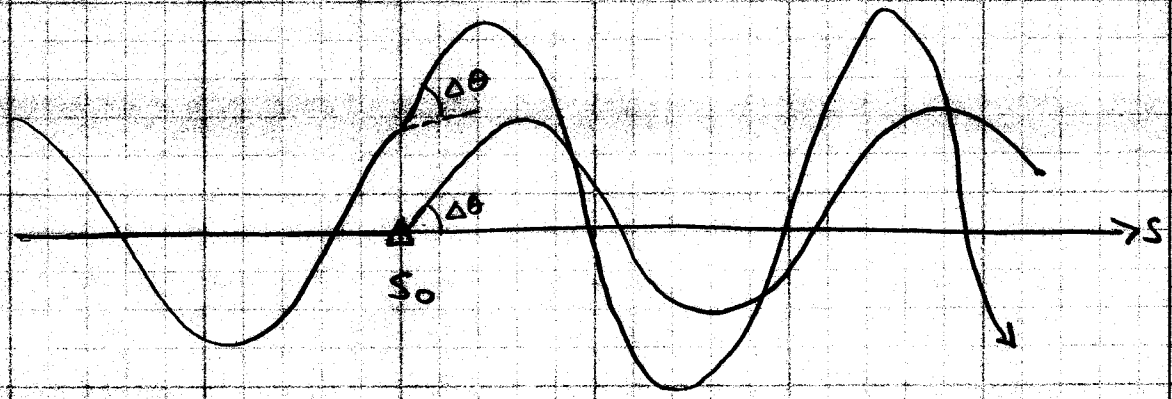
quadrupole alignment error:



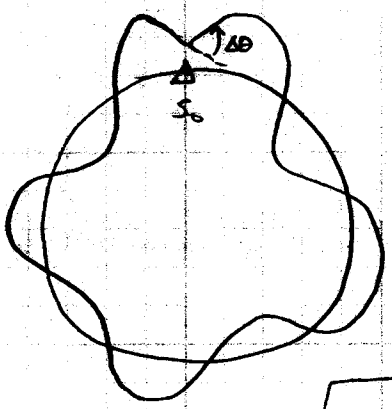
$$\Delta\theta_{x,y} = \frac{d_{x,y}}{f}$$

These are not one-time kicks; they affect the particle motion each revolution...

(Computer Demo)



However, a closed path, or "closed orbit" still exists...



Want to find initial conditions @ s_0, (x_0, x_0') so that after one revolution we come back to s_0, pass through the dipole error, and end up with (x_0, x_0') again....

$$M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \left(I - M_0\right)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} \qquad M_0 = e^{J_0 \mu} = e^{J_0 2\pi\nu}$$

$$= \left(I - e^{J_0\mu}\right)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \left[e^{J_0\mu/2} \left(e^{-J_0\mu/2} - e^{+J_0\mu/2} \right) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \left[e^{J_0\mu/2} \left(-2 J_0 \sin\frac{\mu}{2} \right) \right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= -\frac{1}{2\sin\pi\nu} J_0^{-1} e^{-J_0\mu/2} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$= \frac{1}{2\sin\pi\nu} J_0 \left(I\cos\pi\nu - J_0 \sin\pi\nu \right) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

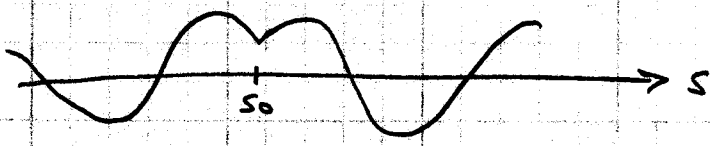
$$= \frac{1}{2\sin\pi\nu} \left(I\sin\pi\nu + J_0 \cos\pi\nu \right) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0 \cos\pi\nu \end{pmatrix}$$

given these initial conditions, can propagate around the ring to find the orbit distortion everywhere...

$$\Delta X(s) = \frac{\Delta\theta \sqrt{\beta_0 \beta(s)}}{2 \sin \pi\nu} \cos\left[|\psi(s) - \psi_0| - \pi\nu \right]$$

a) symmetric about $s = s_0$, ∴ will see a "cusp" in the orbit @ s_0



b) denominator has $\sin\pi\nu \Rightarrow$ avoid ν = integer!

Of course, will be many errors in the accelerator, + closed orbit will be due to their superposition...

Should see (if uncorrected)

$$\Delta X_{rms} \approx \frac{\Delta\theta_{rms}}{2|\sin\pi\nu|} \sqrt{\beta_0 \langle\beta\rangle} \sqrt{\frac{N}{2}}$$

@ β_0 ↗ due to N errors

TEVATRON, if uncorrected, + if $\Delta\theta_{rms} = \frac{d_{rms}}{f} = \frac{0.5\,mm}{25\,m} = 2\cdot10^{-5}$

$$\Rightarrow \Delta X_{rms} \approx \frac{2\cdot10^{-5}}{2} \sqrt{(100\,m)(55\,m)} \sqrt{\frac{200}{2}}$$

$$\approx 7\ mm$$

note: $2.5 \cdot \Delta X_{rms} = 18mm$ = edge of pipe!

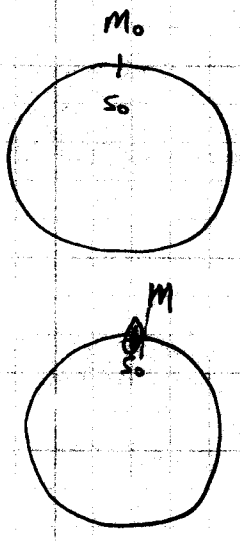
note: $\Delta\beta, \phi \Rightarrow$ even larger

Gradient errors

$$x'' + Kx = 0$$

spring constant determines frequency

$$\therefore \boxed{\Delta K \rightarrow \Delta \nu}$$



Suppose have ring w/ 1-turn transfer matrix M_0

introduce a thin gradient error with strength

$$q = \frac{\Delta B' l}{(Bp)} = \Delta K l \quad (\text{focal length } f = 1/q)$$

New matrix is $M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$

if $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $M = \begin{pmatrix} a & b \\ c-aq & d-bq \end{pmatrix}$

$$\therefore \text{trace } M = 2\cos 2\pi\nu = a + d - bq$$

$$= \text{trace } M_0 - bq$$

$$= 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0) q$$

if q is small enough, and $\nu = \nu_0 + \Delta\nu$,

then $\cos 2\pi\nu_0 \cos 2\pi\Delta\nu - \sin 2\pi\nu_0 \sin 2\pi\Delta\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0$

$$\Rightarrow \quad 2\pi\Delta\nu \approx \frac{1}{2} q\beta_0$$

or,

$$\boxed{\Delta\nu = \frac{1}{4\pi} \beta q} \quad \text{***}$$

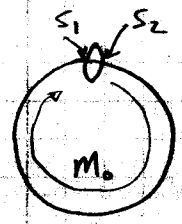
note:

$$\left.\begin{aligned} \Delta\nu_x &= \frac{1}{4\pi} \beta_x q \\ \Delta\nu_y &= -\frac{1}{4\pi} \beta_y q \end{aligned}\right\} \text{ due to some error, } q.$$

ex: need 2 quads (or quad circuits) to independently adjust ν_x + ν_y

Effect of ΔK on $\beta(s)$ ...

look @ either side of a quadrupole magnet, strength $q = \frac{1}{f}$



$$M_1 = M_0 \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} a - bq & b \\ c - dq & d \end{pmatrix}$$

let $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

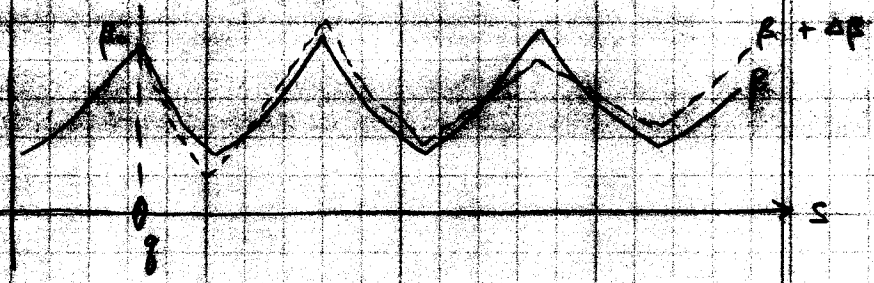
$$M_2 = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0 = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$M_2 - M_1 = J_2 \sin 2\pi\nu - J_1 \sin 2\pi\nu$$

$$= \begin{pmatrix} \Delta\alpha & \Delta\beta \\ -\Delta\gamma & -\Delta\alpha \end{pmatrix} \sin 2\pi\nu = \begin{pmatrix} bq & 0 \\ -(a-d)q & -bq \end{pmatrix}$$

$$= \begin{pmatrix} \beta q & 0 \\ \text{---} & -\beta q \end{pmatrix} \sin 2\pi\nu \quad \therefore \boxed{\Delta\alpha = \beta q = \beta \frac{\Delta B' l}{B\rho}}$$

can show that an error $\Delta\alpha = \beta_0 q$ @ s_0, will generate a $\frac{\Delta\beta}{\beta}$ downstream of the error according to -



$$\frac{\Delta\beta}{\beta}(s) = \underbrace{-q\beta_0 \sin 2(\psi - \psi_0)}_{\text{---}} + \frac{1}{2}(q\beta_0)^2 (1 - \cos 2(\psi - \psi_0))$$

so, to lowest order, introduces a "β-wave" in the accelerator:

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{q\beta_0}{2\sin 2\pi\nu} \cos[2|\psi - \psi_0| - 2\pi\nu]$$

avoid $\frac{1}{2}$-integer tune!

Off-momentum particles (again) -- Transverse effects

Let p_0 = ideal momentum, but consider a particle with $p = p_0 + \Delta p$, $\frac{\Delta p}{p_0} \ll 1$.

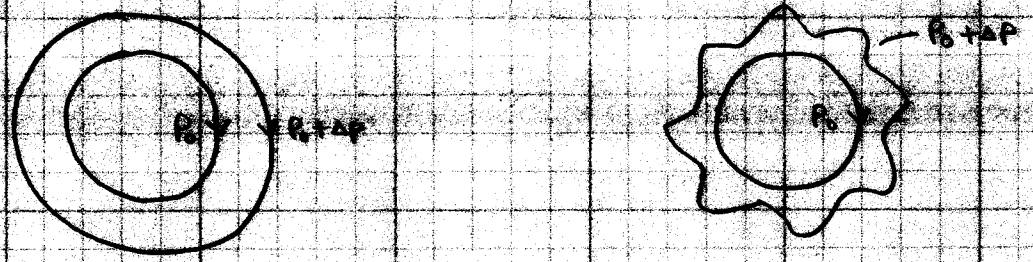
To first order,

a) dipole magnets will not "bend" as much ($p > p_0$), and so will introduce steering "errors"
→ new closed orbit, which is $\propto \frac{\Delta p}{p_0}$

b) quadrupoles will not "focus" as much, and so will introduce gradient "errors"
→ new tune, which is $\propto \frac{\Delta p}{p_0}$

Dispersion

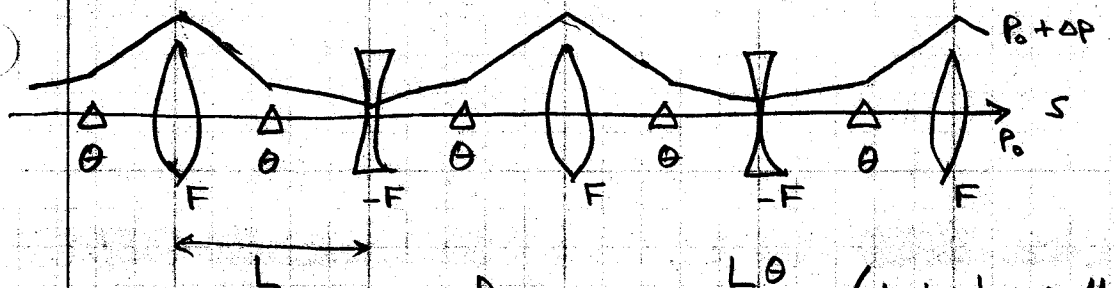
uniform bend field: add gradients...



the orbit of an off-momentum particle which closes on itself is given by the Dispersion Function,

$$D(s,p) = \frac{\Delta x(s,p)}{\Delta p/p_0}$$

ex: FODO cells w/ bending...



$$D_{max,min} = \frac{L\theta}{2\sin^2\frac{\mu}{2}}\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)$$

$\left(\sin\frac{\mu}{2} = \frac{L}{2F}\right)$

~ few meters

effect on transverse particle distribution... $A\sqrt{\beta}\sin\phi$

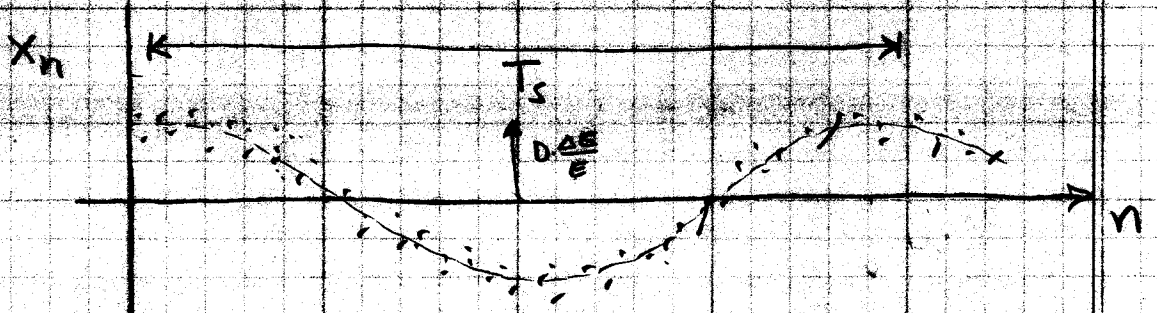
$$x = x_\rho + x_\beta = D\frac{\Delta\rho}{\rho_0} + x_\beta$$

so, $$x^2 = D^2\left(\frac{\Delta\rho}{\rho_0}\right)^2 + 2D\,x_\beta\frac{\Delta\rho}{\rho_0} + x_\beta^2$$

$$\Rightarrow \langle x^2\rangle = D^2\left\langle\left(\frac{\Delta\rho}{\rho}\right)^2\right\rangle + \langle x_\beta^2\rangle \qquad \left(x_\beta,\ \frac{\Delta\rho}{\rho}\ \text{uncorrelated}\right)$$

or,
$$\boxed{\sigma_{TOT} = \sqrt{D^2\left(\frac{\sigma_\rho}{\rho}\right)^2 + \frac{\epsilon_N\beta}{6\pi(\gamma\beta)}}}$$

ex: injection oscillation w/ energy offset, ΔE: T-B-T display...



note: particle with $p = p_0 + \Delta p$ will travel different path, and hence will have different path length according to

$$\frac{\Delta C}{C} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p} = \underbrace{\frac{1}{\gamma_t^2} \frac{\Delta p}{p}}_{\text{by definition}}$$

So,

$$\boxed{\frac{1}{\gamma_t^2} = \left\langle \frac{D}{\rho} \right\rangle}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

can show that, for FODO-type lattices, $\boxed{\gamma_t \approx \nu}$

Chromaticity

$$x'' + \underbrace{K(s)}_{= \frac{B'}{B\rho} = \frac{eB'}{p}} x = 0$$

$\therefore \Delta p \rightarrow \Delta K \rightarrow \Delta \nu$!

we saw that $\Delta \nu = \frac{1}{4\pi} \beta g$ due to gradient error, g

if "g" is due to momentum error: $g \rightarrow -\frac{eB'}{p} \frac{\Delta p}{p} = -K \frac{\Delta p}{p}$

then,

$$\Delta \nu = -\frac{1}{4\pi} \oint K \beta \, ds \cdot \frac{\Delta p}{p}$$

chromaticity $\xi \equiv \Delta \nu / (\Delta p / p)$

So, the natural chromaticity of a synchrotron is

$$\xi_{nat} = -\frac{1}{4\pi} \oint K(s) \beta(s) \, ds$$

can show that for a FODO-type lattice,

$$\boxed{\xi_{nat} \approx -\nu}$$

ex: Tevatron, $\nu \approx 20$; if $\frac{\sigma_p}{p} = 10^{-5}$, then will have a tune spread of $\pm 2 \cdot 10^{-4}$ about the nominal tune

may not want a tune spread (also that <u>some</u> tunes are bad!)

can adjust using <u>sextupoles</u>...

Sextupole field: $B_y = \frac{1}{2} B'' (x^2 - y^2)$

$B_x = B'' x y$

in horizontal plane ($y=0$), beam spreads out due to dispersion...

$$\frac{\partial B_y}{\partial x} = B'' x \rightarrow B'' D \frac{\Delta p}{p}$$

$$\Delta\nu = \frac{1}{4\pi} \beta \frac{B''\ell}{Bp} D \frac{\Delta p}{p} \rightarrow \boxed{\Delta\xi = \frac{1}{4\pi} \beta D \frac{B''\ell}{Bp}}$$

due to sextupole field

$$\Delta\xi_x = \frac{1}{4\pi} \beta_x D \frac{B''\ell}{Bp}$$

$$\Delta\xi_y = -\frac{1}{4\pi} \beta_y D \frac{B''\ell}{Bp}$$

} both due to D in the horiz. plane!

note: sextupole field $\Rightarrow$ NONLINEARITY !

Problems:

① Suppose a quadrupole in the Main Injector is displaced transversely by 1 mm. What would be the resulting orbit distortion maximum in the arcs? (Assume the offset quad is in a FODO arc cell.)

② Suppose a quadrupole in the Main Injector is wired backwards. Would the resulting transverse motion be stable?

③ Show that the natural chromaticity in a synchrotron is given by

$$\xi_{nat} = -\frac{K}{2}\langle\gamma\rangle$$

where $2\pi R$ is the ring circumference, and $\gamma = \frac{1+\alpha^2}{\beta}$.

From this, show that $\xi_{nat} \propto -\nu$ for a FODO accelerator. [Hint: γ = constant in a drift region; can you prove this?]

④ Suppose we have 2 Flying Wire located where

$\beta_1 = 100\,m$, $D_1 = 6\,m$ and

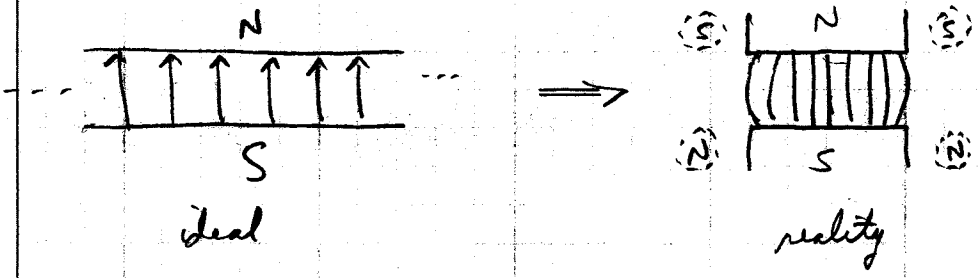
$\beta_2 = 180\,m$, $D_2 = 2\,m$

and we measure $\sigma_1 = 2.5\,mm$ and $\sigma_2 = 2\,mm$ @ 150 GeV.

What are ϵ_N and $\frac{\sigma_p}{p}$?

Other sources of chromaticity ...

major contribution can be from the dipole magnets ...



real "dipole" magnet: $B_y = B_0 (1 + b_2 x^2 + b_4 x^4 + \dots)$

↑ sextupole coefficient

Suppose the bending magnets are constructed with $b_2 \neq 0$ (which they will be); then,

$$\frac{\partial B_y}{\partial x} = 2 B_0 b_2 x$$

$B\rho = B_0 \cdot \rho$ in dipoles

$$\Rightarrow \Delta\xi = -\frac{1}{4\pi} \oint \frac{2 B_0 b_2}{B\rho} D \beta_x \, ds$$

$$= -\frac{1}{2\pi} \oint \frac{b_2 D \beta_x}{\rho} ds = \frac{\oint b_2 D \beta_x \, ds}{2\pi\rho}$$

$$= \langle b_2 D \beta \rangle$$ averaged over the bend magnets

let's estimate: $\langle D \rangle \approx 4\,m$, $\langle \beta \rangle \approx 55\,m$, and for TEV dipoles @ 150 GeV current, $\langle b_2 \rangle \approx 1/m^2$

$\Rightarrow \xi$ due to b_2 is ~ 200 units!

also, it drifts with time!

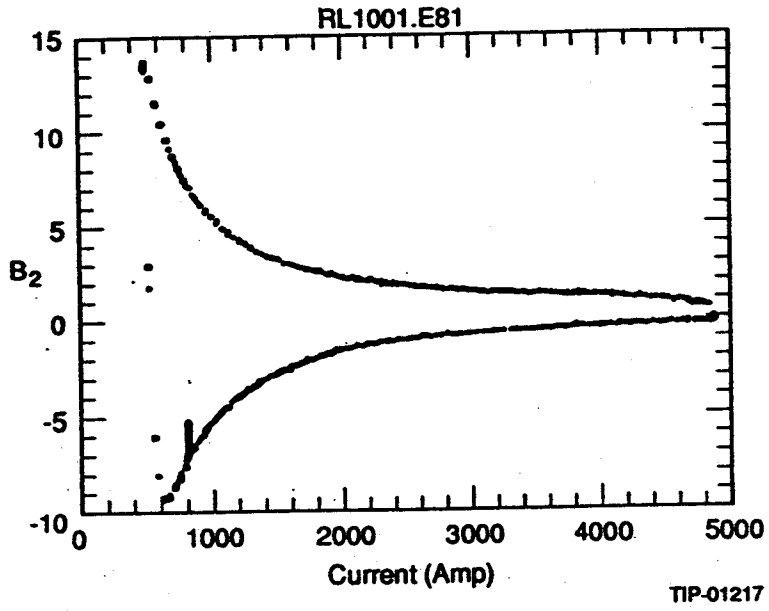


Figure 10: Sextupole moment in a model Tevatron dipole magnet as function of excitation current. Here, the current was held constant at 800 A for 15 min.; the sextupole moment drifted by more than 2 units.

↳ $\Delta\xi = (4m)(55m)(0.3/m^2) = 66$

@ FNAL, "1 unit" = 1 part in 10^4, measured @ x = 1 inch

(during TEVATRON construction days...)

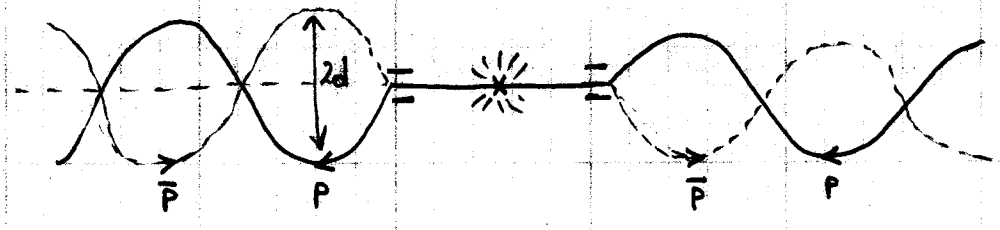
$\Rightarrow$ 2 units = $.2 \cdot 10^{-4} \cdot \left(\frac{1}{0.0254m}\right)^2$

$= 0.3/m^2$

So, if $\frac{\sigma_p}{p} = 10^{-4}$, say, and ξ changes by 66 units, then a typical particle's tune will change by $\Delta\nu \approx 0.007$

Use of sextupoles for Feed-down tune correctors

$\bar{p}p$ → use Electrostatic Separators to split p, $\bar{p}$ orbits



If have distribution of sextupoles correctors around the ring, then

$$\Delta\nu_x^p = +\frac{1}{2\pi}\sum_i \beta_{xi} S_i d_i \qquad (d>0) \qquad S \equiv \frac{\beta'' l}{2(B\rho)}$$

$$\Delta\nu_y^p = -\frac{1}{2\pi}\sum_i \beta_{yi} S_i d_i$$

$$\Delta\nu_x^{\bar{p}} = -\frac{1}{2\pi}\sum_i \beta_{xi} S_i d_i$$

$$\Delta\nu_y^{\bar{p}} = +\frac{1}{2\pi}\sum_i \beta_{yi} S_i d_i$$

∴ use 4 circuits of S's to independently control p, $\bar{p}$ tunes

Note: if helix changes, then the "coefficients" of the circuits change ⇒ will need to retune the algorithm

Sextupoles necessary for ξ control, ν control on helix, etc.

but, introduces nonlinearity into our "ideal" accelerator

Just as in case of longitudinal dynamics,

- ν will vary with particle amplitude
- regions of stability / instability will exist

⇒ dynamic aperture

Phase Space due to single sextupole...

@ sextupole location: $\Delta x' = -\frac{B_y \ell}{(B\rho)} = -\frac{B'' \ell}{2(B\rho)} x^2 \equiv -S x^2$ $(y=0)$

∴ going thru the sextupole, then once around, looks like:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} \begin{pmatrix} x_n \\ x'_n - S x_n^2 \end{pmatrix}$$

transform to x, $p = \alpha_0 x + \beta_0 x'$ coordinates:

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x_n \\ p_n - \beta_0 S x_n^2 \end{pmatrix}$$

transform again: $u \equiv \beta_0 S x$, $v \equiv \beta_0 S p$ note: $\mu = 2\pi\nu$

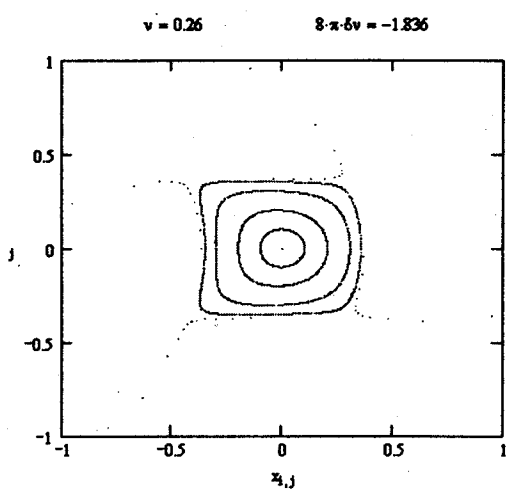
$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \begin{pmatrix} u_n \\ v_n - u_n^2 \end{pmatrix}$$

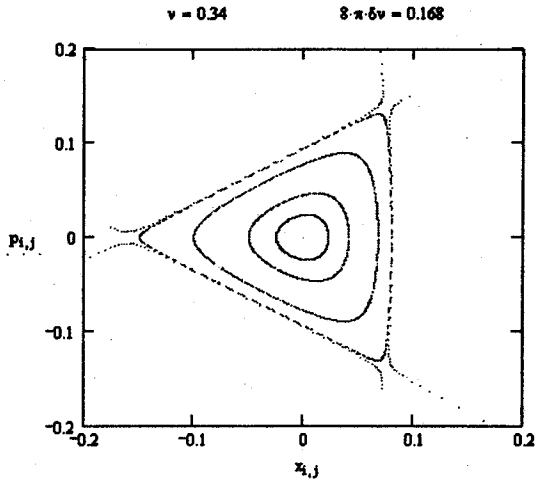
→ the phase space scales like $\beta \cdot S$.
i.e., if S is 2x stronger, things happen 2x closer to the origin

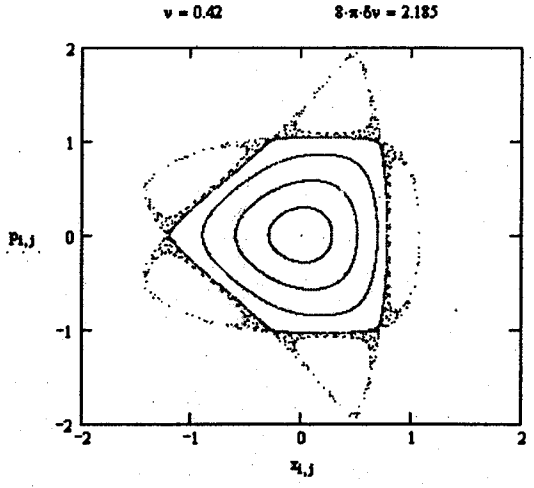
the topology of the phase space only depends upon the tune, ν

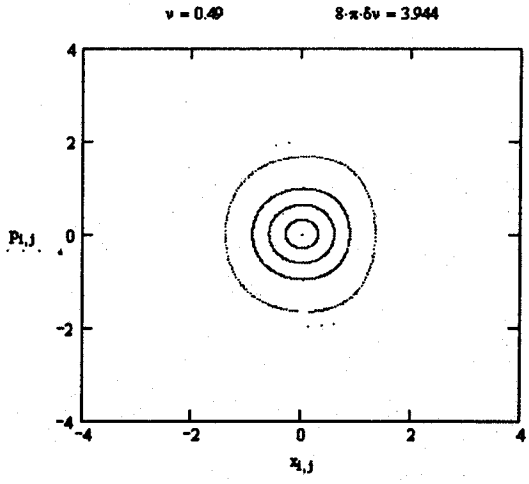
look @ what happens: track particles

movies: resonant extraction


ν = 0.26
8·π·δν = −1.836
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
x1,j


ν = 0.34
8·π·δν = 0.168
0.2
0.1
0
−0.1
−0.2
p1,j
−0.2
−0.1
0
0.1
0.2
x1,j


ν = 0.42
8·π·δν = 2.185
2
1
0
−1
−2
p1,j
−2
−1
0
1
2
x1,j


ν = 0.49
8·π·δν = 3.944
4
2
0
−2
−4
p1,j
−4
−2
0
2
4
x1,j

Dipole field errors → $\nu = n$ is bad!

Quadrupole field errors → $\nu = n/2$ is bad

Sextupole field errors → $\nu = n/3$ is bad

⋮

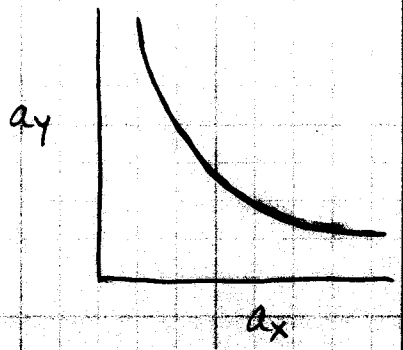
in general,

$$m\nu_x \pm n\nu_y = k$$

coupling resonances: particle motion shared / exchanged: hor / ver

sum resonance:

$$m\nu_x + n\nu_y = k$$

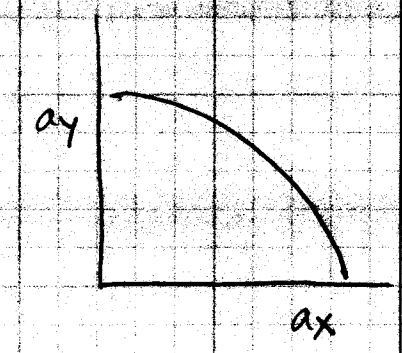


difference resonance

$$m\nu_x - n\nu_y = k$$

amplitudes are bounded,

∴ "not as bad"



tune plots ...