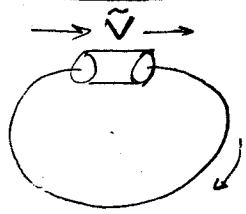
## Longitudinal Motion  circular accelerator

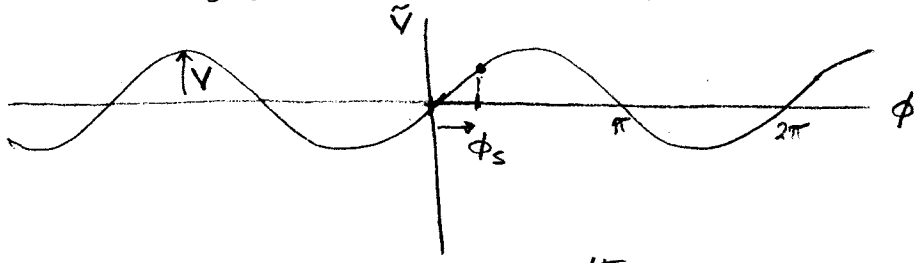



$$\tilde{V} = V \sin \omega_{RF} t$$

assumes resonant cavity

$$C = 2\pi R = v T_0 = v/f_0$$

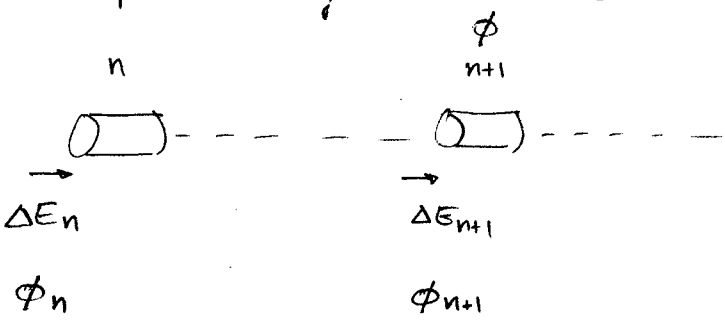
let $\omega_{RF} T_0 = h\, \omega_0 T_0 = 2\pi h$  $h$ = harmonic #




synchronous particle: $E_s$  $\frac{dE_s}{dn} = eV \sin \phi_s$

[ note: must adjust Bend field so that $\int B\, ds = \frac{p_s}{e} \cdot 2\pi$ ]

look at particle nearby: $E = E_s + \Delta E$




easy to see $\Delta E_{n+1} = \Delta E_n + eV(\sin \phi_n - \sin \phi_s)$

what about $\phi$ ?

particle will take more/less time to go around, depending upon $E$ (or, $p$)

$$T = \frac{C}{v} \qquad \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2} \frac{\Delta p}{p}$$

$$\frac{\Delta p}{p} = \frac{1}{\beta^2} \frac{\Delta E}{E}$$

$$\frac{\Delta C}{C} \equiv \frac{1}{\gamma_t^2} \frac{\Delta p}{p} \text{ (more later)}$$

$$\Rightarrow \frac{\Delta T}{T} = \underbrace{\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)}_{\eta} \frac{\Delta p}{p}$$

($\eta$ = slip factor)

so, going once around, the phase will slip by an amount

$$\frac{\Delta\phi}{2\pi h} = \frac{\Delta T}{T} = \eta \frac{\Delta p}{p} = \frac{\eta}{\beta^2}\frac{\Delta E}{E}$$

$$\Rightarrow \qquad \phi_{n+1} = \phi_n + 2\pi h \eta \left(\frac{\Delta p}{p_s}\right)_{n+1}$$

or, we have

$$\boxed{\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \\ \phi_{n+1} &= \phi_n + \frac{2\pi h \eta}{\beta^2 E_s}\Delta E_{n+1} \end{aligned}}$$

(BASIC program)

TRACK, or approximate w/ diff eq's ...

$$\frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s)$$

$$\frac{d\phi}{dn} = \frac{2\pi h \eta}{\beta^2 E_s}\Delta E$$

$$\Rightarrow \quad \frac{d^2\phi}{dn^2} = \frac{2\pi h \eta e V}{\beta^2 E_s}(\sin\phi - \sin\phi_s)$$

suppose $\phi = \phi_s + \Delta\phi$, $\Delta\phi \ll 1$, then

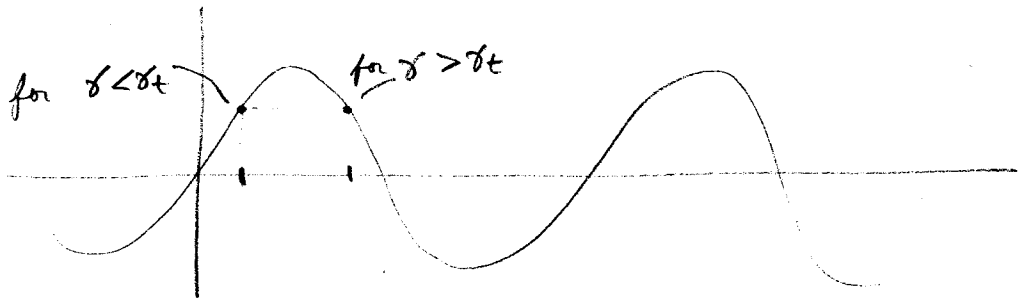
$$\begin{aligned} &\sin\phi - \sin\phi_s \\ &= \sin\phi_s \cancel{\cos\Delta\phi} + \sin\Delta\phi\cos\phi_s - \cancel{\sin\phi_s} \\ &\approx \Delta\phi\cos\phi_s \end{aligned}$$

$$\Rightarrow \quad \boxed{\frac{d^2\Delta\phi}{dn^2} - \frac{2\pi h \eta e V \cos\phi_s}{\beta^2 E_s}\Delta\phi = 0}$$

then motion is stable if $\boxed{\eta\cos\phi_s < 0}$

So, when $\eta < 0$ $[\gamma < \gamma_t]$, we want $\cos\phi_s > 0$

when $\eta > 0$ $[\gamma > \gamma_t]$, we want $\cos\phi_s < 0$




$\therefore$ if $\gamma_t$ exists, need "phase jump @ transition"

$\gamma mc^2$ = transition energy

| | | |
|---|---|---|
| BOO: | $\gamma_t \approx 4.5$ | $\therefore$ cross |
| MI: | $\gamma_t \approx 20$ | $\therefore$ cross |
| TEV: | $\gamma_t \approx 18$ | $\therefore$ do NOT cross |

small oscillation period...

$$\omega_s = \sqrt{-\frac{2\pi h \eta e V \cos\phi_s}{\beta^2 E_s}}$$

$$\nu_s \equiv \sqrt{-\frac{h \eta e V \cos\phi_s}{2\pi \beta^2 E_s}} = \text{synchrotron tune}$$

$$\frac{1}{\nu_s} = \text{synchrotron period (\# turns)} \gg 1$$

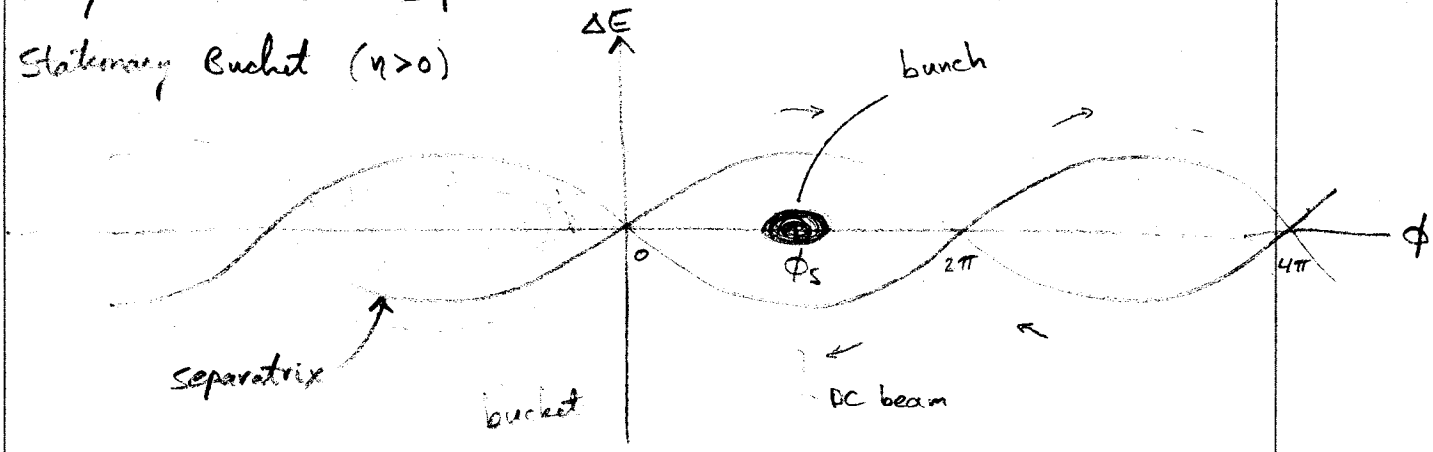
ex: TEV @ 150 GeV

$$\nu_s = \sqrt{-\frac{(1113)(\frac{1}{18^2})(10^6\,eV)(-1)}{2\pi \quad 150\cdot 10^9\,eV}} \approx \frac{1}{500}$$

synchrotron "frequency" $= \nu_s f_0 = \dfrac{\nu_s c\beta}{C} \approx \dfrac{3\cdot 10^8\,m/s}{500\cdot 2\pi\cdot 10^3\,m} \approx 100\,Hz$

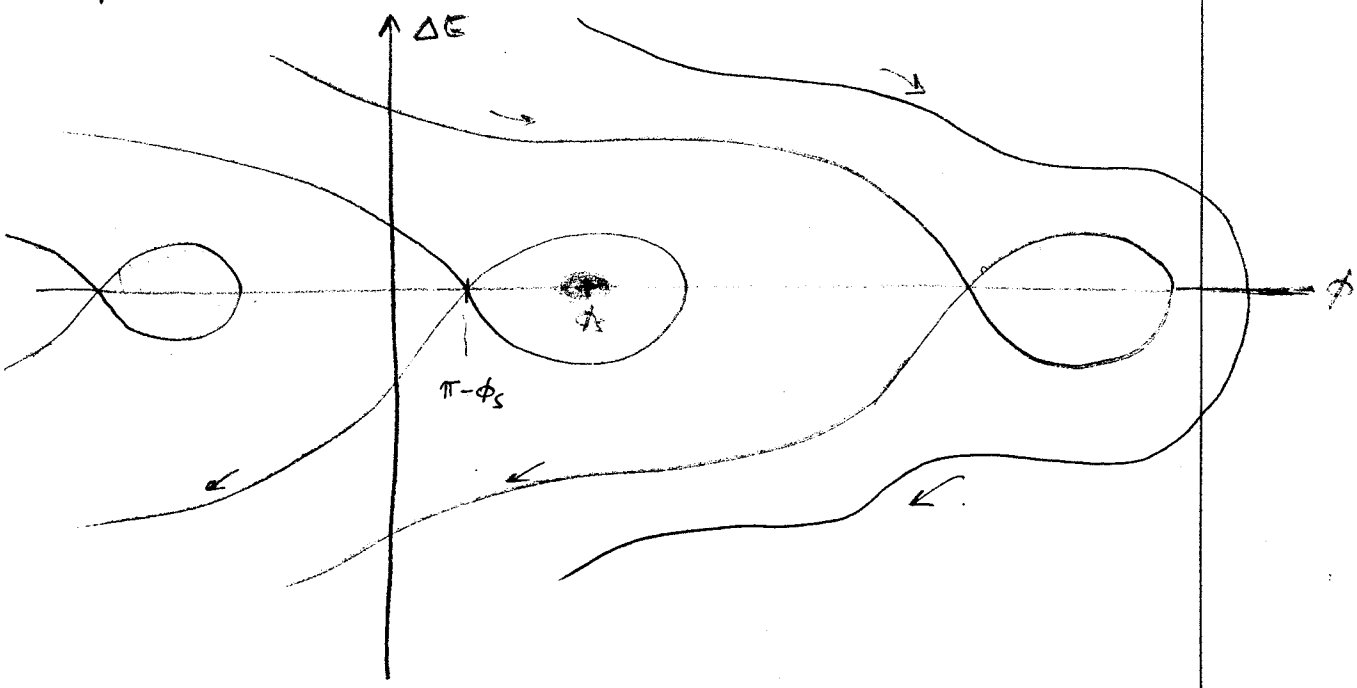
Longitudinal Phase Space

Stationary Bucket ($\eta > 0$)

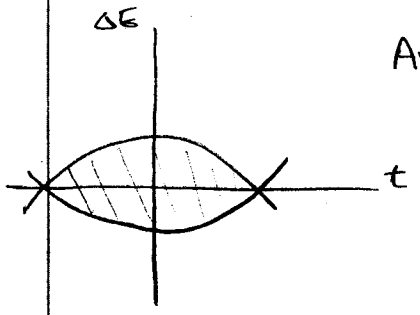



$\nu_s$ depends on amplitude of motion; near separatrix, $\nu_s \to 0$

Accelerating Bucket ($\eta > 0$)




Change to canonical variables: $\oint \Delta E \cdot dt$ = adiabatic invariant




Area of Stationary bucket:

$$\boxed{A = \frac{8C}{\pi h c}\sqrt{\frac{eV \cdot E_s}{2\pi h |\eta|}}}$$

Longitudinal Phase Space Area of beam = Longitudinal Emittance

ex: 95% longitudinal emittance of Gaussian Beam...

<u>assume</u>: "small" oscillations, $\Delta\phi = \phi - \phi_s \ll 1$

then,

$$\boxed{S = 2\pi^2 h \sqrt{-\frac{18\beta^2 E_s eV \cos\phi_s}{\pi h \eta}}\; f_o \sigma_t^2}$$

$$= \frac{3}{h f_o}\sqrt{-\frac{2\pi h \eta E_s^3 \beta^2}{eV \cos\phi_s}}\left(\frac{\sigma_p}{p}\right)^2$$

ex: TEVATRON @ 150

$$A = \frac{8(2\pi \cdot 10^3 m)}{\pi(1113)(3\cdot 10^8 m/s)}\sqrt{\frac{(10^6 eV)(150\cdot 10^9 eV)}{2\pi(1113)\left(\frac{1}{18^2}\right)}} = \boxed{4 \text{ eV}\cdot\text{sec}}$$

$$S = 2\pi^2(1113)\sqrt{\frac{18(150\cdot 10^9)(10^6)}{\pi(1113)\left(\frac{1}{18^2}\right)}}\,(417.7\cdot 10^{-3})\,(2\cdot 10^{-9})^2 \text{ eV}\cdot\text{sec}$$

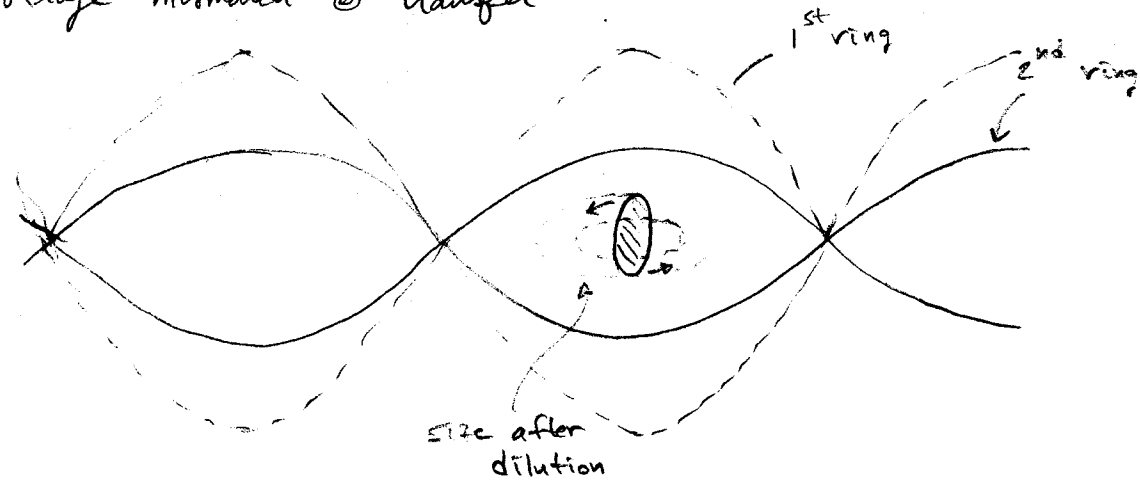
$$= \boxed{2 \text{ eV}\cdot\text{sec}}$$

note: $2 \text{ eV}\cdot\text{sec} \Rightarrow \frac{\sigma_p}{p} = 0.36\cdot 10^{-3}$

Machine matching: $\left(\frac{V}{h|\eta|}\right)_1 = \left(\frac{V}{h|\eta|}\right)_2$ on transfer

(assumming same $f_{RF}$)

Voltage mismatch @ transfer

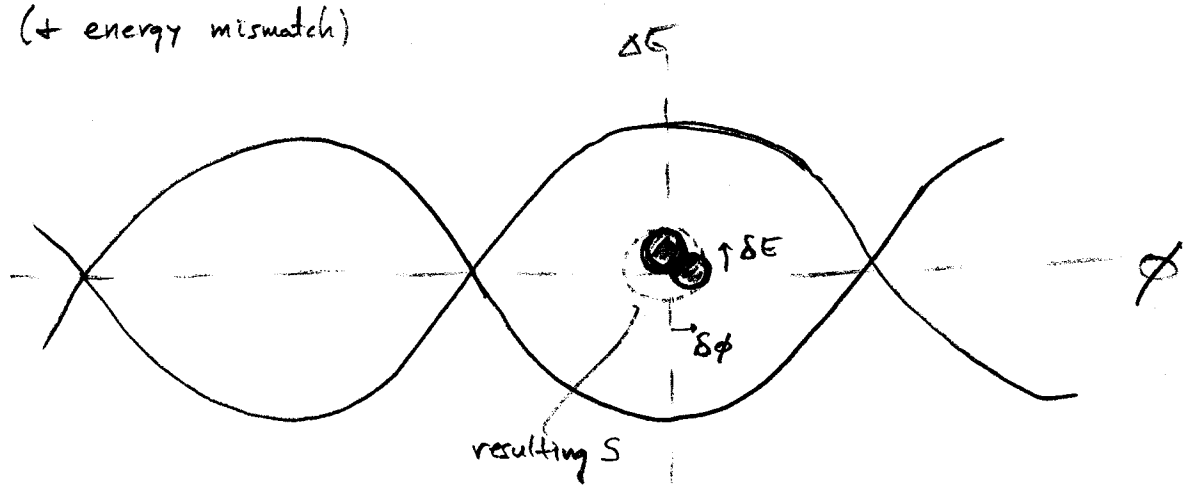



increase in longitudinal emittance:

$$S/S_0 = \frac{1}{2} \cdot \frac{(1 + K/K_0)}{\sqrt{K/K_0}} \qquad K = \frac{V}{h|\eta|}$$

Phase mismatch @ transfer is similar:

(+ energy mismatch)




$$S/S_0 = 1 + \frac{1}{2}\left(\frac{\delta E}{\sigma_E}\right)^2 \quad , \text{ or } \quad = 1 + \frac{1}{2}\left(\frac{\delta \phi}{\sigma_\phi}\right)^2$$

[all these assume $S \ll A$]

Problems :

a) From $$\frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s)$$

$$\frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E_s}\Delta E$$

Show that

$$\Delta E^2 + \frac{2\beta^2 E_s eV}{2\pi h\eta}(\cos\phi + \phi\sin\phi_s) = \text{constant}$$

b) From a) above, and for $\phi_s = 180°$, find an expression for the maximum $\Delta E$ on the separatrix (i.e., the "bucket height").

[ ANS: for a stationary bucket,

$$\Delta E_{max} = \sqrt{\frac{2\beta^2 E_s eV}{\pi h|\eta|}}$$ ]

c) Evaluate the bucket height for the Tevatron at i) 150 GeV

ii) 980 GeV