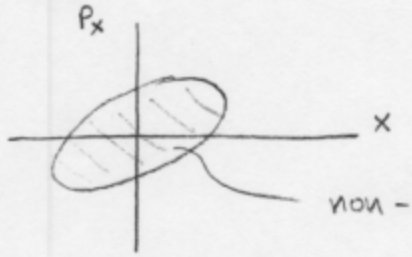
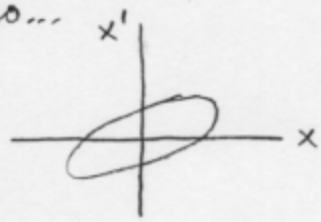
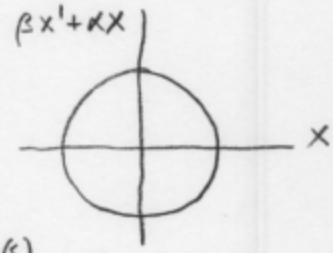


non-dissipative process $\Rightarrow$ area is adiabatic invariant

accelerator lingo...



or



$\langle x^2 \rangle$ is averaged over particle distribution

$$\epsilon_n \equiv \frac{6\pi \langle x^2 \rangle (s)}{\beta(s)} (\beta\gamma)$$

Will look at 2 types of disruptive processes:

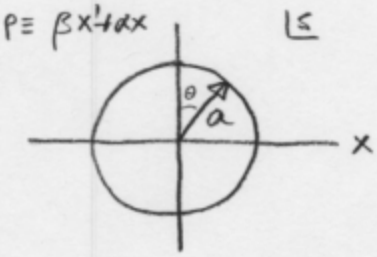
a) Single non-adiabatic disturbance of distribution

ex's:
- injection errors: steering, focusing
- separator spark
- single pass through a vacuum window
- pinger excitation
- flying wire measurement

b) Random disturbances of individual particles leading to diffusion

ex's:
- rf noise
- beam-gas scattering
- power supply noise
- mechanical vibration

Examine a <u>single</u> particle (easiest in $x - (\beta x' + \alpha x)$ space)



$$a^2 = x^2 + p^2$$

$$\langle x \rangle = 0 \; ; \quad \langle x^2 \rangle = \langle p^2 \rangle = \tfrac{1}{2} a^2$$

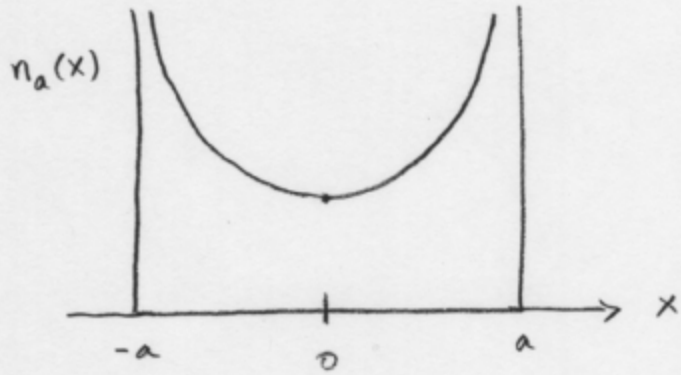
$$\therefore \; x_{rms} = \frac{\sqrt{2}}{2} a$$

prob. of finding particle between x + $x + dx$: $n_a(x)$

$$n_a(x)\, dx = \underset{\text{top, bott.}}{2} \cdot \frac{d\theta}{2\pi}$$

$$x = a \sin\theta$$
$$dx = a \cos\theta \, d\theta = a \sqrt{1 - \left(\tfrac{x}{a}\right)^2}\, d\theta$$

$$\boxed{n_a(x) = \frac{1}{\pi a} \frac{1}{\sqrt{1 - (x/a)^2}}}$$



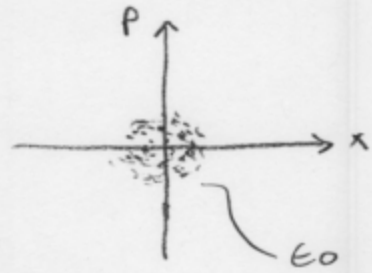
to study result of a disturbance to the distribution, look at how amplitudes a are affected, then average over time to determine the net equilibrium result

Remember: $\nu = \nu(a)$ due to nonlinearities

$\therefore$ motion will tend to decohere...

ex: injection steering mismatch...

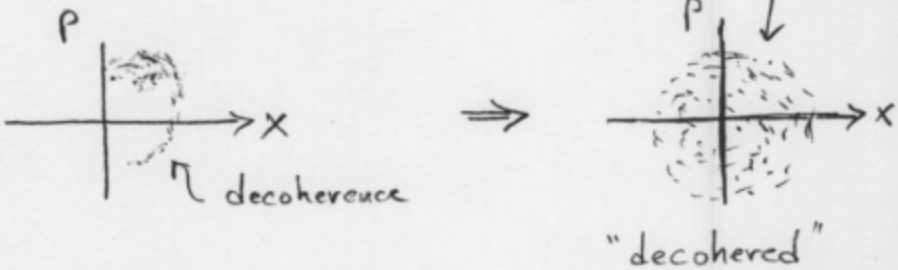
ideal distribution at injection:



steering error: $t=0$

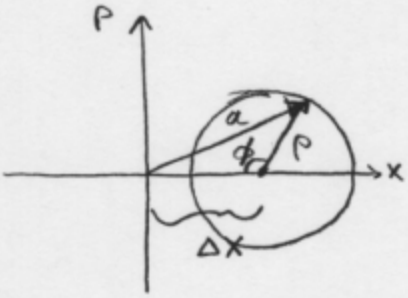
later t...

much later t...



(movie)

how does rms evolve?



if $\Delta x = 0$, would have

$$\langle x^2 \rangle = \frac{1}{2}\rho^2 \equiv \sigma_0^2$$

but here,

$$a^2 = \rho^2 + \Delta x^2 - 2\Delta x\, \rho \cos\phi$$

average over all particles... $\langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2 - 2\Delta x \underbrace{\langle \rho \cos\phi \rangle}_{\sim \phi}$

$$\Rightarrow \langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2 \quad \text{after decoherence}$$

$$\Rightarrow \langle x^2 \rangle = \sigma_0^2 + \frac{1}{2}\Delta x^2 \quad \text{"}$$

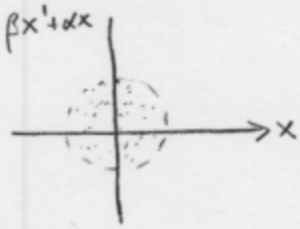
if $\epsilon \equiv \frac{6\pi\sigma^2}{\beta}\cdot(\gamma\beta) \longrightarrow \boxed{\Delta\epsilon_N = \frac{3\pi\,\Delta x^2}{\beta}(\gamma\beta)}$ or, $\epsilon/\epsilon_0 = 1 + \frac{1}{2}\left(\frac{\Delta x}{\sigma_0}\right)^2$

β @ location of Δx

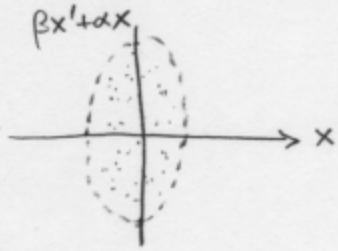
another ex.: passage through vacuum window/foil in beamline...

$\Rightarrow$ multiple Coulomb scattering: $\Delta\theta_{rms}^{(x,y)} \approx \frac{13.6\,\text{MeV}}{p v} \sqrt{\frac{l}{L_{RAD}}}$

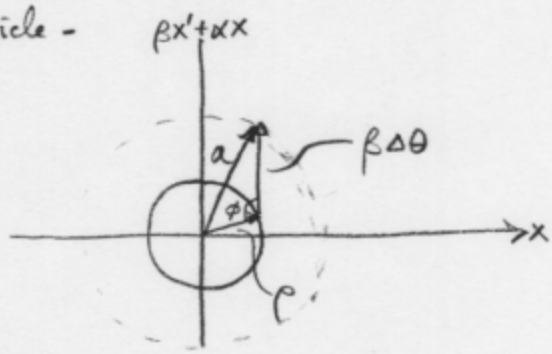
Just before foil:



Just after foil



individual particle -



$$a^2 = \rho^2 + (\beta\Delta\theta)^2 - 2\rho\beta\Delta\theta\cos\phi$$

$$\Downarrow$$

$$\langle a^2\rangle = \langle\rho^2\rangle + \beta^2\langle\Delta\theta^2\rangle$$

$\therefore$ after resulting decoherence, $\underbrace{\langle x^2\rangle}_{\sigma^2} = \frac{1}{2}\langle a^2\rangle = \underbrace{\frac{1}{2}\langle\rho^2\rangle}_{\sigma_0^2} + \frac{1}{2}\beta^2\Delta\theta_{rms}^2$

or,

$$\Delta\epsilon_N = 3\pi\beta\cdot\left(\frac{13.6\,\text{MeV}}{p v}\right)^2 \frac{l}{L_{RAD}}(\gamma\beta)$$

Note:

injection errors - $\Delta\epsilon_N \propto (\gamma\beta)$, $\therefore$ more important @ higher energies

let $\Delta X = 1\,\text{mm}$ @ injection to

	$\hat{\beta}$	$\gamma\beta$	$\Delta\epsilon_N$
BOO	30m	1.017	0.10 π mm-mr
MI	50m	9.48	0.57 π mm-mr
TEV	100m	160	4.8 π mm-mr ☆

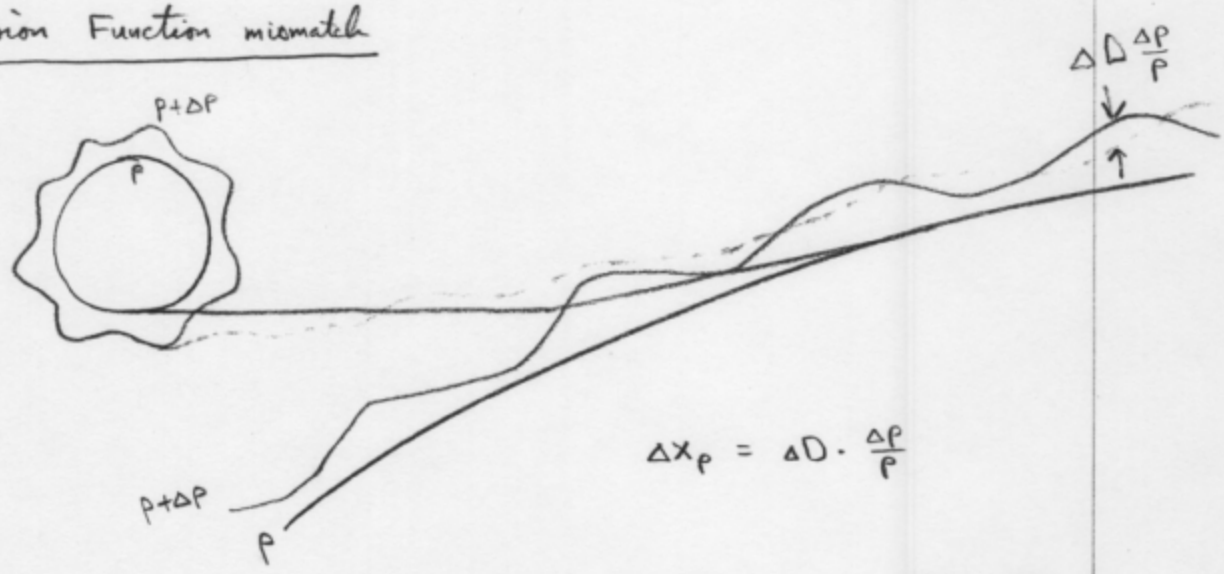
Ti vacuum window between MI + RR:

$$\Delta\epsilon_N = 3\pi(50\,\text{m})\left[\frac{13.6}{938}\right]^2 \cdot \frac{(0.005'')(2.54\,\text{cm/in.})}{3.56\,\text{cm}}\,\frac{1}{9.48} = \underline{\underline{12\pi\ \text{mm-mr}}}$$

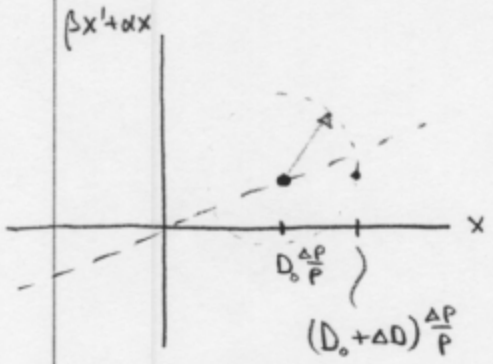
(since removed!)

Optics mismatches ...

Dispersion Function mismatch



$$\Delta x_p = \Delta D \cdot \frac{\Delta p}{p}$$



each particle of momentum $\frac{\Delta p}{p}$ will see an injection steering error of amplitude $x_p = \Delta D \frac{\Delta p}{p}$

$$\Rightarrow \quad a^2 = \rho^2 + \Delta D^2 \left(\frac{\Delta p}{p}\right)^2$$

$$\Rightarrow \quad \langle a^2 \rangle = \langle \rho^2 \rangle + \Delta D^2 \left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle$$

and so,

$$\boxed{\Delta \epsilon_N = \frac{3\pi \, \Delta D^2(s)}{\beta(s)} \left(\frac{\sigma_p}{p}\right)^2 (\gamma\beta)}$$

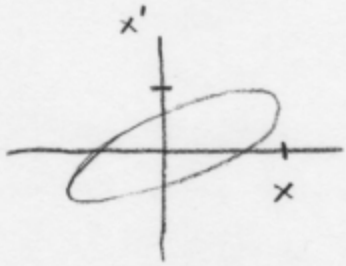
worse for larger σ_p/p (coalesced!)

ex: coalesced beam, w/ $\frac{\sigma_p}{p} = 5 \times 10^{-4}$, and $\Delta D \approx 1\,\text{m}$ from MI → TEV

$$\Rightarrow \Delta \epsilon_N \approx \frac{3\pi \, (1\,\text{m})^2}{100\,\text{m}} \left(0.5 \times 10^{-3}\right)^2 (160) = 1.2\ \pi \text{ mm-mrad}$$

Amplitude Function mismatch

remember:

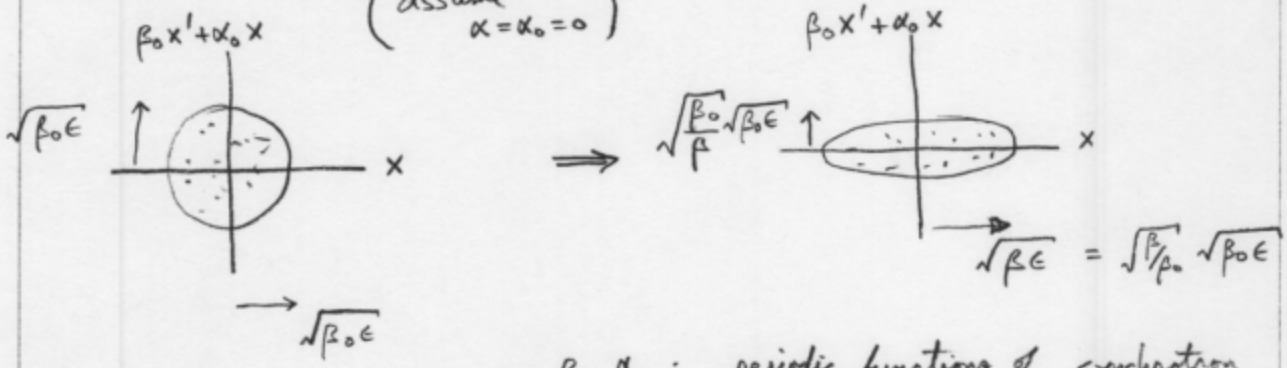


$$\hat{x} \sim \sqrt{\beta\epsilon}$$

$$\hat{x}' \sim \sqrt{\gamma\epsilon} = \sqrt{(1+\alpha^2)\frac{\epsilon}{\beta}}$$

∴ in "circular" phase space ...

(assume $\alpha = \alpha_0 = 0$)



β_0, α_0 : periodic functions of synchrotron

β, α : delivered by the beamline

then, after the distribution tumbles and filaments in phase space, the emittance will grow ...

$$a^2 = x^2 + (\beta_0 x' + \alpha_0 x)^2 \equiv x^2 + p^2$$

let $b \equiv \beta/\beta_0 \Rightarrow$

$$\langle a^2 \rangle = \langle x^2 \rangle + \langle p^2 \rangle$$
$$= (\sqrt{b}\,\sigma_0)^2 + \left(\frac{1}{\sqrt{b}}\sigma_0\right)^2$$

$$\Rightarrow 2\sigma^2 = \frac{b^2+1}{b}\sigma_0^2 \Rightarrow \boxed{\epsilon/\epsilon_0 = \frac{b^2+1}{2b}}$$

(movie)

Let $\beta = \beta_0 + \Delta\beta$ (remember, assume $\alpha = \alpha_0 = 0$),

then $b = \beta/\beta_0 = 1 + \Delta\beta/\beta_0$

$$\Rightarrow \quad \epsilon/\epsilon_0 = \sigma^2/\sigma_0^2 = \frac{1 + \frac{\Delta\beta}{\beta_0} + \frac{1}{2}\left(\frac{\Delta\beta}{\beta_0}\right)^2}{1 + \frac{\Delta\beta}{\beta_0}}$$

or,
$$\boxed{\epsilon/\epsilon_0 = 1 + \frac{1}{2}\frac{(\Delta\beta/\beta_0)^2}{1 + \Delta\beta/\beta_0}}$$

or,
$$\boxed{\Delta\epsilon = \frac{\epsilon_0}{2}\frac{(\Delta\beta/\beta_0)^2}{1 + \Delta\beta/\beta_0}}$$

note: $\Delta\epsilon \propto \epsilon_0$!

Suppose β coming from beamline is off by

$$10\% \longrightarrow \frac{\Delta\epsilon}{\epsilon_0} = 0.45\%$$

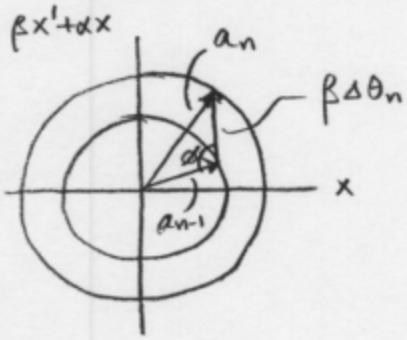
$$50\% \longrightarrow \quad = 8.3\%$$

$\therefore$ not as sensitive as steering (+ dispersion) errors, typically

---

Next, consider diffusive mechanisms...

each particle's amplitude is changed "continuously" by a random process...



$$a_1^2 = a_0^2 + \beta^2\Delta\theta_1^2 - 2a_0\beta\theta_1\cos\phi_1$$

$$a_n^2 = a_{n-1}^2 + \beta^2\Delta\theta_n^2 - 2a_{n-1}\beta\theta_n\cos\phi_n$$

average over all particles...

$$\langle a^2\rangle_n = \langle a^2\rangle_{n-1} + \beta^2\langle\Delta\theta^2\rangle_n - \cancel{2\beta\langle a\rangle\langle\theta\rangle_n\langle\cos\phi\rangle_n} \to 0$$

$$\therefore \; d\langle a^2\rangle = \beta^2\theta_{rms}^2\, dn$$

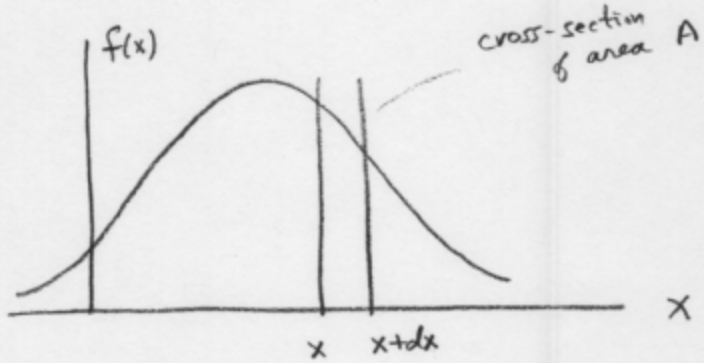
$$\Rightarrow \quad \frac{d\langle x^2\rangle}{dn} = \frac{1}{2}\beta^2 \Delta\theta_{rms}^2 \qquad \text{or} \qquad \frac{d\epsilon_N}{dn} = 3\pi\beta\,\Delta\theta_{rms}^2 \cdot (\gamma\beta)$$

assumes $\Delta\theta$ is random turn-to-turn,

and particle-to-particle

Diffusion Equation

Look @ 1-D first:



$$\frac{\partial}{\partial t}\left(f \cdot A \cdot dx\right) = \underbrace{A \cdot J(x)}_{\#\text{ into}} - \underbrace{A \cdot J(x+dx)}_{\#\text{ out of}}$$

J = average # particles per unit time passing position x

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x}$$

if f uniform, then $J = 0$; otherwise $J \propto -\frac{\partial f}{\partial x}$

$$\therefore \quad \boxed{\frac{\partial f}{\partial t} = C \cdot \frac{\partial^2 f}{\partial x^2}} \qquad C = \text{constant}$$

more general 3-D case : $\boxed{\frac{\partial f}{\partial t} = C\,\nabla^2 f}$

For us, a 2-D problem ...

$$\frac{\partial f}{\partial t} = C \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)$$

if define $\epsilon_p \equiv r^2/\beta$ and $R \equiv \frac{\partial}{\partial t}\langle r^2/\beta \rangle$

then we get

$$\boxed{\frac{\partial f}{\partial t} = R \frac{\partial}{\partial \epsilon_p}\left(\epsilon_p \frac{\partial f}{\partial \epsilon_p}\right)}$$

So, apply boundary conditions...

$f = f(r,t) \longrightarrow f(r,0) = f_0(r)$ ex: $f = \frac{1}{\sigma^2} e^{-r^2/2\sigma^2}$

and $f(a,t) = 0$ i.e., an aperture at $r = a$

define dimensionless variables $Z \equiv \epsilon_p/w$

$\tau \equiv (R/w)\,t$, $w \equiv a^2/\beta$ (~admittance)

$$\Rightarrow \boxed{\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial z}\left(z \frac{\partial f}{\partial z}\right) \quad \text{and} \quad \begin{array}{l} f(z,0) = f_0(z) \\ f(1,\tau) = 0 \end{array}}$$

<u>Solution</u>:

$$f(z,\tau) = \sum_n c_n J_0(\lambda_n \sqrt{z})\, e^{-\lambda_n^2 \tau/4}$$

$$\text{where } c_n = \frac{1}{J_1(\lambda_n)^2} \int_0^1 f_0(z) J_0(\lambda_n \sqrt{z})\, dz$$

$\lambda_n = n^{th}$ zero of $J_0(z)$

So, for arbitrary (but cylindrically symmetric) initial distribution:

<u>INTENSITY vs. time</u> :

$$N(\tau) = \int_0^1 f(z,\tau)\,dz$$

$$= 2 \sum_n \frac{c_n}{\lambda_n} J_1(\lambda_n)\, e^{-\lambda_n^2 \tau/4}$$

<u>LIFETIME vs. time</u> :

$$\tau_L(\tau) \equiv -\frac{N(\tau)}{dN/d\tau} \longrightarrow \frac{4}{\lambda_1^2} \quad \text{as } t \to \infty$$

$\therefore$ asymptotic lifetime : $t_L = \frac{4}{\lambda_1^2}\left(\frac{W}{R}\right)$

<u>PARTICLE DISTRIBUTION vs. time</u> :

$$f(z,\tau) = \sum_n c_n J_0(\lambda_n \sqrt{z})\, e^{-\lambda_n^2 \tau/4}$$

$$\longrightarrow f_\infty(z) = c_1 J_0(\lambda_1 \sqrt{z})\, e^{-\lambda_1^2 \tau/4} \quad \text{as } \tau \to \infty$$

$$\therefore \langle z \rangle^\infty = \frac{\int_0^1 J_0(\lambda_1\sqrt{z})\, z\, dz}{\int_0^1 J_0(\lambda_1\sqrt{z})\, dz} = \frac{\int_0^1 u^3 J_0(\lambda_1 u)\, du}{\int_0^1 u\, J_0(\lambda_1 u)\, du} = 0.308$$

and so $\left\langle \frac{x^2}{\beta} \right\rangle^\infty = \frac{1}{2} \cdot W \cdot \langle z \rangle^\infty = 0.154\, W$

or $\langle x^2 \rangle^{1/2} \longrightarrow \sqrt{0.154}\, a = \underline{\underline{0.393\, a}}$

$\therefore$ <u>start</u> w/ $\epsilon_0 = \frac{6\pi\sigma_0^2}{\beta} \longrightarrow \epsilon_f = \frac{6\pi\langle x^2 \rangle}{\beta} = \frac{0.924\,\pi a^2}{\beta}$

$(\sigma_0 \ll a)$

So, look at some examples...

a) f_0 is Gaussian: i.e., $f_0 r dr = \frac{1}{\sigma_0^2} e^{-r^2/2\sigma_0^2} r dr$

$$= \frac{1}{2} e^{-r^2/2\sigma_0^2} d(r^2/\sigma_0^2)$$

$$\Rightarrow \quad f_0(z)\,dz = \underbrace{\left(\frac{a^2}{2\sigma_0^2}\right)}_{\equiv \alpha} e^{-\left(\frac{a^2}{2\sigma_0^2}\right)z} dz$$

$$\Rightarrow \quad c_n = \frac{\alpha}{J_1(\lambda_n)^2} \int_0^1 e^{-\alpha z} J_0(\lambda_n \sqrt{z})\, dz$$

will look at $a = 10\,\sigma_0$

$a = 5\,\sigma_0$

$a = 2.5\,\sigma_0$

$a = \sigma_0$

b) f_0 is Uniform: $f_0\, r dr = \frac{2}{a^2} r dr = d(r^2/a^2)$

$$\Rightarrow \quad f_0(z)\, dz = dz$$

$$\Rightarrow \quad c_n = \frac{1}{J_1(\lambda_n)^2} \int_0^1 J_0(\lambda_n \sqrt{z})\, dz = \frac{1}{J_1(\lambda_n)^2} \cdot \frac{2}{\lambda_n^2} \int_0^{\lambda_n} J_0(u)\, u\, du$$

$$\Rightarrow N(\tau) = \sum_n c_n \int_0^1 J_0(\lambda_n \sqrt{z})\, dz \cdot e^{-\lambda_n^2 \tau/4}$$

$$\boxed{N(\tau) = \sum_n \left\{ \frac{2 \int_0^1 J_0(\lambda_n \sqrt{z})\, dz}{\lambda_n J_1(\lambda_n)} \right\} e^{-\lambda_n^2 \tau/4}}$$

Diffusion and Lifetime...

Zeroes of Bessel Function J_0: $n := 1..100$

$$\lambda_n := \left| \begin{array}{l} a \leftarrow 0 \\ b \leftarrow 3 \\ x \leftarrow \mathrm{root}(J0(y), y, a, b) \\ x \ \text{ if } \ n = 1 \\ \text{otherwise} \\ \quad \left| \begin{array}{l} a \leftarrow \lambda_{n-1} + 0.05 \\ b \leftarrow \lambda_{n-1} + 3.5 \\ x \leftarrow \mathrm{root}(J0(y), y, a, b) \end{array} \right. \\ x \end{array} \right.$$

$\lambda =$

	0
0	0
1	2.405
2	5.52
3	8.654
4	11.792
5	14.931
6	18.071
7	21.212
8	24.352
9	27.493
10	30.635
11	33.776
12	36.917
13	40.058
14	43.2
15	46.341

Max order of summations used: $n_{max} := 17$

Ratio of aperture to beam sigma: $r_{ap} := 10$ $\qquad \alpha := \dfrac{r_{ap}^{\,2}}{2}$

Initial Distribution: $f_0(Z) := \alpha \cdot e^{-\alpha \cdot Z}$

Series coefficients:

$$c_n := \frac{1}{J1(\lambda_n)^2} \cdot \int_0^1 f_0(Z) \cdot J0\left(\lambda_n \cdot \sqrt{Z}\right) dZ$$

Beam Intensity evolution:

$$N(\tau) := 2 \sum_{n=1}^{n_{max}} \frac{c_n}{\lambda_n} \cdot J1(\lambda_n) \cdot e^{\frac{-(\lambda_n)^2 \cdot \tau}{4}}$$

Intensity lifetime evolution:

$$\tau_L(\tau) := \frac{-N(\tau)}{\frac{d}{d\tau} N(\tau)}$$

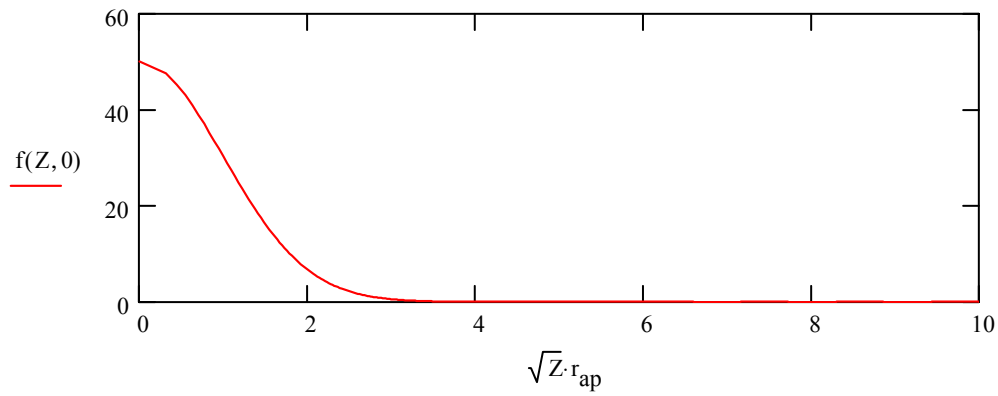
Asymptotic lifetime:

$$\tau_{asy} := \frac{4}{(\lambda_1)^2}$$

Distribution function evolution: $$f(Z,\tau) := \sum_{n=1}^{n_{max}} c_n \cdot J0(\lambda_n \cdot \sqrt{Z}) \cdot e^{\frac{-(\lambda_n)^2 \cdot \tau}{4}}$$

Plot Ranges: $\tau := 0, 0.01 .. 2$

$Z := 0, .001 .. 1$



Scaled distribution function: (for plot comparisons) $$f_{scld}(Z,\tau) := f(Z,\tau) \cdot \frac{f(0,0.01)}{f(0,\tau)}$$

rms beam size evolution: $$u_{rms}(\tau) := \sqrt{\frac{\left(\int_0^1 Z \cdot f(Z,\tau)\, dZ\right)}{2 \cdot \left(\int_0^1 f(Z,\tau)\, dZ\right)}}$$

$u_{rms}(0) = 0.1$

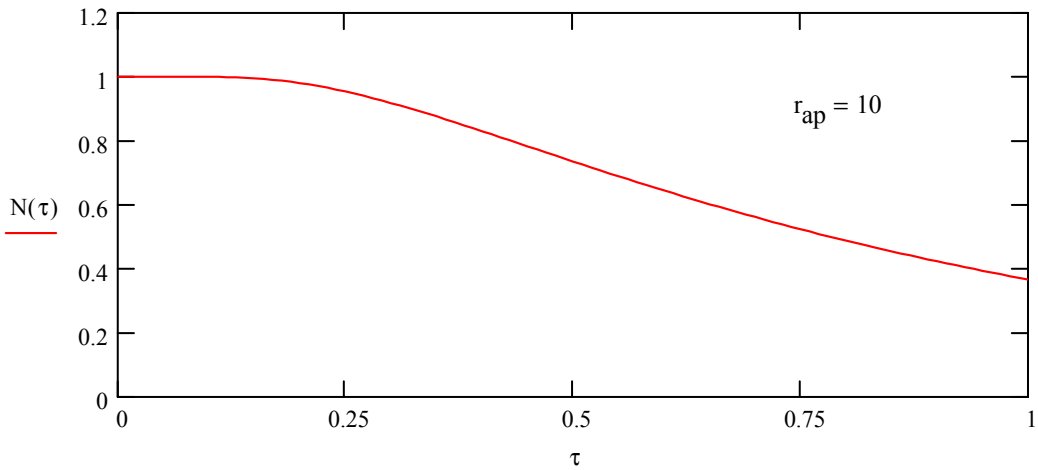
$u_{rms}(4) = 0.393$

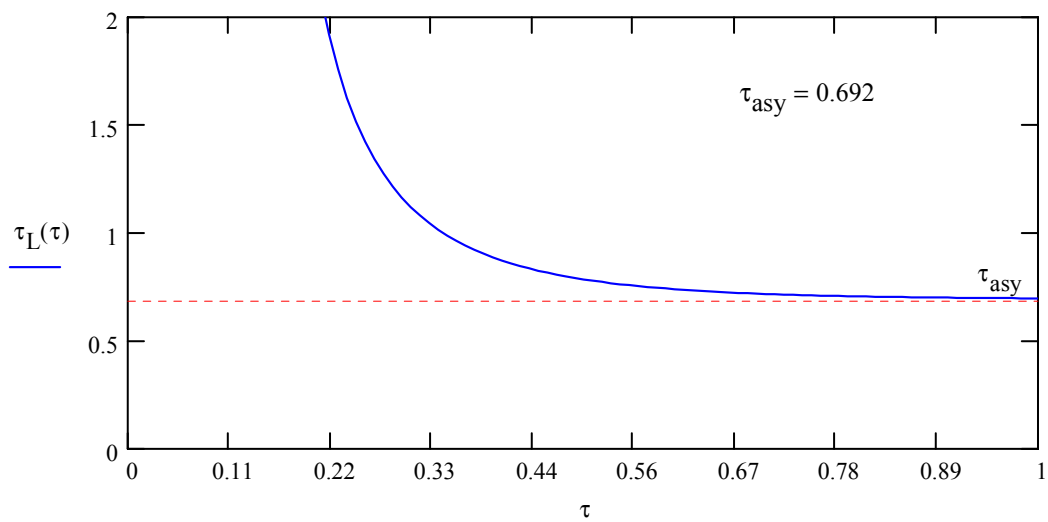
asymptotic beam size: $$u_{rms_asy} := \sqrt{\frac{1}{2} \cdot \frac{\int_0^1 u^3 \cdot J0(\lambda_1 \cdot u)\, du}{\int_0^1 u \cdot J0(\lambda_1 \cdot u)\, du}} \cdot r_{ap}$$

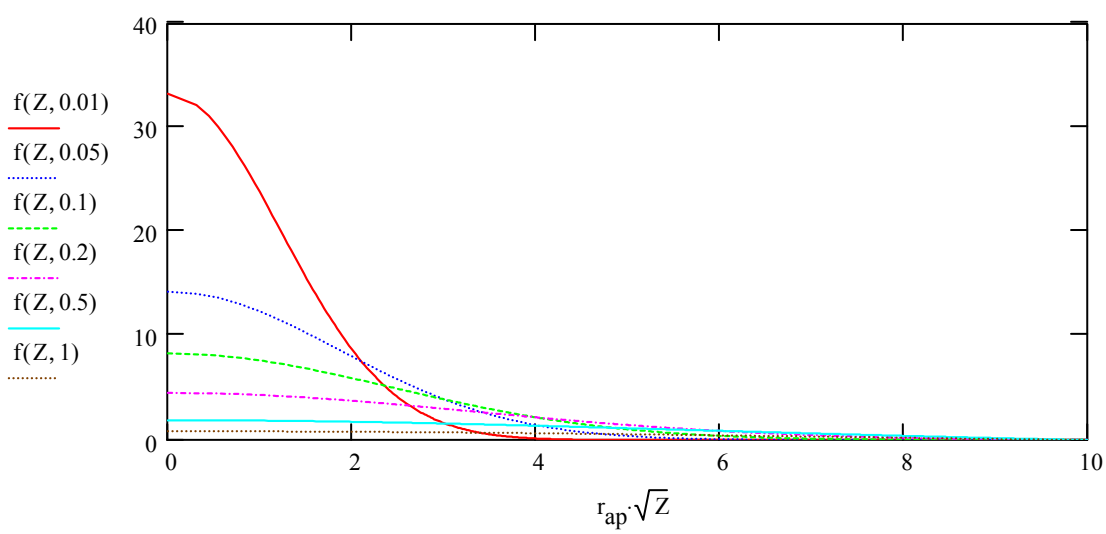
$u_{rms_asy} = 3.926$

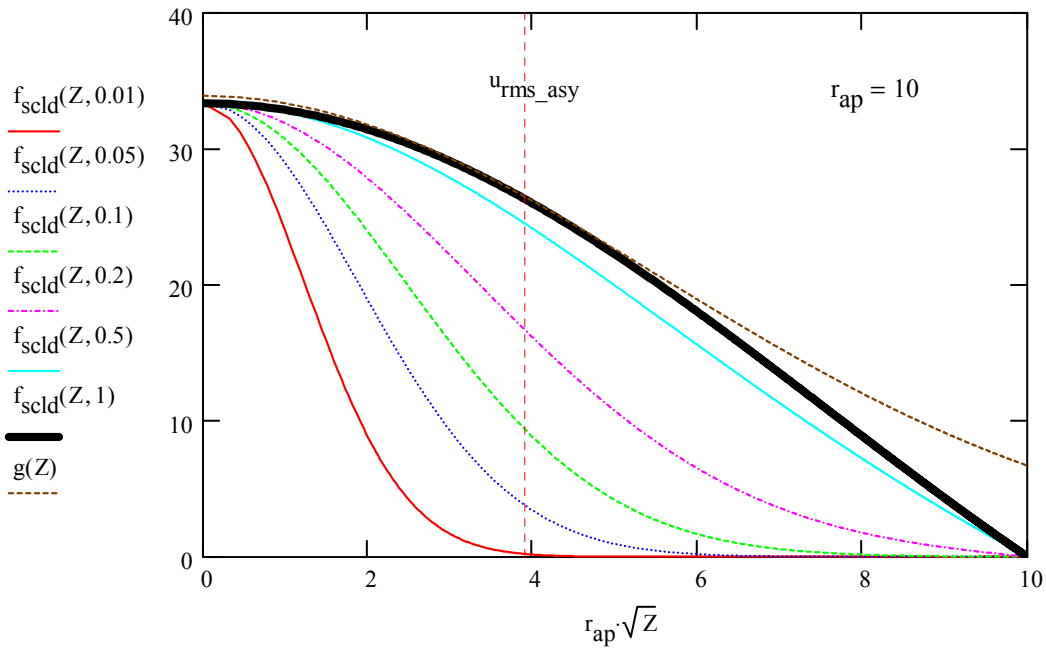
$\frac{u_{rms_asy}}{r_{ap}} = 0.393$

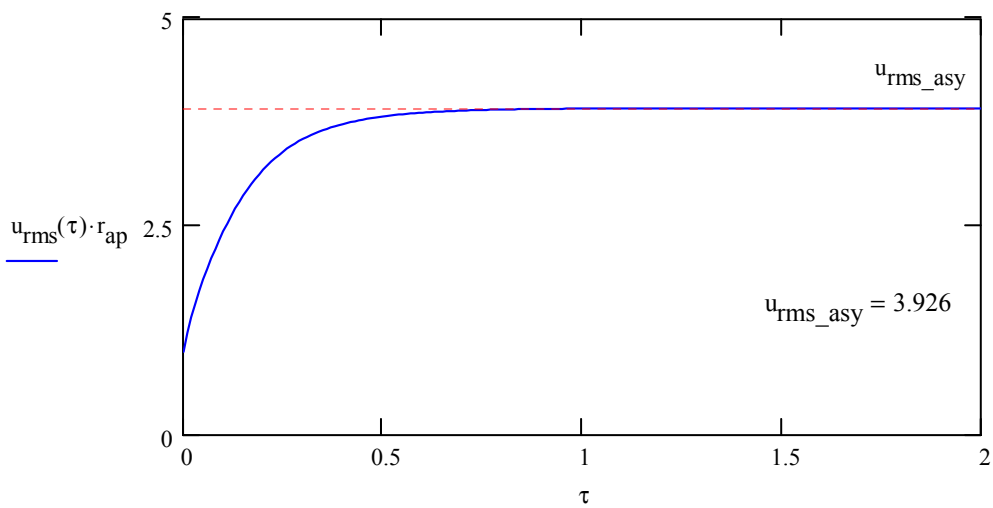
Guassian distribution, for comparison: $$s_{asy} := \frac{u_{rms_asy}}{r_{ap}} \cdot \sqrt{2} \qquad g(z) := e^{-\left(\frac{z-0.01}{2 \cdot s_{asy}^2}\right)} \cdot f(0,0.01)$$

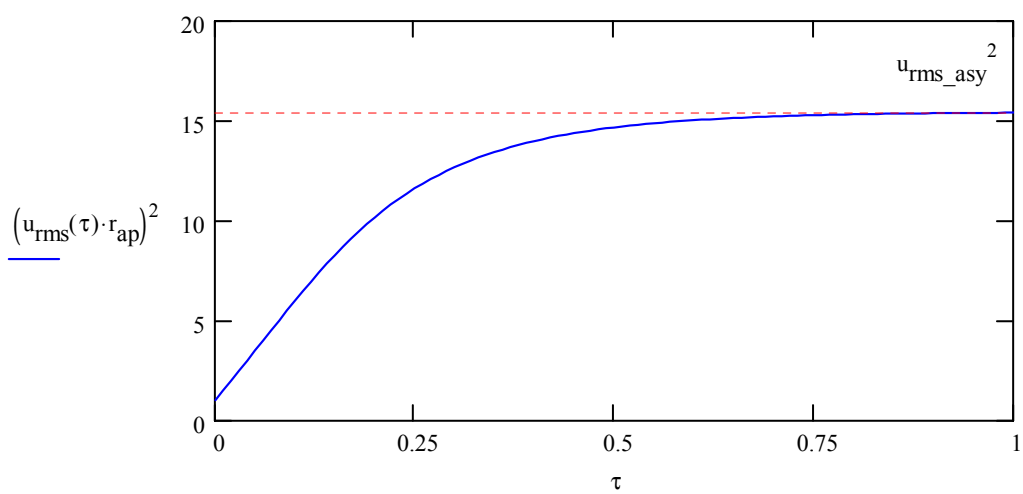

1.2
1
0.8
0.6
0.4
0.2
0
N(τ)
r_ap = 10
0
0.25
0.5
0.75
1
τ

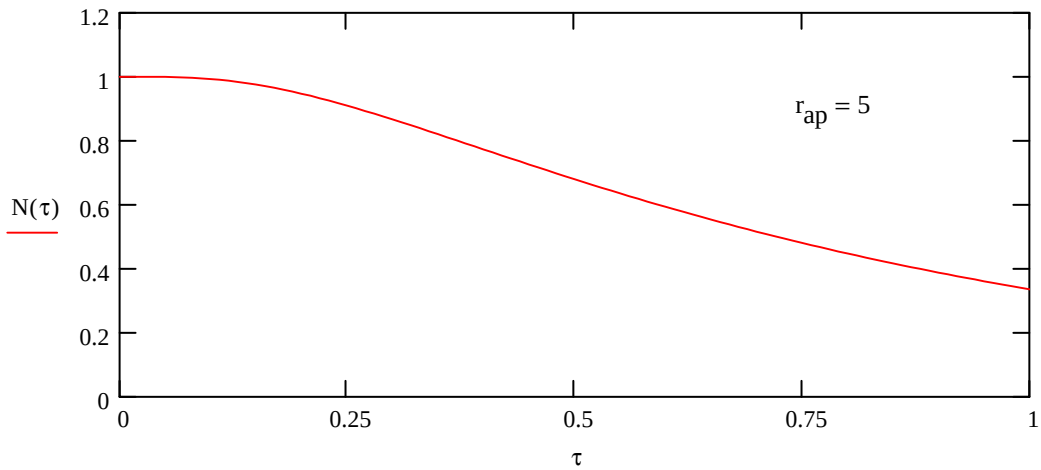

2
1.5
1
0.5
0
τ_L(τ)
τ_asy = 0.692
τ_asy
0
0.11
0.22
0.33
0.44
0.56
0.67
0.78
0.89
1
τ

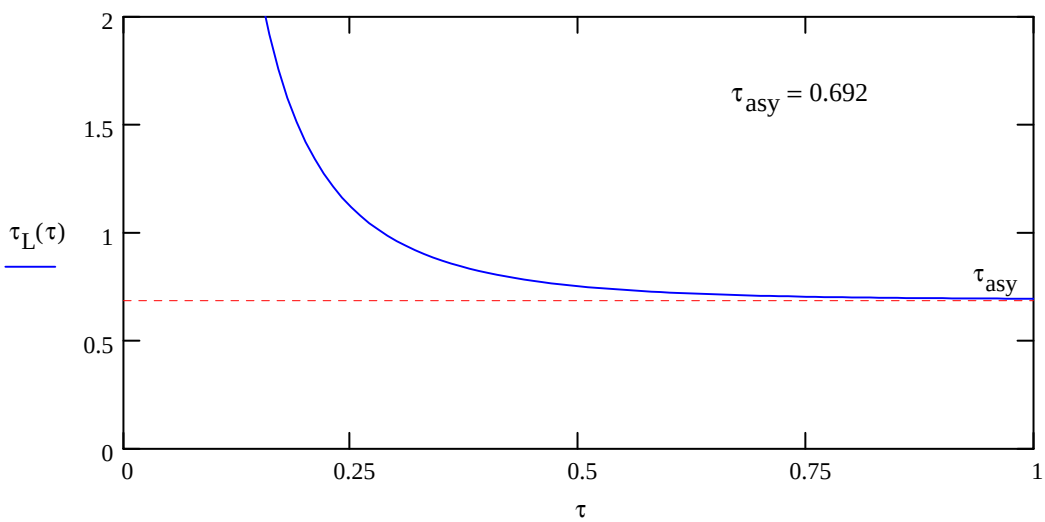

f(Z, 0.01)
f(Z, 0.05)
f(Z, 0.1)
f(Z, 0.2)
f(Z, 0.5)
f(Z, 1)
40
30
20
10
0
0
2
4
6
8
10
r_ap·√Z

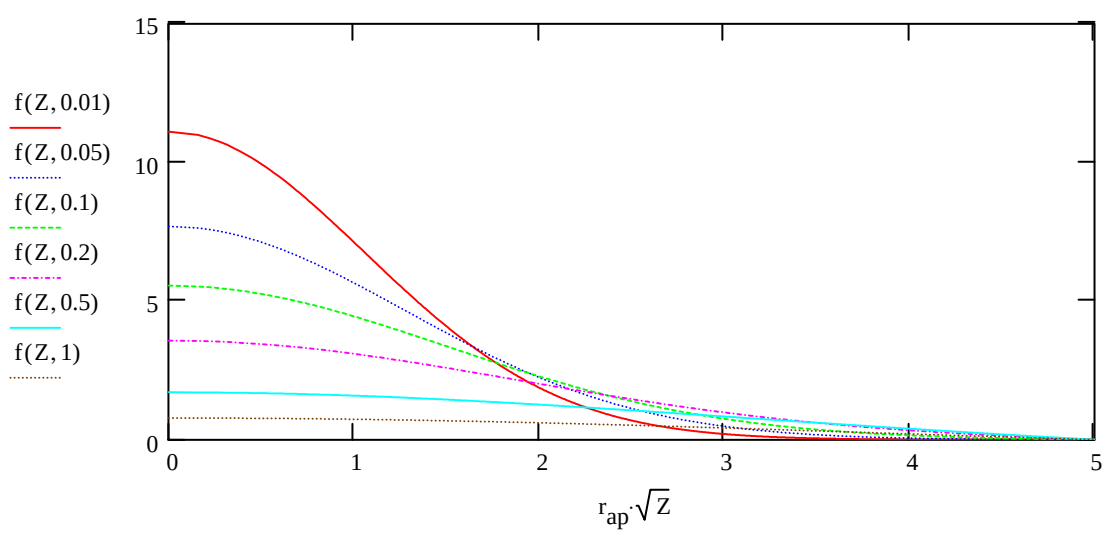

$f_{scld}(Z, 0.01)$
$f_{scld}(Z, 0.05)$
$f_{scld}(Z, 0.1)$
$f_{scld}(Z, 0.2)$
$f_{scld}(Z, 0.5)$
$f_{scld}(Z, 1)$
$g(Z)$
u_{rms_asy}
$r_{ap} = 10$
$r_{ap} \cdot \sqrt{Z}$

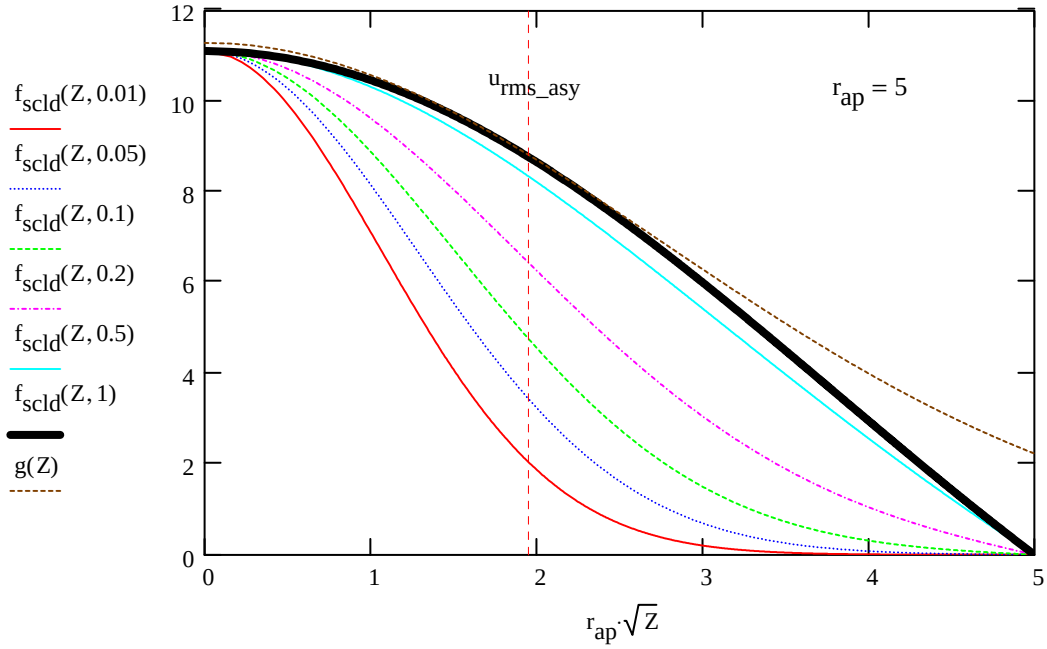

$u_{rms}(\tau) \cdot r_{ap}$
u_{rms_asy}
$u_{rms_asy} = 3.926$
τ

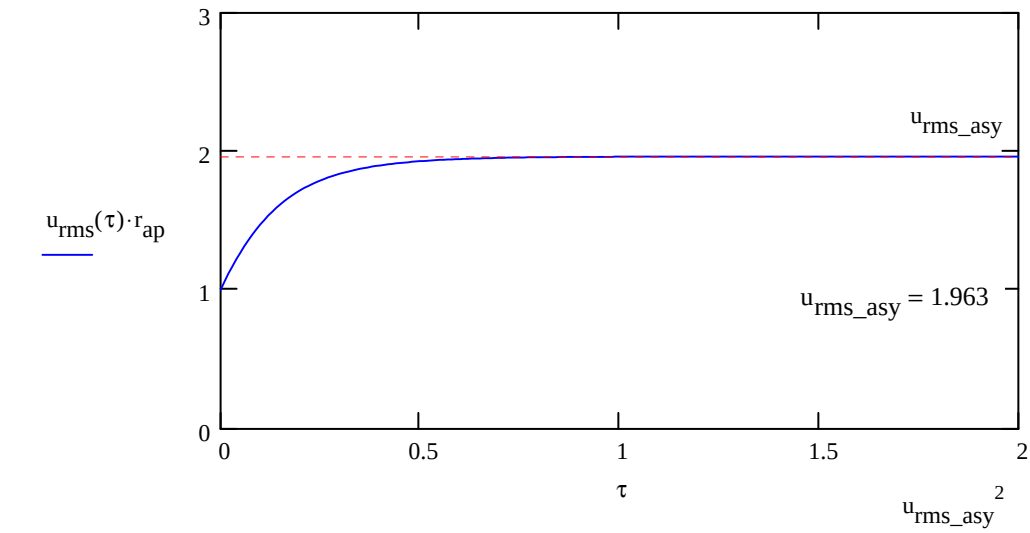

$(u_{rms}(\tau) \cdot r_{ap})^2$
$u_{rms_asy}^2$
τ

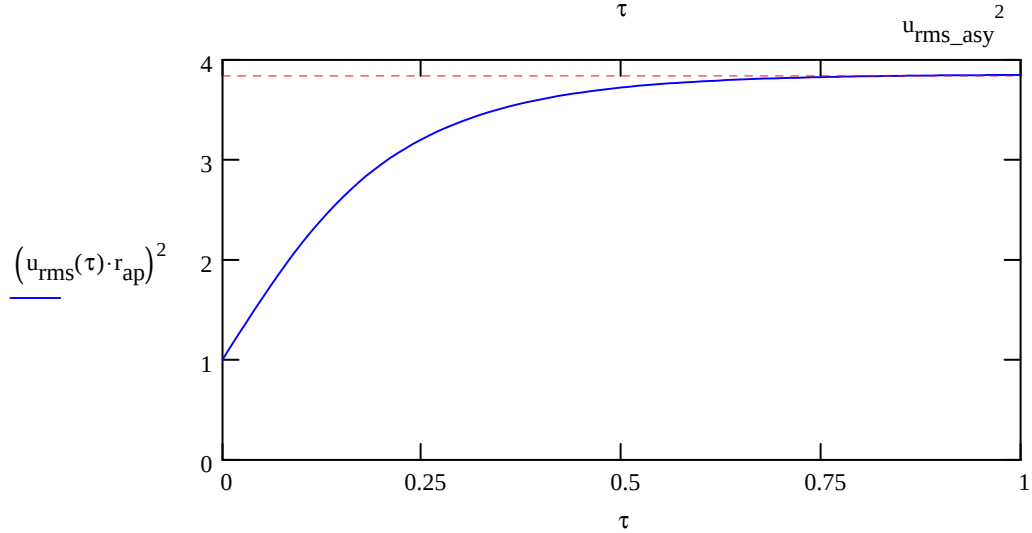

1.2
1
0.8
0.6
0.4
0.2
0
N(τ)
r_ap = 5
0
0.25
0.5
0.75
1
τ

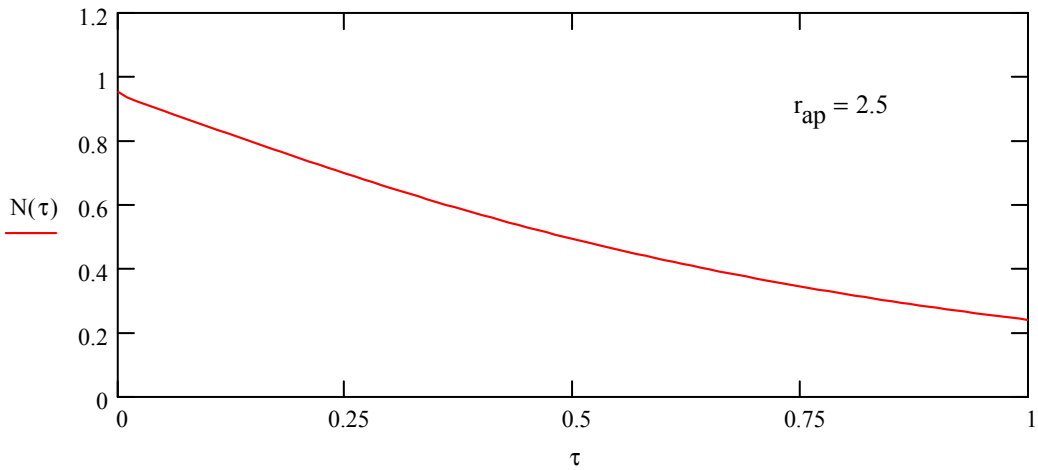

2
1.5
1
0.5
0
τ_L(τ)
τ_asy = 0.692
τ_asy
0
0.25
0.5
0.75
1
τ

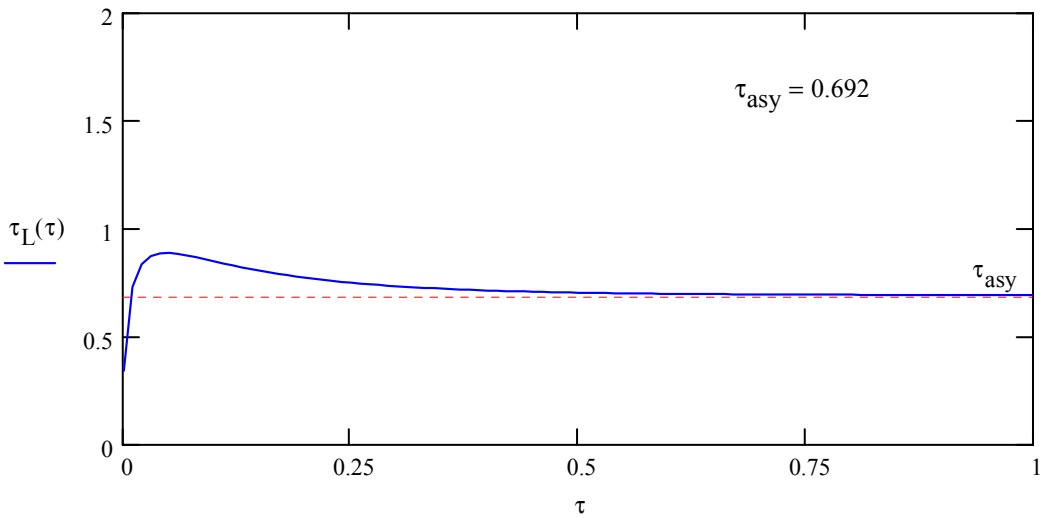

15
10
5
0
f(Z, 0.01)
f(Z, 0.05)
f(Z, 0.1)
f(Z, 0.2)
f(Z, 0.5)
f(Z, 1)
0
1
2
3
4
5
r_ap·√Z

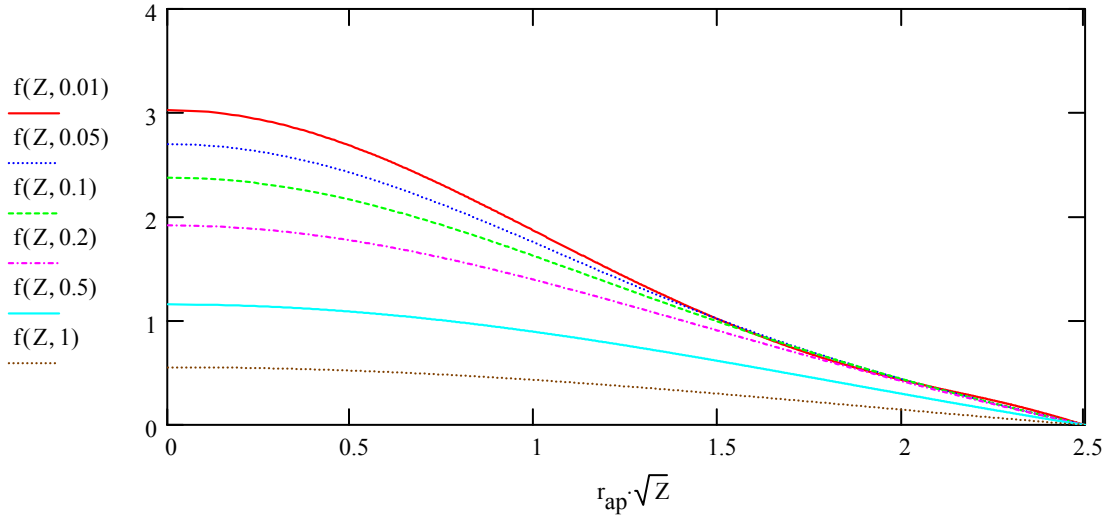

$f_{scld}(Z, 0.01)$
$f_{scld}(Z, 0.05)$
$f_{scld}(Z, 0.1)$
$f_{scld}(Z, 0.2)$
$f_{scld}(Z, 0.5)$
$f_{scld}(Z, 1)$
$g(Z)$
u_{rms_asy}
$r_{ap} = 5$
$r_{ap} \cdot \sqrt{Z}$

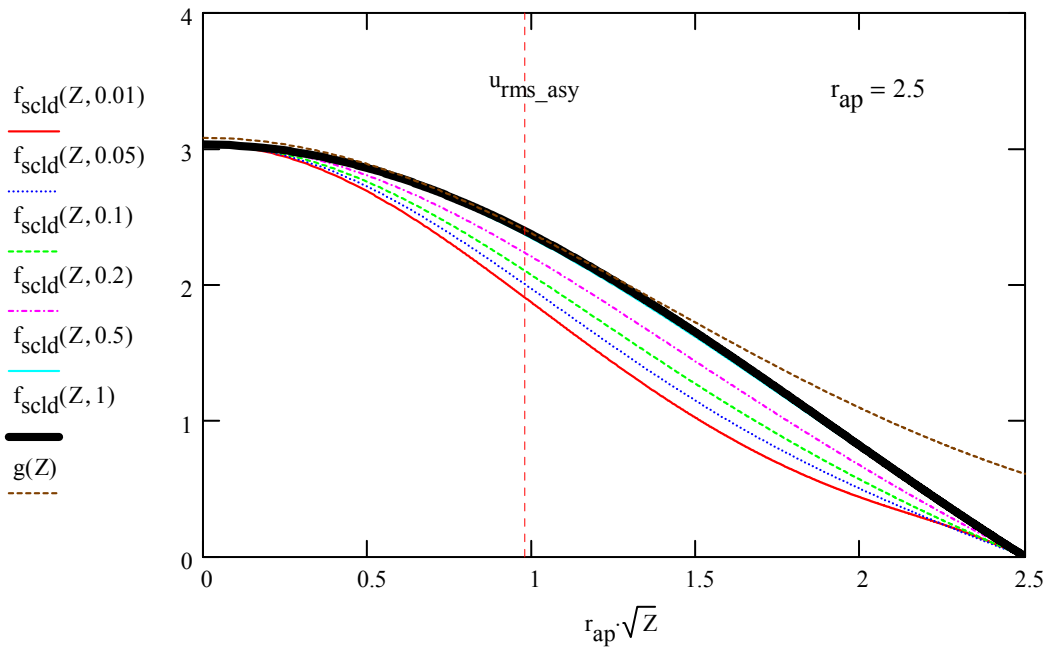

$u_{rms}(\tau) \cdot r_{ap}$
u_{rms_asy}
$u_{rms_asy} = 1.963$
τ

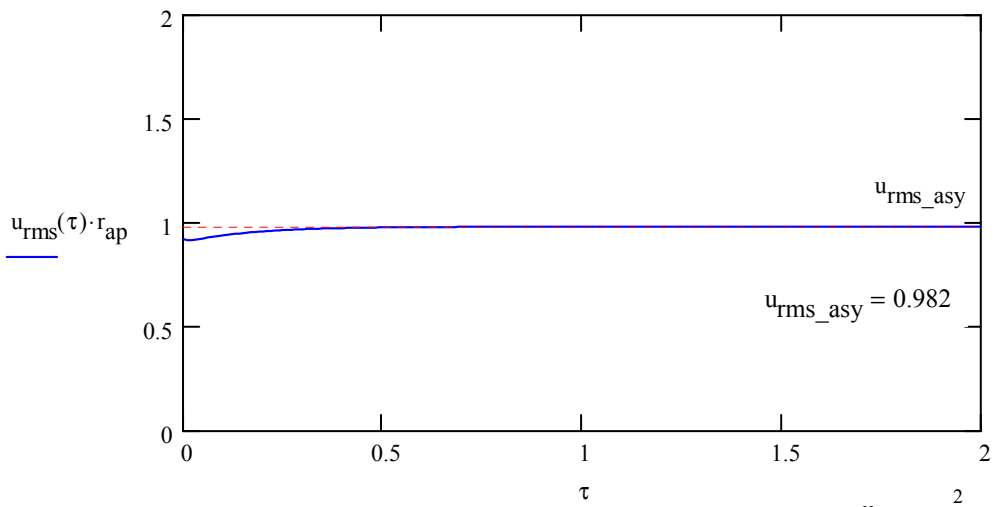

$u_{rms_asy}^2$
$(u_{rms}(\tau) \cdot r_{ap})^2$
τ

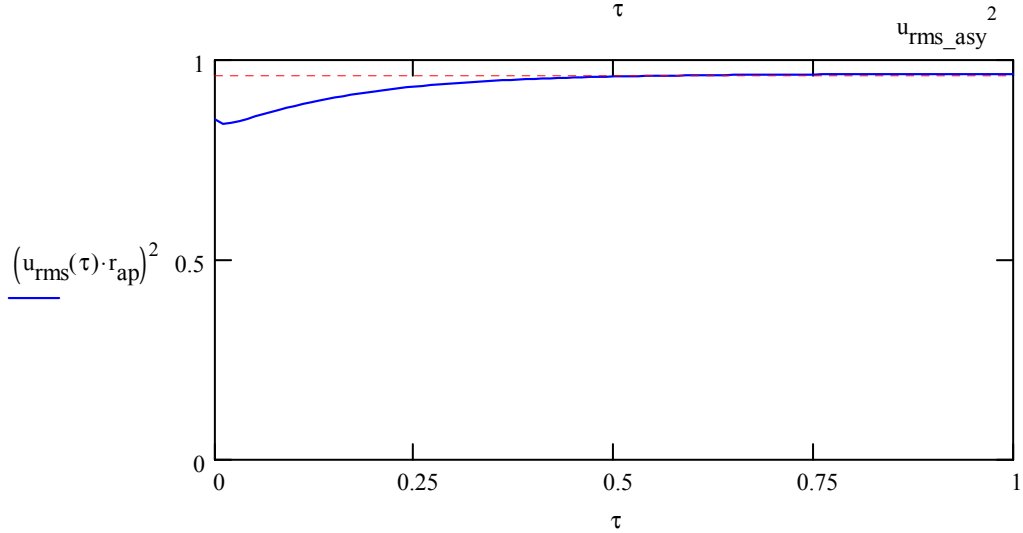

1.2
1
0.8
0.6
0.4
0.2
0
N(τ)
r_ap = 2.5
0
0.25
0.5
0.75
1
τ

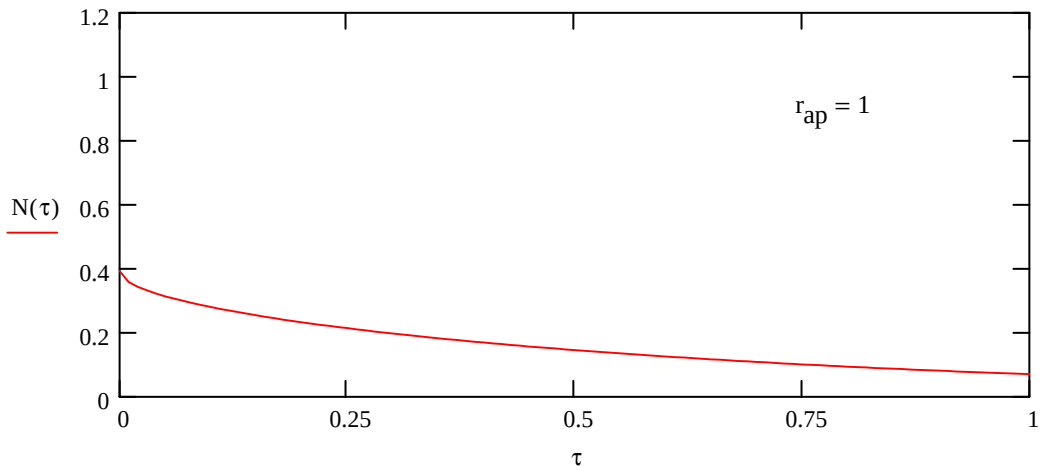

2
1.5
1
0.5
0
τ_L(τ)
τ_asy = 0.692
τ_asy
0
0.25
0.5
0.75
1
τ

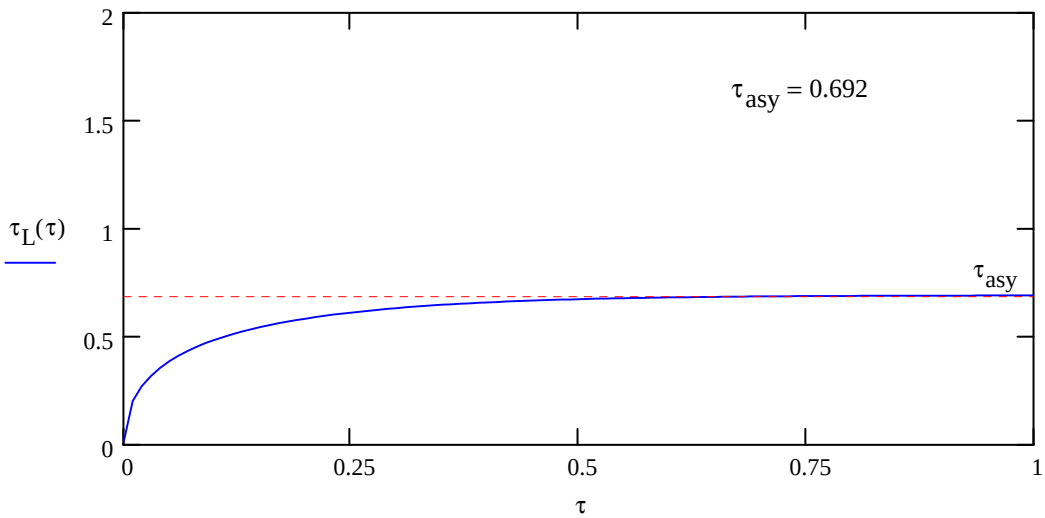

f(Z, 0.01)
f(Z, 0.05)
f(Z, 0.1)
f(Z, 0.2)
f(Z, 0.5)
f(Z, 1)
4
3
2
1
0
0
0.5
1
1.5
2
2.5
r_ap·√Z

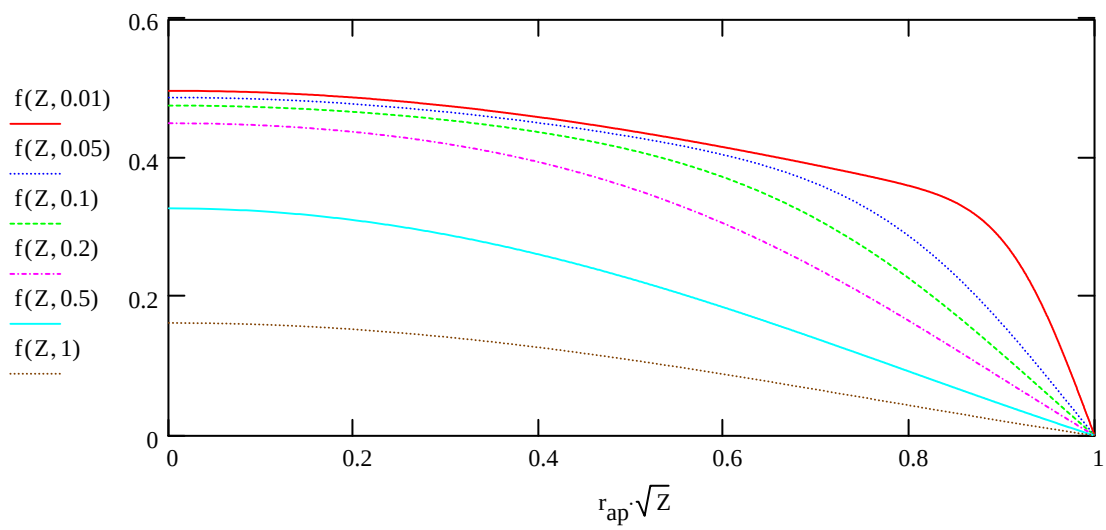

$f_{scld}(Z, 0.01)$
$f_{scld}(Z, 0.05)$
$f_{scld}(Z, 0.1)$
$f_{scld}(Z, 0.2)$
$f_{scld}(Z, 0.5)$
$f_{scld}(Z, 1)$
$g(Z)$
u_{rms_asy}
$r_{ap} = 2.5$
$r_{ap} \cdot \sqrt{Z}$

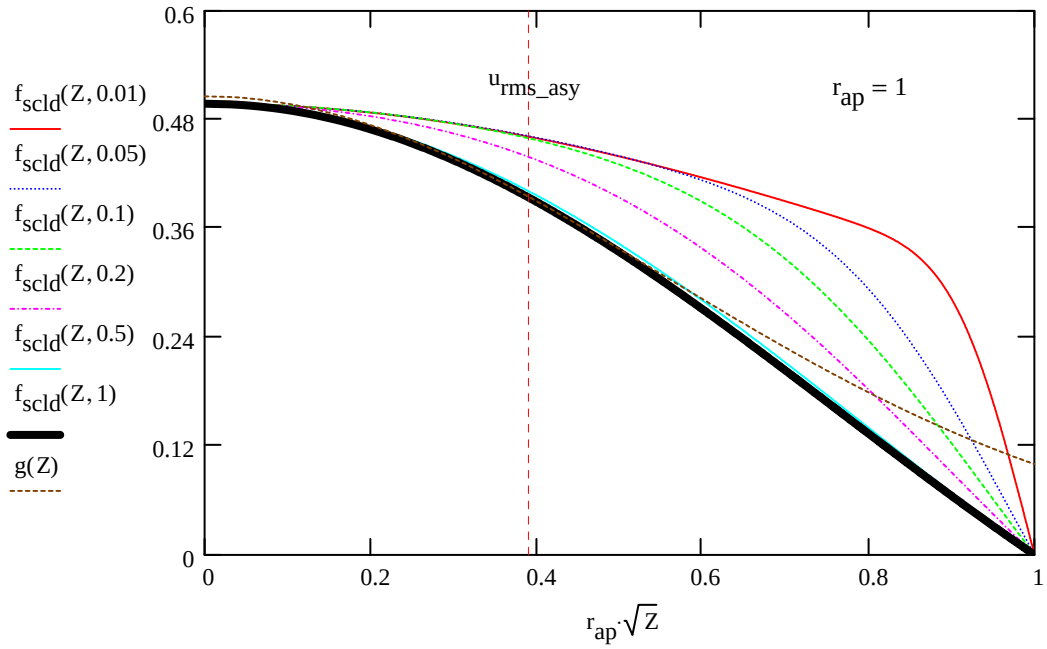

$u_{rms}(\tau) \cdot r_{ap}$
u_{rms_asy}
$u_{rms_asy} = 0.982$
τ

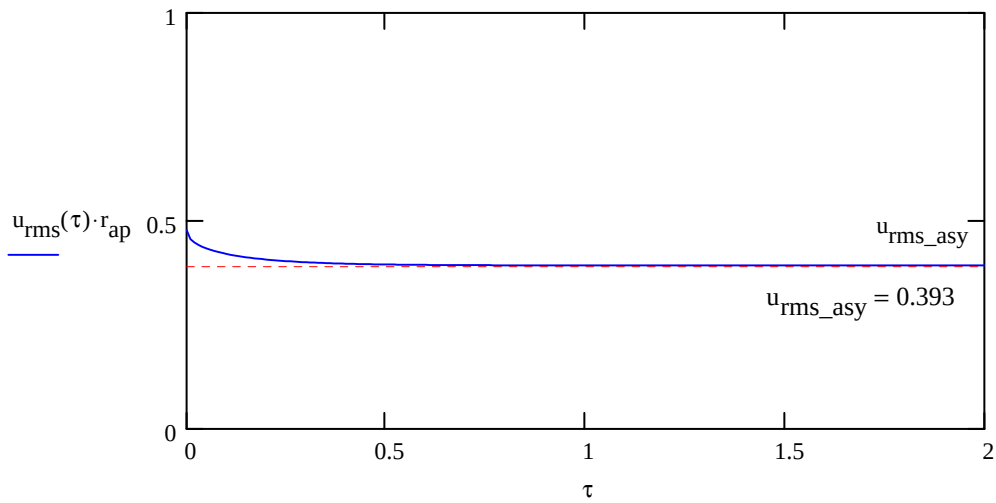

$u_{rms_asy}^2$
$(u_{rms}(\tau) \cdot r_{ap})^2$
τ

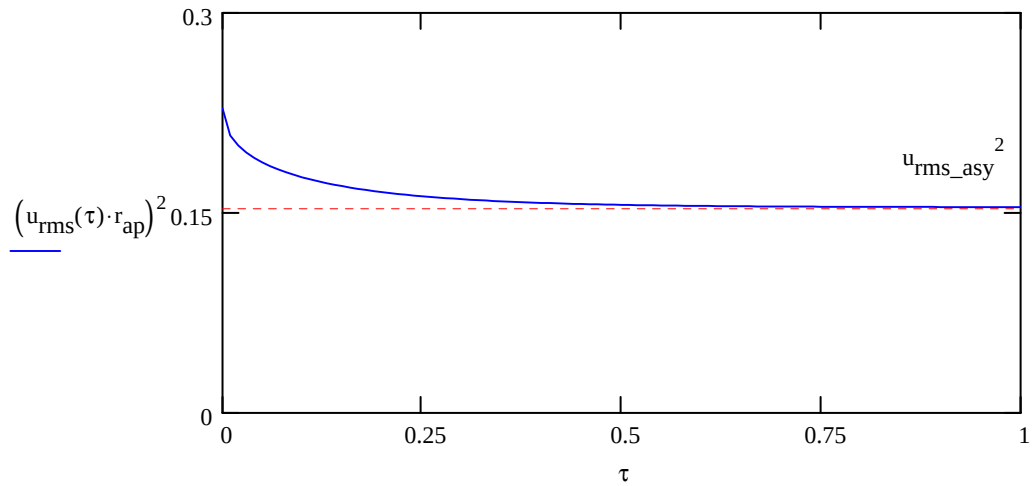

1.2
1
0.8
0.6
0.4
0.2
0
N(τ)
r_ap = 1
0
0.25
0.5
0.75
1
τ


2
1.5
1
0.5
0
τ_L(τ)
τ_asy = 0.692
τ_asy
0
0.25
0.5
0.75
1
τ

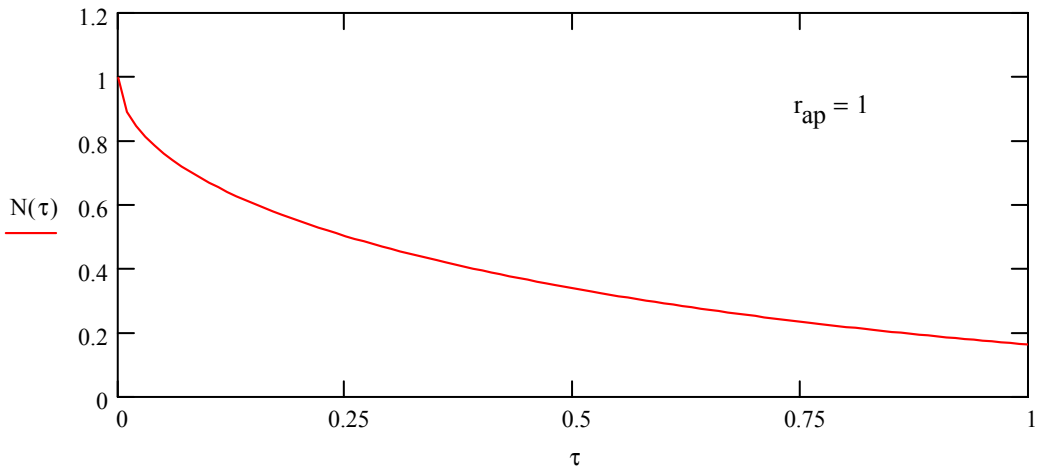

0.6
0.4
0.2
0
f(Z, 0.01)
f(Z, 0.05)
f(Z, 0.1)
f(Z, 0.2)
f(Z, 0.5)
f(Z, 1)
0
0.2
0.4
0.6
0.8
1
r_ap·√Z

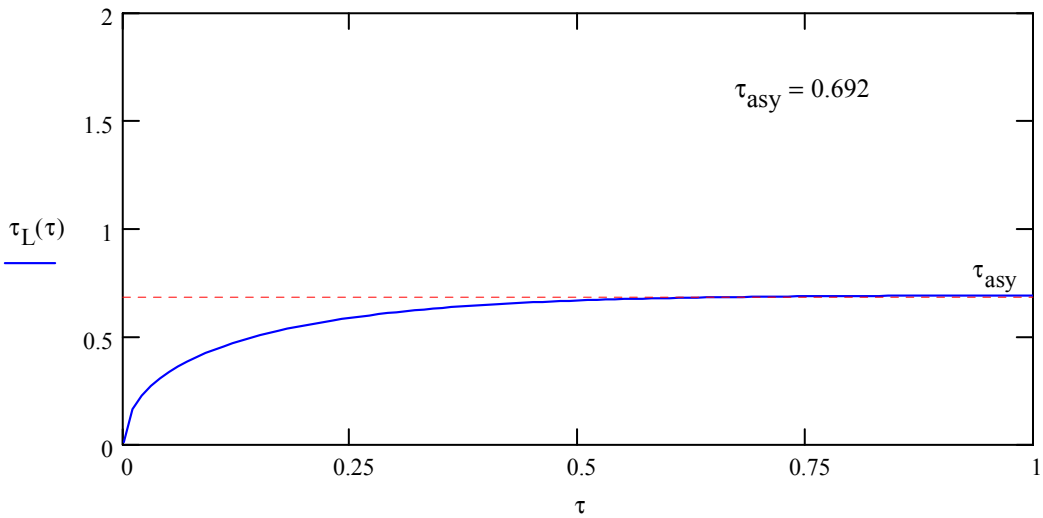

0.6
0.48
0.36
0.24
0.12
0
0
0.2
0.4
0.6
0.8
1
$f_{scld}(Z, 0.01)$
$f_{scld}(Z, 0.05)$
$f_{scld}(Z, 0.1)$
$f_{scld}(Z, 0.2)$
$f_{scld}(Z, 0.5)$
$f_{scld}(Z, 1)$
g(Z)
u_{rms_asy}
$r_{ap} = 1$
$r_{ap} \cdot \sqrt{Z}$

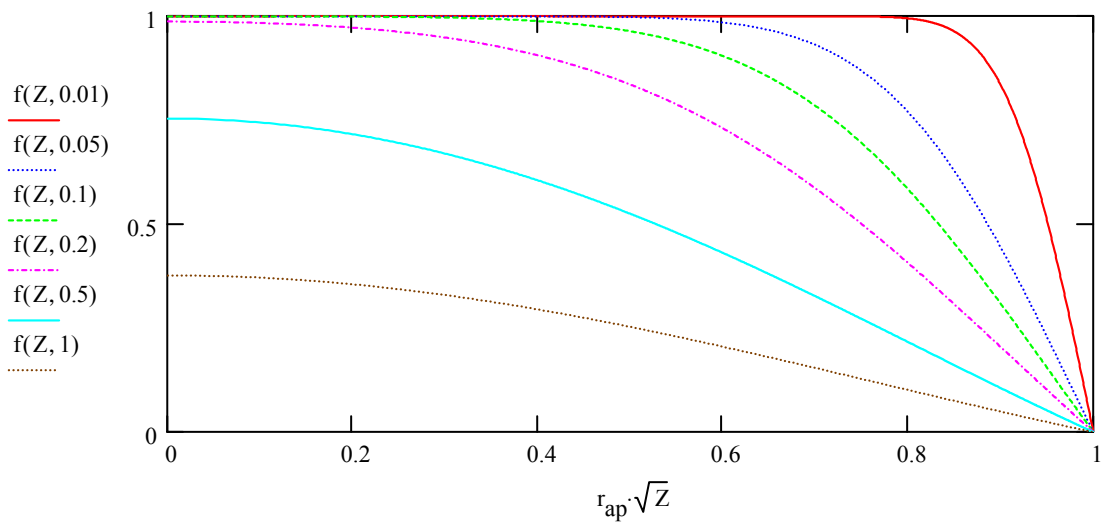

1
0.5
0
0
0.5
1
1.5
2
$u_{rms}(\tau) \cdot r_{ap}$
u_{rms_asy}
$u_{rms_asy} = 0.393$
τ


0.3
0.15
0
0
0.25
0.5
0.75
1
$(u_{rms}(\tau) \cdot r_{ap})^2$
$u_{rms_asy}^2$
τ

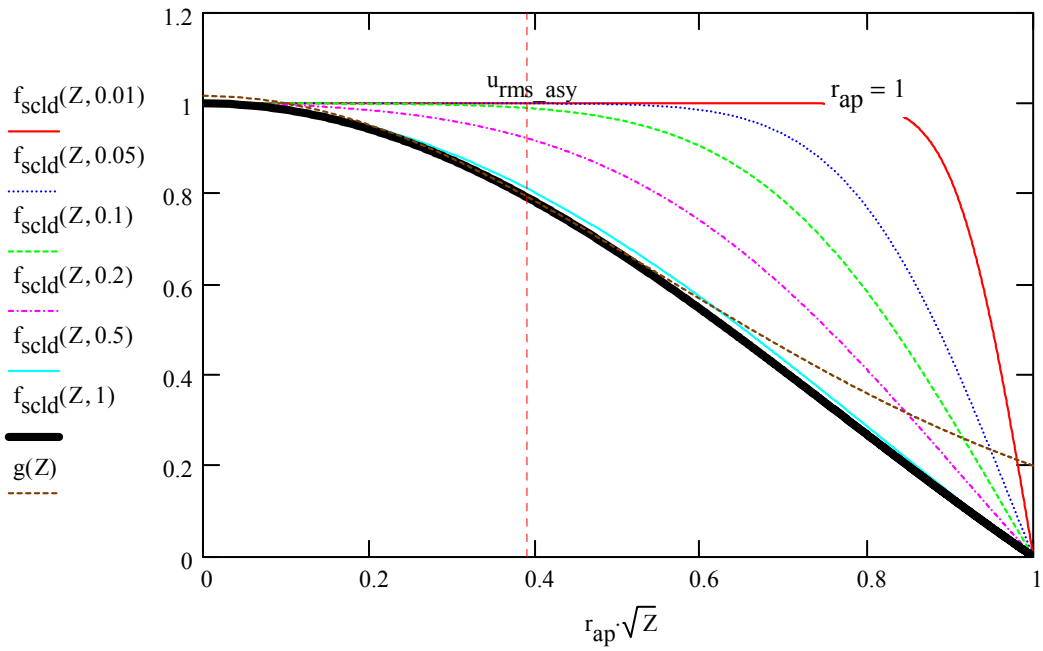
Here, Initial Distribution is Uniform

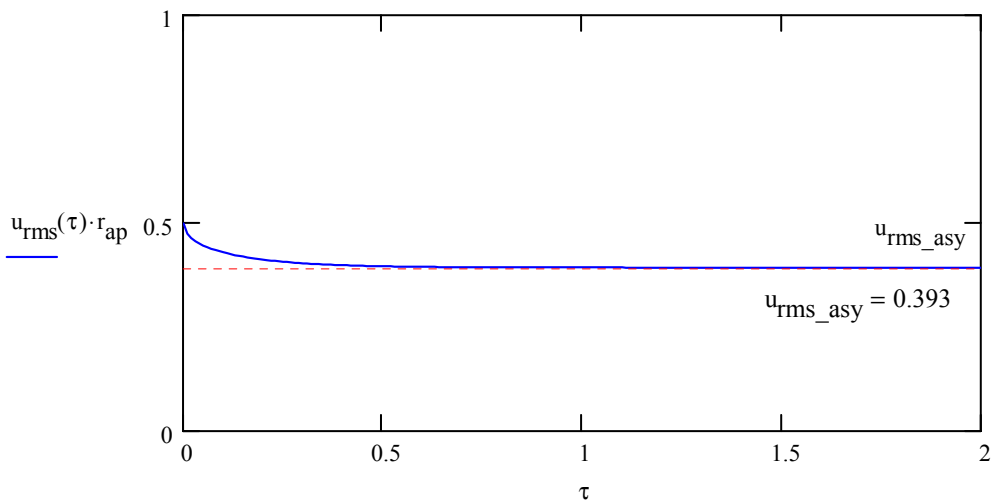


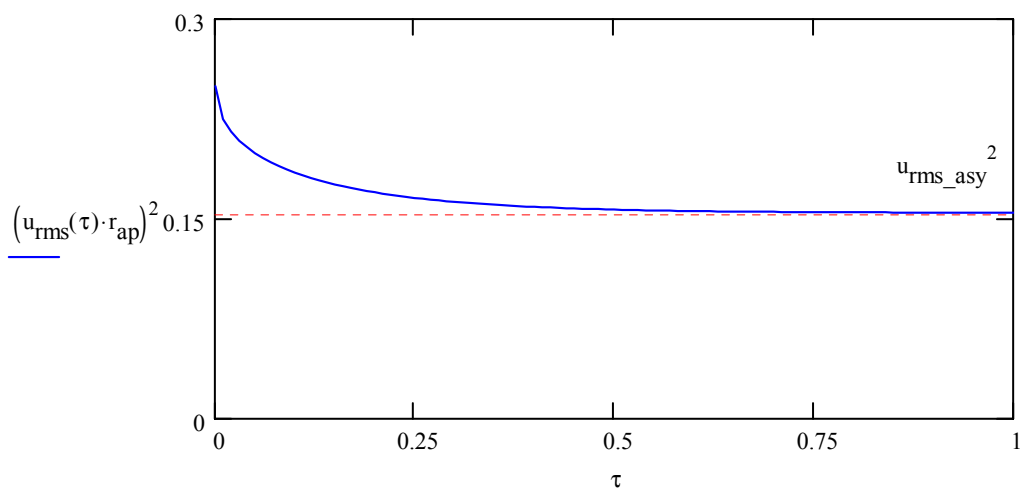




Here, Initial Distribution is Uniform

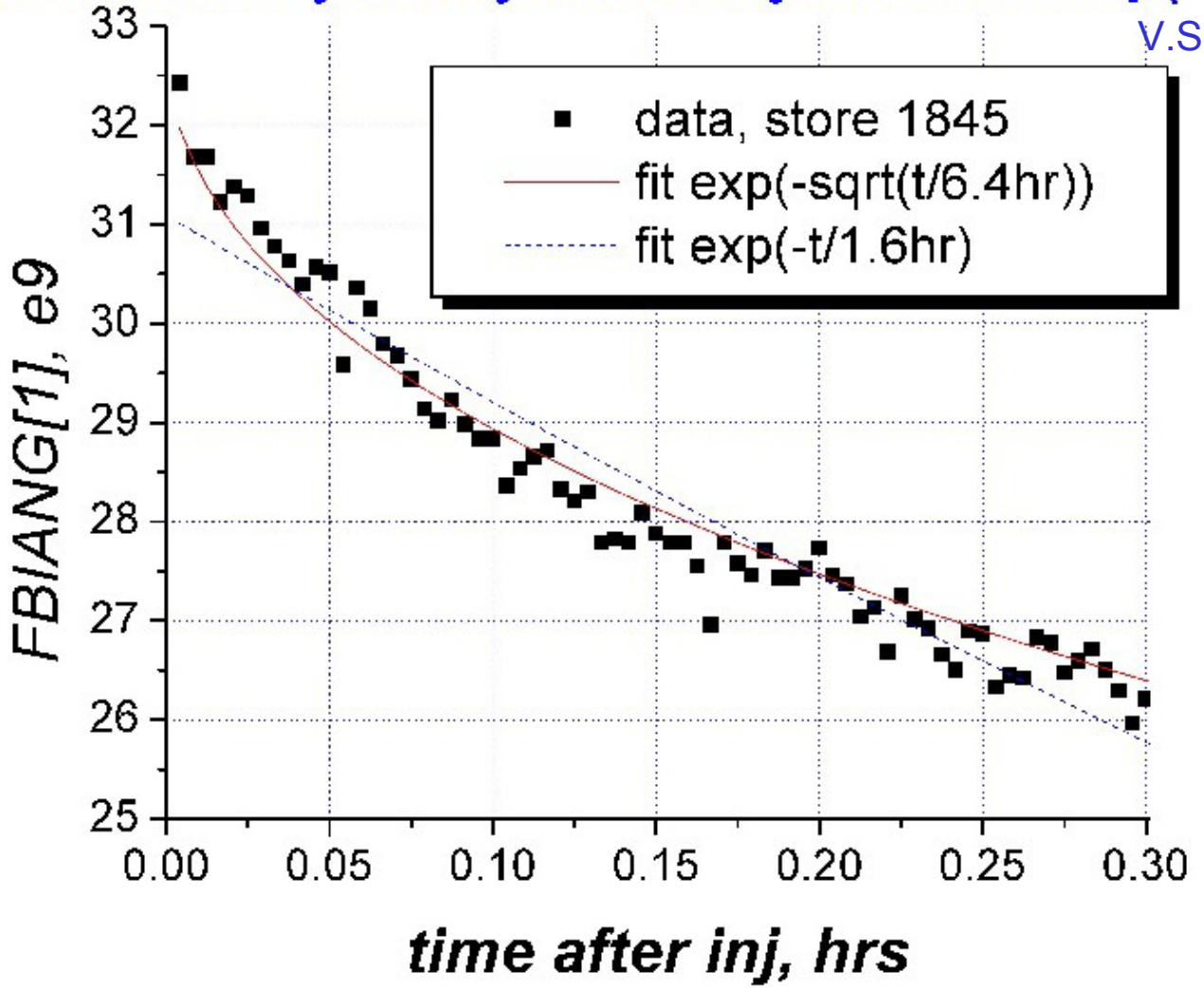






Pbar intensity decays after injection as $\exp(-t^{0.5})$

V.Shiltsev



Diffusion and Lifetime of Antiprotons in the Tevatron ...

Zeroes of Bessel Function J_0: $n := 1..100$

$$\lambda_n := \left| \begin{array}{l} a \leftarrow 0 \\ b \leftarrow 3 \\ x \leftarrow \mathrm{root}(J0(y), y, a, b) \\ x \ \ \mathrm{if}\ \ n = 1 \\ \mathrm{otherwise} \\ \quad \left| \begin{array}{l} a \leftarrow \lambda_{n-1} + 0.05 \\ b \leftarrow \lambda_{n-1} + 3.5 \\ x \leftarrow \mathrm{root}(J0(y), y, a, b) \end{array} \right. \\ x \end{array} \right.$$

$\lambda =$

	0
0	0
1	2.405
2	5.52
3	8.654
4	11.792
5	14.931
6	18.071
7	21.212
8	24.352
9	27.493
10	30.635
11	33.776
12	36.917
13	40.058
14	43.2
15	46.341

Max order of summations used: $n_{max} := 99$

Ratio of aperture to beam sigma: $r_{ap} := 1.2$ $\quad \alpha := \dfrac{r_{ap}^{\,2}}{2}$ *

Initial Distribution: $f_0(Z) := \alpha \cdot e^{-\alpha \cdot Z}$ *

$eV := 1.6 \cdot 10^{-19} \cdot joule$ *

Series coefficients:

$$c_n := \frac{1}{J1(\lambda_n)^2} \cdot \int_0^1 f_0(Z) \cdot J0\left(\lambda_n \cdot \sqrt{Z}\right) dZ$$

Beam Intensity evolution:

$$N(\tau) := 2 \sum_{n=1}^{n_{max}} \frac{c_n}{\lambda_n} \cdot J1(\lambda_n) \cdot e^{\frac{-(\lambda_n)^2 \cdot \tau}{4}}$$

Intensity lifetime evolution:

$$\tau_L(\tau) := \frac{-N(\tau)}{\frac{d}{d\tau} N(\tau)}$$

Asymptotic lifetime:

$$\tau_{asy} := \frac{4}{(\lambda_1)^2}$$

$Data := READPRN("DA-1886-1.txt")$ $\quad t := Data^{\langle 0 \rangle} \cdot hr$ $\quad N_{pbar} := Data^{\langle 1 \rangle}$

Set time scale for diffusion process: $t4tau := 5.76 \cdot hr$

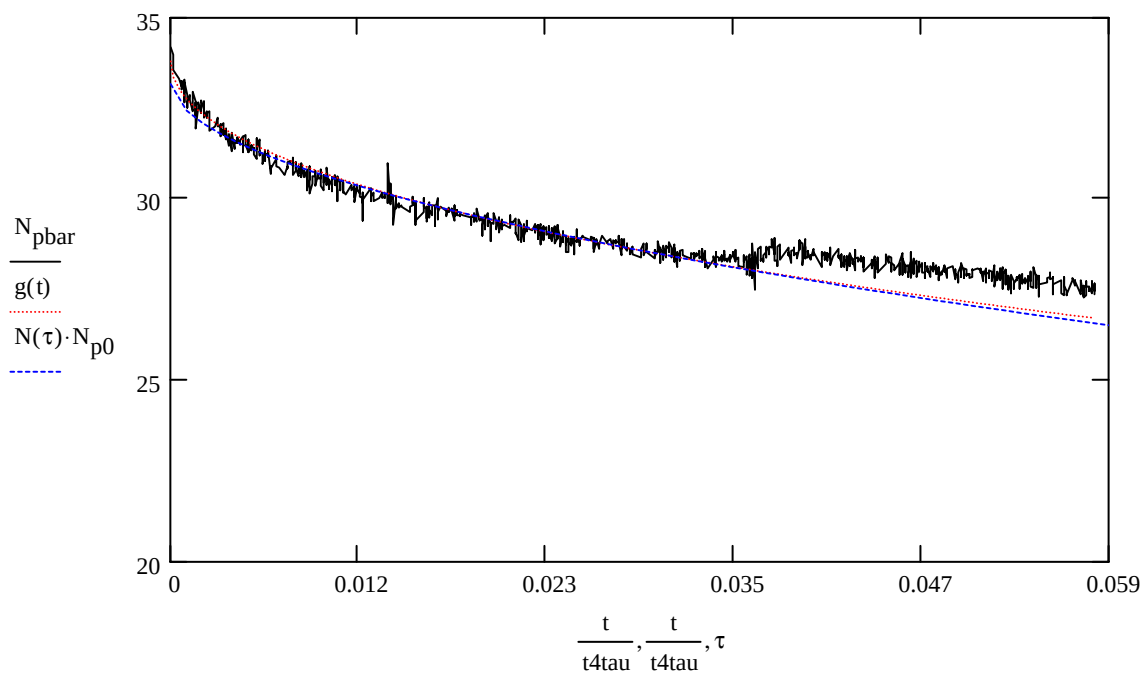
Scale factors to fit to data: $N_{p0} := 64.8$ $\quad N_{g0} := 33.8$

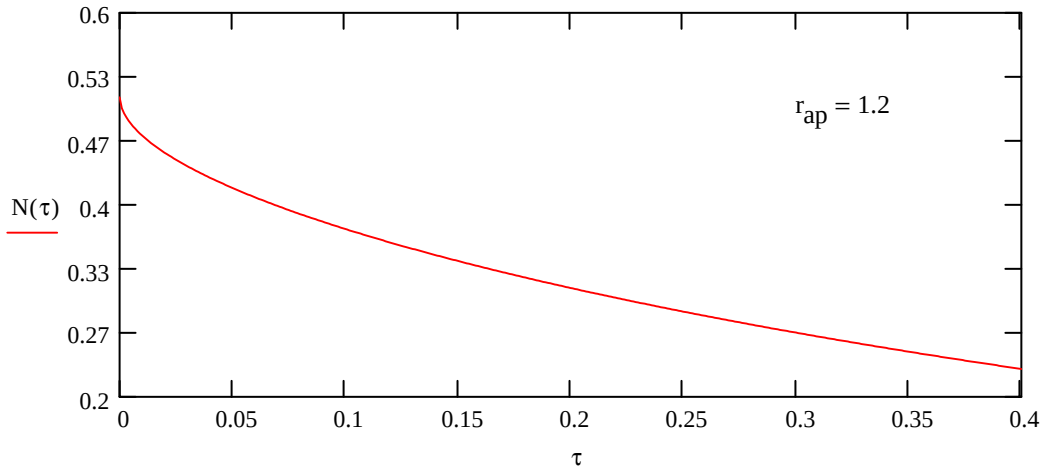
$N_{p0} \cdot N(0) = 33.166$ $\quad N_{pbar_0} = 34.186$

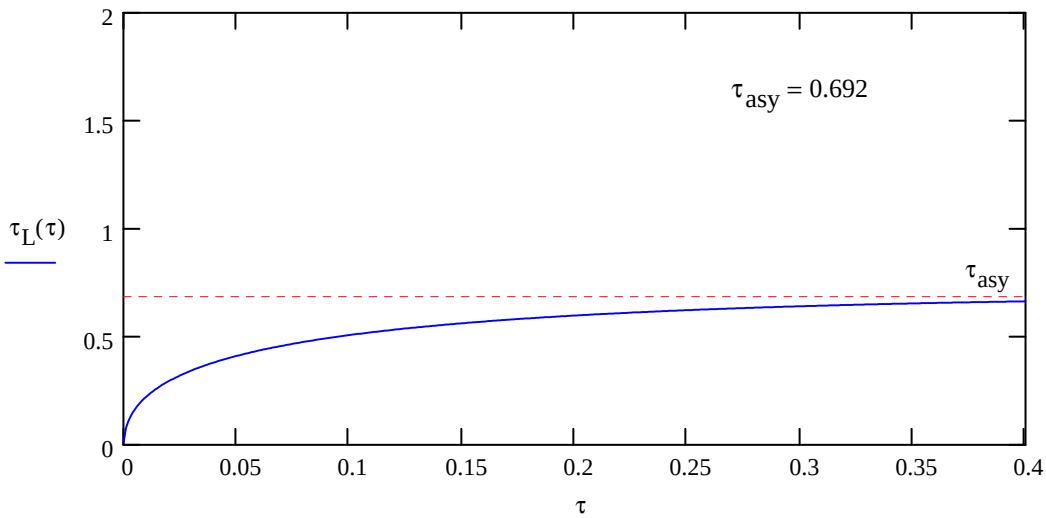
"Root-t" fit (by hand) to data -- $\tau_2 := 6 \cdot hr$

$$g(x) := N_{g0} \cdot e^{-\sqrt{\frac{x+0.001}{\tau_2}}}$$

$r_{ap} = 1.2$ $\quad t4tau = 5.76\ hr$




0.6
0.53
0.47
0.4
0.33
0.27
0.2
N(τ)
r_ap = 1.2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
τ


2
1.5
1
0.5
0
τ_L(τ)
τ_asy = 0.692
τ_asy
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
τ

Thus, the outcome is that the diffusion taking place for this store is consistent with time scales given by $\tau \sim 0.04$ corresponding to ~ 15 minutes

$$(.0435)\cdot t4tau = 15.034\,\mathrm{min}$$

and simultaneously is consistent with a ratio $a/\sigma_0 \sim 1.2$. Thus, we look at mechanisms which would correspond to these conditions...

$$\tau = 2\,[\,(d\varepsilon/dt)/\varepsilon\,]\,(\sigma_0/a)^2\,t$$

$$= 2\cdot\frac{2.5\cdot\frac{\pi}{\mathrm{hr}}}{20\cdot\pi}\cdot\left(\frac{1}{1.2}\right)^2\cdot\left(\frac{1}{4}\cdot\mathrm{hr}\right) = 0.043$$

However, if the transverse distribution were Gaussian with an aperture $a \sim \sigma_0$, then would see very dramatic beam loss (~60%) immediately at injection

$$= 2\cdot\frac{16\cdot\frac{\pi}{\mathrm{hr}}}{20\cdot\pi}\cdot\left(\frac{1}{3}\right)^2\cdot\left(\frac{1}{4}\cdot\mathrm{hr}\right) = 0.044$$

Thus, if transverse distribution were Gaussian with $a \sim 3\sigma_0$, then would need to have a VERY large emittance growth rate in order to be consistent with the time scale of the data

$$= 2\cdot\frac{0.33\cdot\frac{\mathrm{eV\cdot sec}}{\mathrm{hr}}}{4\cdot\mathrm{eV\cdot sec}}\cdot\left(\frac{1}{1}\right)^2\cdot\left(\frac{1}{4}\cdot\mathrm{hr}\right) = 0.041$$

So, perhaps the mechanism is not transverse, but longitudinal. Beam loss could be due to a 4 eV-sec bucket (as we have at 150 GeV) which was more or less uniformly populated (coalescing) and a 0.6 eV-sec/hr longitudinal emittance growth rate forcing particles out of the buckets (which we know to be happening)

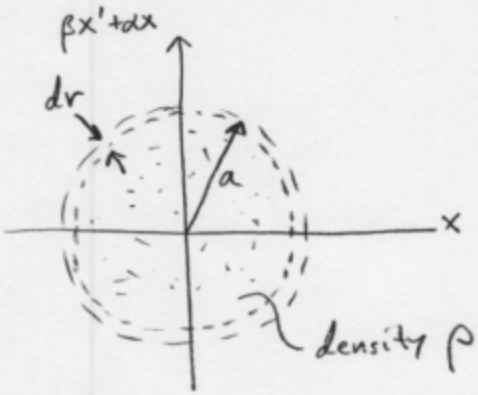
a check: $W/R = \dfrac{4\cdot\mathrm{eV\cdot sec}}{\left(0.66\,\frac{\mathrm{eV\cdot sec}}{\mathrm{hr}}\right)} = 6.061\,\mathrm{hr}$ $\qquad \tau_2 = 6\,\mathrm{hr}$

$e^{-\sqrt{t}}$?

let $k = f\langle\beta^2\rangle\theta_{rms}^2 = \frac{d}{dt}\langle r^2\rangle$

$$\therefore\ r^2 \sim k\cdot t\ ,\quad \text{or}\quad r \sim \sqrt{kt}$$

Suppose start with uniform density out to the aperture @ $r = a$



uniform $\Longrightarrow$ equal # particles flow into and out of a region

consider region at $r=a$, with thickness dr; particle amplitudes either grow or shrink during a time step dt; $\therefore$ $\frac{1}{2}$ of these are lost

$$\#\text{particles lost} = dN = -\rho\cdot(2\pi a\,dr)\cdot\frac{1}{2}$$

$$= -\frac{N}{\pi a^2}\cdot 2\pi a\,dr\cdot\frac{1}{2}$$

$$= -\frac{N}{a}\,dr$$

$r = \sqrt{kt}$

$dr = \frac{1}{2}\sqrt{k}\,t^{-1/2}\,dt$

$$\Rightarrow\quad \frac{dN}{N} = -\frac{dr}{a} = -\frac{1}{2}\frac{\sqrt{k}}{a}\,t^{-1/2}\,dt$$

$$\Rightarrow\quad \ln N = -\sqrt{\frac{k}{a^2}}\sqrt{t} + \text{const}$$

or, $\boxed{N = N_0\, e^{-\sqrt{t/\tau}}}$ where

$$\tau = \frac{a^2}{k}\quad\left(= \frac{W}{R}\right)$$

Tracking Verification of the Diffusion Equation

nonlinear tune with amplitude: $\nu_0 := 0.581883$ $\quad k := 0.05$ $\quad \nu(a) := \nu_0 + k \cdot a^2$

aperture: $ap := 2$ $\quad$ rms scattering angle: $\sigma_\theta := 0.1$

1-turn matrix: $M(a) := \begin{pmatrix} \cos(2 \cdot \pi \cdot \nu(a)) & \sin(2 \cdot \pi \cdot \nu(a)) \\ -\sin(2 \cdot \pi \cdot \nu(a)) & \cos(2 \cdot \pi \cdot \nu(a)) \end{pmatrix}$

$N_{turns} := 400$ $\quad N_{part} := 2000$ $\quad n := 0..N_{turns}$ $\quad i := 0..N_{part}$

Initialize the distribution (uniform):

$$\begin{pmatrix} x_{i,0} \\ p_{i,0} \end{pmatrix} := \left[\begin{array}{l} a \leftarrow 2 \\ b \leftarrow 2 \\ \text{while } a^2 + b^2 > 1 \\ \quad a \leftarrow \text{rnd}(2) - 1 \\ \quad b \leftarrow \text{rnd}(2) - 1 \\ \quad V \leftarrow \begin{pmatrix} a \\ b \end{pmatrix} \\ V \end{array} \right] \cdot ap$$

Track the particles:

$$\begin{pmatrix} x_{i,n+1} \\ p_{i,n+1} \end{pmatrix} := \begin{cases} \begin{pmatrix} 50 \cdot ap \\ 0 \end{pmatrix} & \text{if } (x_{i,n})^2 + (p_{i,n})^2 \geq ap^2 \\ \text{otherwise} & \\ \quad \theta \leftarrow \text{rnorm}(1, 0, \sigma_\theta)_0 & \\ \quad pt \leftarrow p_{i,n} + \theta & \\ \quad a \leftarrow \sqrt{(x_{i,n})^2 + pt^2} & \\ \quad M(a) \cdot \begin{pmatrix} x_{i,n} \\ pt \end{pmatrix} & \end{cases}$$

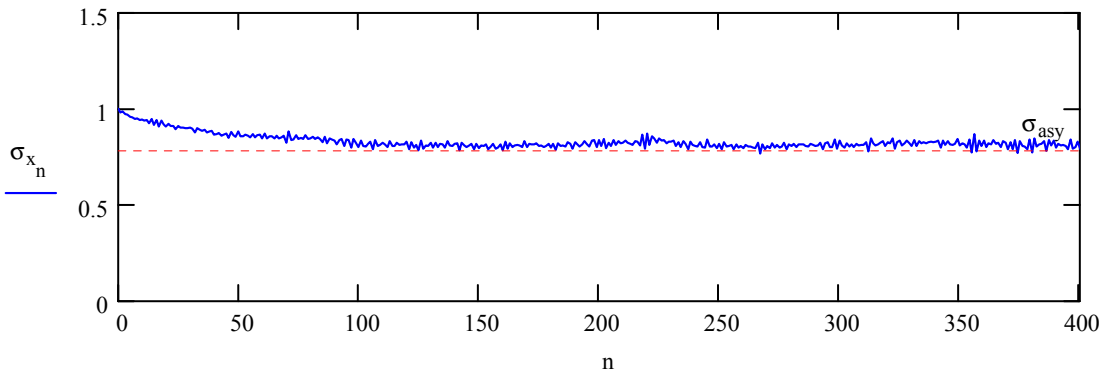
Calculate number of particles within the aperture: $N_{left_n} := \sum_i \left[(x_{i,n})^2 + (p_{i,n})^2 < ap^2 \right]$

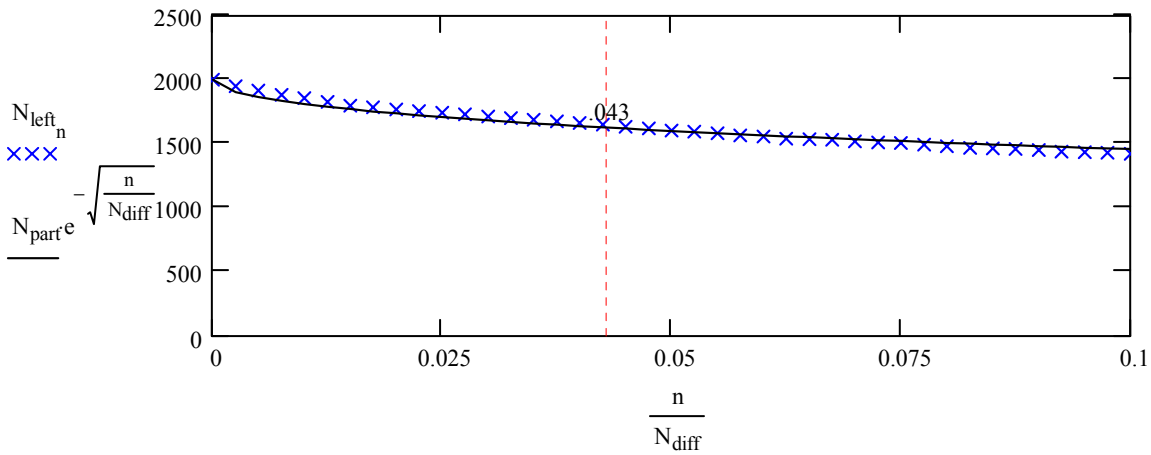
Compute rms of distribution at each step: $\sigma_{x_n} := \sqrt{\frac{1}{N_{left_n}} \cdot \left[\sum_i (x_{i,n})^2 \cdot \left[(x_{i,n})^2 + (p_{i,n})^2 < ap^2 \right] \right]}$

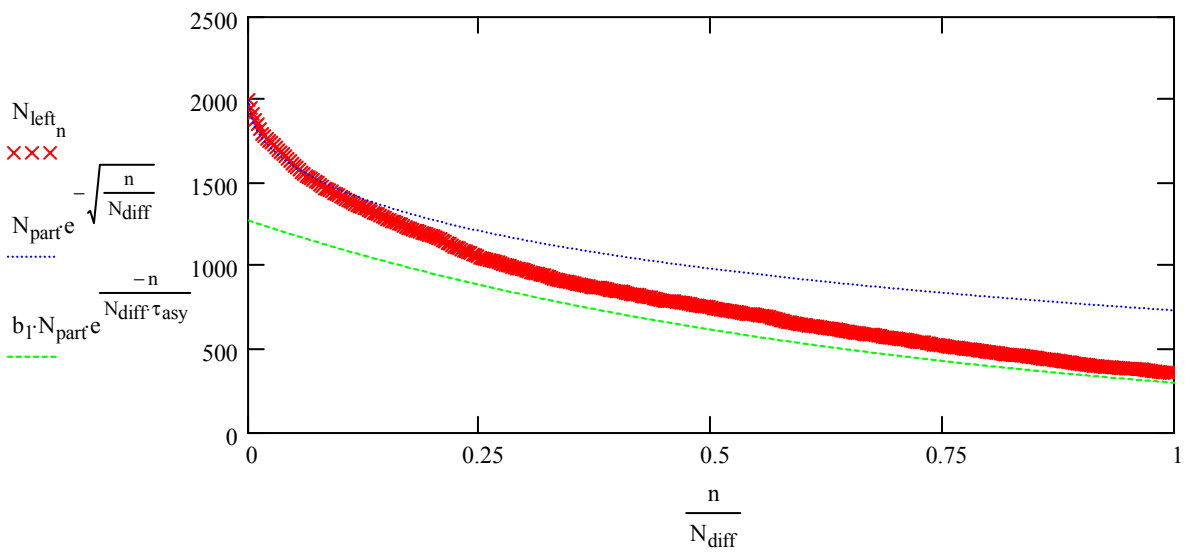
Expected rms of Asymptotic distribution: $\sigma_{asy} := 0.393 \cdot ap$ $\quad \sigma_{asy} = 0.786$

$\tau_{asy} := \left(\frac{2}{2.405} \right)^2$ $\quad$ Diffusion lifetime: $N_{diff} := \frac{ap^2}{\sigma_\theta^2}$ $\quad N_{diff} = 400$

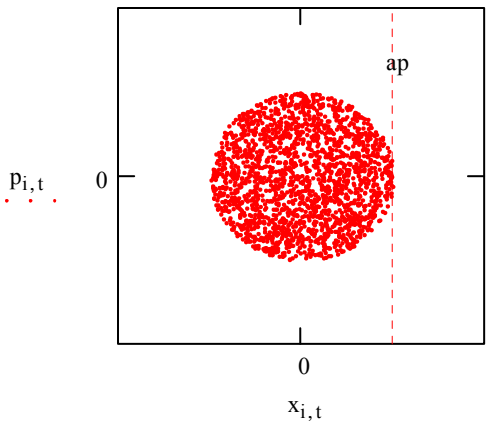
$\lambda := 2.504$ $\quad b_1 := \frac{2}{\lambda \cdot J1(\lambda)} \cdot \int_0^1 J0(\lambda \cdot \sqrt{Z})\, dZ$

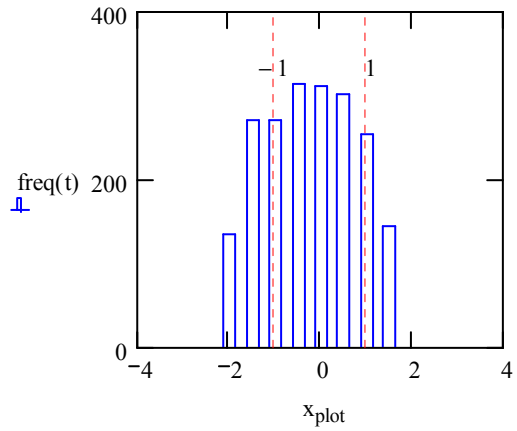

1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
n
σx_n
σasy


2500
2000
1500
1000
500
0
0
0.025
0.05
0.075
0.1
n / Ndiff
Nleft_n
Npart·e^(−√(n/Ndiff))
043

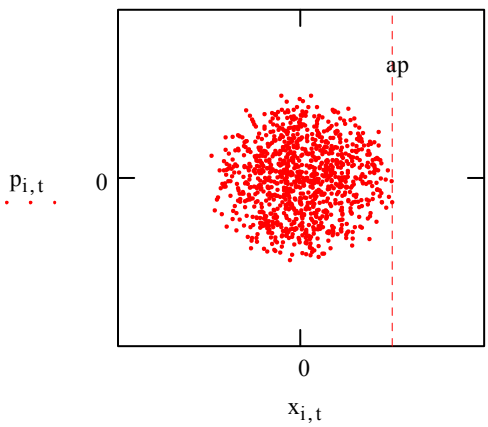

2500
2000
1500
1000
500
0
0
0.25
0.5
0.75
1
n / Ndiff
Nleft_n
Npart·e^(−√(n/Ndiff))
b1·Npart·e^(−n / (Ndiff·τasy))

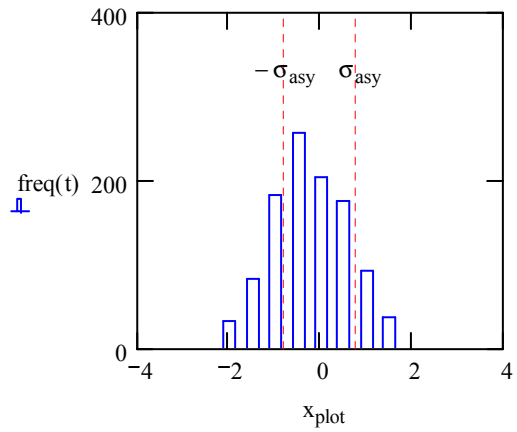
$t := 0$



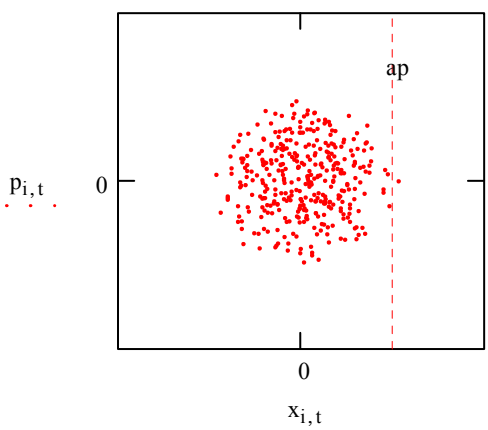


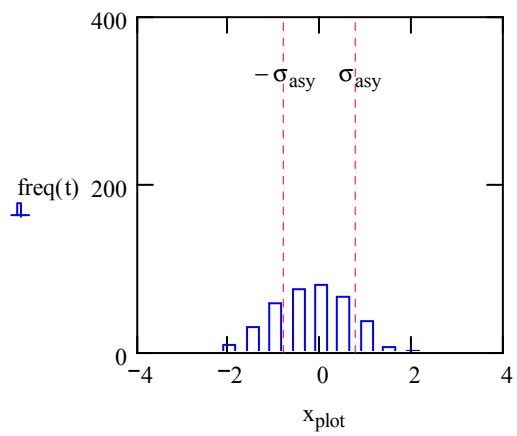
$t := 100$





$t := 400$





$\sigma_{asy} = 0.786$ $\sigma_{x_t} = 0.79$