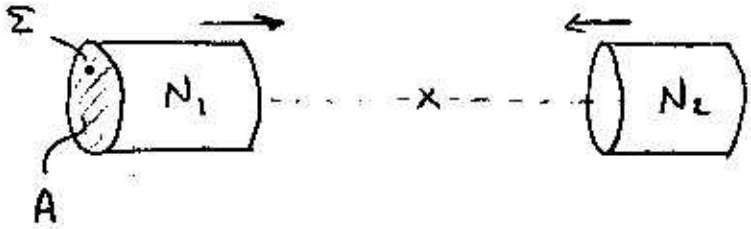
LUMINOSITY

a) uniform beams, cross sectional area A, interaction cross section Σ

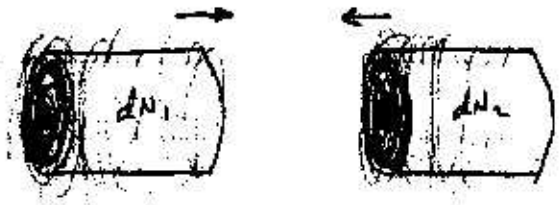


interaction rate

$$R = \underbrace{\frac{\Sigma}{A}}_{\text{fractional area obscured by each of particles in beam 1}} \cdot \underbrace{N_1}_{\text{\# of "targets" in beam 1}} \cdot \underbrace{N_2}_{\text{\# of incident particles in beam 2}} \cdot \underbrace{f}_{\text{frequency of bunch passages}} = \underbrace{\frac{f N_1 N_2}{A}}_{\mathcal{L}} \cdot \Sigma$$

$$\boxed{\mathcal{L} = \frac{f N_1 N_2}{A}}$$

b) round, Gaussian beams



$$dN = \frac{N_1}{2\pi\sigma^2} e^{-x^2/2\sigma^2} e^{-y^2/2\sigma^2} dx\,dy \quad , \quad \frac{N_2}{2\pi\sigma^2} e^{-x^2/2\sigma^2} e^{-y^2/2\sigma^2} dx\,dy$$

$$= \frac{N_1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta \quad , \quad \frac{N_2}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta$$

shell @ radius r, thickness dr

$$\therefore R = \iint_{\text{all shells}} \frac{\Sigma}{r\,dr\,d\theta} \cdot \left(\frac{N_1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta\right)\left(\frac{N_2}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta\right) \cdot f$$

$$\Rightarrow R = \sum \int_0^{2\pi}\int_0^{\infty} \frac{N_1 N_2}{4\pi^2\sigma^4} e^{-r^2/\sigma^2} \, r\,dr\,d\theta \cdot f$$

$$\Rightarrow \mathcal{L} = \frac{f N_1 N_2}{2\pi\sigma^2} \cdot \frac{1}{2}\int_0^{\infty} e^{-r^2/\sigma^2} \frac{2r\,dr}{\sigma^2}$$

$$= \frac{f N_1 N_2}{4\pi\sigma^2} \underbrace{\int_0^{\infty} e^{-u}\,du}_{\longrightarrow \; -e^{-u}\Big|_0^{\infty} \; = \; 1}$$

$$\therefore \boxed{\mathcal{L} = \frac{f N_1 N_2}{4\pi\sigma^2}} = \frac{(48\times10^3 \cdot 36 \text{ /sec})(2\cdot10^{11})(10^{10})}{4\pi\,(25)^2 \cdot 10^{-8}\text{ cm}^2} = 4\times10^{31}\text{ cm}^{-2}\text{sec}^{-1}$$

c) What if the beams are not round? i.e., $\sigma_x \neq \sigma_y$

need to replace with $\frac{N_1}{2\pi\sigma_x\sigma_y} e^{-x^2/2\sigma_x^2} e^{-y^2/2\sigma_y^2} \cdot \frac{N_2}{2\pi\sigma_x\sigma_y} e^{-x^2/2\sigma_x^2} e^{-y^2/2\sigma_y^2}\,dx\,dy$

$$= N_1 N_2 \frac{1}{2\pi\sigma_x^2} e^{-x^2/\sigma_x^2} \frac{1}{2\pi\sigma_y^2} e^{-y^2/\sigma_y^2}\,dx\,dy$$

note: $\frac{1}{\sqrt{2\pi}\,\sigma_x}\int_{-\infty}^{\infty} e^{-x^2/\sigma_x^2}\,dx = \frac{\sqrt{2}}{\sqrt{2\pi}\,\sigma_x}\int_{-\infty}^{\infty} e^{-x^2/2(\sigma_x/\sqrt{2})^2}\,dx = \sqrt{2}$

$\vdots$

$$\Rightarrow \boxed{\mathcal{L} = \frac{f N_1 N_2}{4\pi\sigma_x\sigma_y}}$$

d) What if round, but different sizes for the 2 beams? i.e., $\sigma_1 \neq \sigma_2$

need to replace with $\frac{N_1}{2\pi\sigma_1^2} e^{-r^2/2\sigma_1^2} \frac{N_2}{2\pi\sigma_2^2} e^{-r^2/2\sigma_2^2}\, r\,dr$

$$\Rightarrow \quad N_1 N_2 \frac{1}{2\pi\sigma_1^2\sigma_2^2} e^{-r^2\left(\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}\right)} r\,dr$$

$$= \frac{N_1 N_2}{2\pi(\sigma_1^2+\sigma_2^2)} \underbrace{\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right) e^{-\frac{r^2}{2}\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)} r\,dr}_{\text{integrate} \to 1 \text{ as in case (b)}}$$

$$\therefore \quad \mathcal{L} = \frac{f N_1 N_2}{2\pi(\sigma_1^2+\sigma_2^2)}$$

e) Put it together: $\sigma_1 \neq \sigma_2$, $\sigma_x \neq \sigma_y$ for either beam

$$\Rightarrow \quad \boxed{\mathcal{L} = \frac{f N_1 N_2}{2\pi\sqrt{(\sigma_{1x}^2+\sigma_{2x}^2)(\sigma_{1y}^2+\sigma_{2y}^2)}}}$$

$\sigma^2 = \beta\epsilon_N/6\pi\gamma$, and $f = f_0 \cdot B$

revolution freq ↗ ↖ # bunches (36, say)

$$\Rightarrow \quad \boxed{\mathcal{L} = \frac{3 f_0 B N_p N_{\bar{p}} \gamma}{\beta^* \sqrt{(\epsilon_x^p+\epsilon_x^{\bar{p}})(\epsilon_y^p+\epsilon_y^{\bar{p}})}}}$$

assumes a) $\beta_x^* = \beta_y^*$

b) σ_x, σ_y independent of s

f) Consider LONGITUDINAL distribution...

note: design is for beam to be at a "waist" at the collision point:

$$\left.\begin{aligned}\beta &= \beta^* \\ \alpha &= \alpha^* = 0\end{aligned}\right\} \text{ at } s=0$$

we know that $K\beta = \alpha' + \gamma$, and so in a drift region:

$$\alpha' = -\gamma$$

$$\therefore \quad \alpha' = -\tfrac{1}{2}\beta'' = -\frac{1+\alpha^2}{\beta} = -\frac{1}{\beta^*} \quad @\ s=0$$

$$\text{or, } \beta'' = \frac{2}{\beta^*} \quad @\ s=0$$

$$\Rightarrow \quad \beta' = \frac{2}{\beta^*}s + \underbrace{\beta'(s=0)}_{=0} = \frac{2}{\beta^*}s$$

$$\Rightarrow \quad \boxed{\beta(s) = \beta^* + \frac{s^2}{\beta^*}} \quad \text{if } \alpha^* = 0$$

if bunch length $\sigma_z \ll \beta^*$, then $\beta \approx \beta^*$ over the luminous region + above equation(s) holds true.

if $\sigma_z \gtrsim \beta^*$, then can adversely affect total luminosity...

So, if consider Gaussian in all degrees of freedom, and watch as the 2 bunches move toward each other, then

$$R = f \cdot \iiint_{z \text{ over time}} \frac{\Sigma}{dA} \underbrace{\left[\rho_1 \quad dA \cdot (v_1 - v_2)\, dt\right]}_{\substack{\#\text{ particles in beam ① in} \\ \text{the volume } dA\cdot(\Delta v\, dt) \\ \text{where the 2 beams overlap}}} \rho_2 \quad dA\, dz$$

here,

$$\rho_1 = \frac{N_1}{(2\pi)^{3/2}\,\sigma_z\,\sigma(z)^2}\; e^{-(x^2+y^2)/2\sigma(z)^2}\; e^{-(z-vt)^2/2\sigma_z^2}$$

$$\rho_2 = \frac{N_2}{(2\pi)^{3/2}\,\sigma_z\,\sigma(z)^2}\; e^{-(x^2+y^2)/2\sigma(z)^2}\; e^{-(z-vt)^2/2\sigma_z^2}$$

$$\text{where} \quad \sigma(z)^2 = \frac{\epsilon_N \beta^*}{6\pi(\gamma\beta)}\left[1+\left(\frac{z}{\beta^*}\right)^2\right]$$

and so, since $v_2 = -v_1 = -v$,

$$R = \Sigma \cdot f \cdot 2v \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \frac{N_1 N_2}{(2\pi)^3\,\sigma_z^2}\,\frac{1}{\sigma(z)^4}\; e^{-r^2/\sigma(z)^2}\; e^{-[(z-vt)^2+(z+vt)^2]/2\sigma_z^2} \cdot dt\; r\,dr\,d\theta\,dz$$

so,

$$\mathcal{L} = \frac{f N_1 N_2}{(2\pi)^{3/2}\,\sigma_z} \cdot \sqrt{2}\,\frac{\sqrt{2}\,v}{\sqrt{2\pi}\,\sigma_z}\int_{-\infty}^{\infty} e^{-v^2t^2/\sigma_z^2}\,dt$$

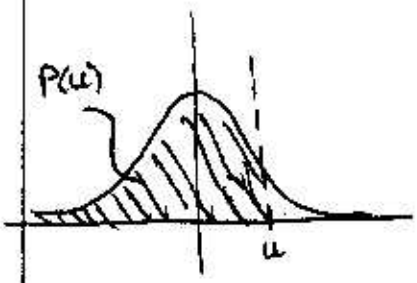
$$\cdot\,\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\sigma(z)^2}\left[\frac{2}{\sigma(z)^2}\int_0^{\infty} e^{-r^2/\sigma(z)^2}\,r\,dr\right] e^{-z^2/\sigma_z^2}\,dz$$

$$\Rightarrow \mathcal{L} = \frac{f N_1 N_2}{(2\pi)^{3/2}\,\sigma_z}\cdot\sqrt{2}\cdot\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\sigma(z)^2}\; e^{-z^2/\sigma_z^2}\,dz$$

$$= \underbrace{\frac{f N_1 N_2}{4\pi\sigma^{*2}}}_{=\mathcal{L}_0} \cdot \underbrace{\frac{\sqrt{2}}{\sqrt{2\pi}\,\sigma_z}\int_{-\infty}^{\infty}\frac{1}{1+(z/\beta^*)^2}\; e^{-z^2/\sigma_z^2}\,dz}_{\text{Hourglass factor}}$$

$$\therefore \quad f \equiv \mathcal{L}/\mathcal{L}_0 = \frac{\sqrt{2}}{\sqrt{2\pi}\,\sigma_z} \int_{-\infty}^{\infty} \frac{1}{1+(z/\beta^*)^2}\, e^{-z^2/\sigma_z^2}\, dz$$

G&R, eq. 3.466 #1: $\displaystyle\int_0^\infty \frac{e^{-\mu^2 x^2}}{x^2+\beta^2}\, dx = \left[1 - \Phi(\beta\mu)\right] \frac{\pi}{2\beta}\, e^{\beta^2\mu^2}$



where $\displaystyle\Phi(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$

$$= 2P(z\sqrt{2}) - 1$$

So, $\displaystyle f = \frac{\sqrt{2}\,\beta^*}{\sqrt{2\pi}\,\sigma_z} \cdot 2 \int_0^\infty \frac{e^{-\mu^2 u^2}}{u^2+1}\, du$, $\mu \equiv \beta^*/\sigma_z$

$u \equiv z/\beta^*$

$$= \frac{2\sqrt{2}}{\sqrt{2\pi}} \left(\frac{\beta^*}{\sigma_z}\right) \cdot \frac{\pi}{2}\, e^{(\beta^*/\sigma_z)^2} \left[1 - \Phi(\beta^*/\sigma_z)\right]$$

or, $$\boxed{f(\beta^*/\sigma_z) = \sqrt{\pi}\,(\beta^*/\sigma_z)\, e^{(\beta^*/\sigma_z)^2} \left[1 - \Phi(\beta^*/\sigma_z)\right]}$$

ex: $\beta^* = 35$ cm, $\sigma_z = 60$ cm (2 nsec),

then $f \approx 0.6$

Luminosity reduction factor, f = L/L_0

$c := 2.9979 \cdot 10^{8} \cdot \frac{m}{sec}$

Note:

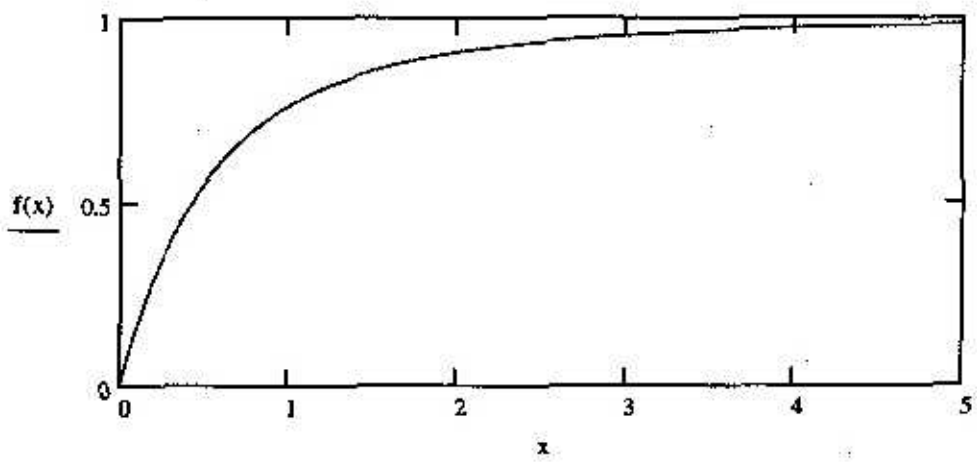
$nsec := 10^{-9} \cdot sec$

Phi(x) from notes, in terms of Mathcad function pnorm(x,mu,sigma) ---

$$\Phi(x) := 2 \cdot pnorm\left(x \cdot \sqrt{2}, 0, 1\right) - 1$$

So,...

$$f(x) := \sqrt{\pi} \cdot x \cdot e^{x^2} \cdot \left(1 - \Phi(x)\right)$$



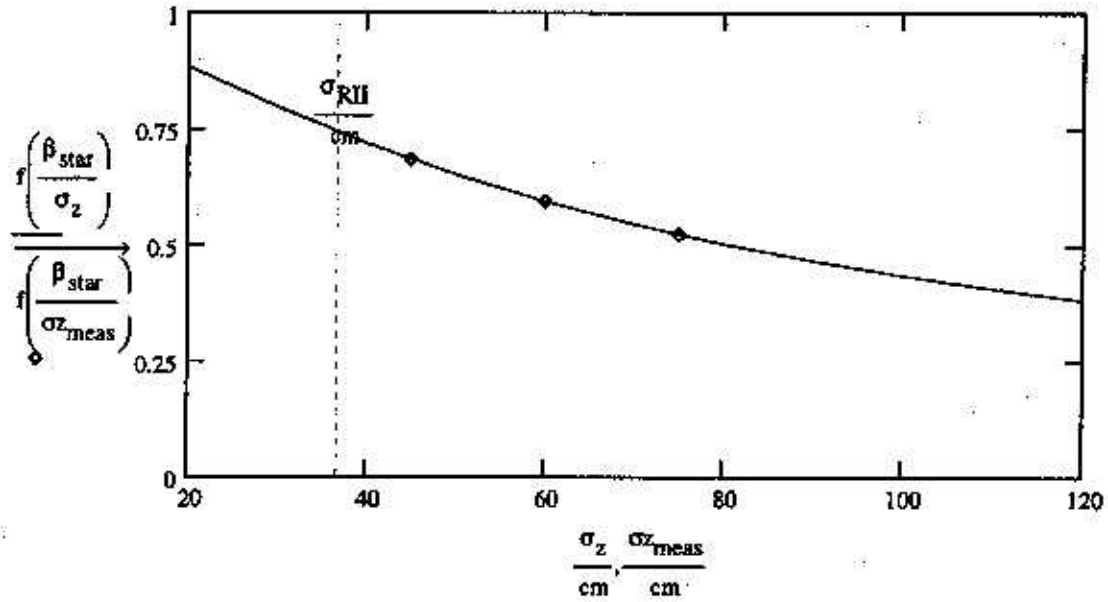
For Tevatron,

$\beta_{star} := 0.35 \cdot m$ Design bunch length: (Runll Handook) $\sigma_{RII} := 37 \cdot cm$ $f\left(\frac{\beta_{star}}{\sigma_{RII}}\right) = 0.742$

Range of measured bunch lengths (rms) at 980 GeV: 1.5 nsec to about 2.5 nsec

$ss := \begin{pmatrix} 1.5 \\ 2 \\ 2.5 \end{pmatrix} \cdot nsec$ $\sigma z_{meas} := ss \cdot c$ $\sigma z_{meas} = \begin{pmatrix} 44.969 \\ 59.958 \\ 74.948 \end{pmatrix} cm$

$\sigma_z := 0.1 \cdot m, 0.2 \cdot m .. 5 \cdot m$



Length of Luminous Region

Round, Gaussian beams of equal emittances, with $\sigma_z \ll \beta^*$:

start w/

$$\mathcal{L} = \frac{f N_1 N_2}{4\pi \sigma^{*2}} \frac{\sqrt{2}}{\sqrt{2\pi}\,\sigma_z} \int_{-\infty}^{\infty} \frac{1}{1+(z/\beta^*)^2} e^{-z^2/\sigma_z^2} dz$$

$$\Rightarrow d\mathcal{L}(z) = \mathcal{L}_0 \frac{1}{\sqrt{2\pi}\,(\sigma_z/\sqrt{2})} e^{-\frac{1}{2}\frac{z^2}{(\sigma_z/\sqrt{2})^2}} dz$$

which is Gaussian w/ st. dev $= \underline{\underline{\sigma_z/\sqrt{2}}}$

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for $\sigma_z \gtrsim \beta^*$, $d\mathcal{L} = \mathcal{L}_0 \frac{\sqrt{2}\,\beta^*}{\sqrt{2\pi}\,\sigma_z} \frac{1}{1+u^2} e^{-\mu^2 u^2} du$, $u = z/\beta^*$, $\mu = \beta^*/\sigma_z$

$$\therefore \langle u^2 \rangle = \frac{\int_0^\infty \frac{u^2}{1+u^2} e^{-\mu^2 u^2} du}{\int_0^\infty \frac{1}{1+u^2} e^{-\mu^2 u^2} du} = \frac{\frac{\sqrt{\pi}}{2\mu} - \frac{\pi}{2} e^{\mu^2}[1-\Phi(\mu)]}{\frac{\pi}{2} e^{\mu^2}[1-\Phi(\mu)]}$$

or,

$$\boxed{z_{rms} = \beta^* u_{rms} = \beta^* \frac{\frac{\sqrt{\pi}}{2}\frac{\sigma_z}{\beta^*} - \frac{\pi}{2} e^{(\beta^*/\sigma_z)^2}[1-\Phi(\beta^*/\sigma_z)]}{\frac{\pi}{2} e^{(\beta^*/\sigma_z)^2}[1-\Phi(\beta^*/\sigma_z)]}}$$

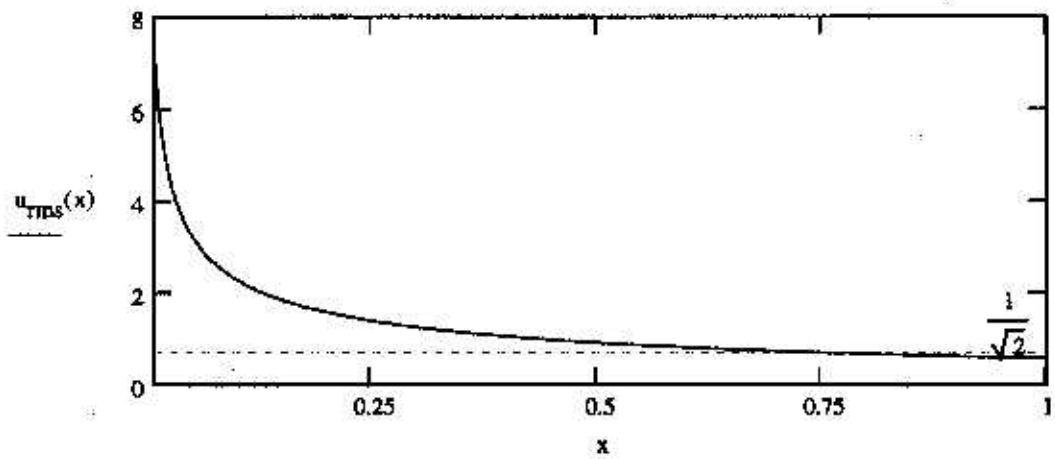
note, $\lim_{\mu\to\infty} \langle u^2 \rangle = \frac{\frac{1}{\sqrt{\pi}}\frac{1}{\mu^2} + 2\frac{1}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}}{\frac{2}{\sqrt{\pi}}} \rightarrow \frac{1}{2}\frac{1}{\mu^2} = \frac{1}{2}\frac{\sigma_z^2}{\beta^{*2}}$

$\Rightarrow z_{rms} \rightarrow \sigma_z/\sqrt{2}$ as $\sigma_z \ll \beta^*$

Luminous Region with Hour Glass Factor:

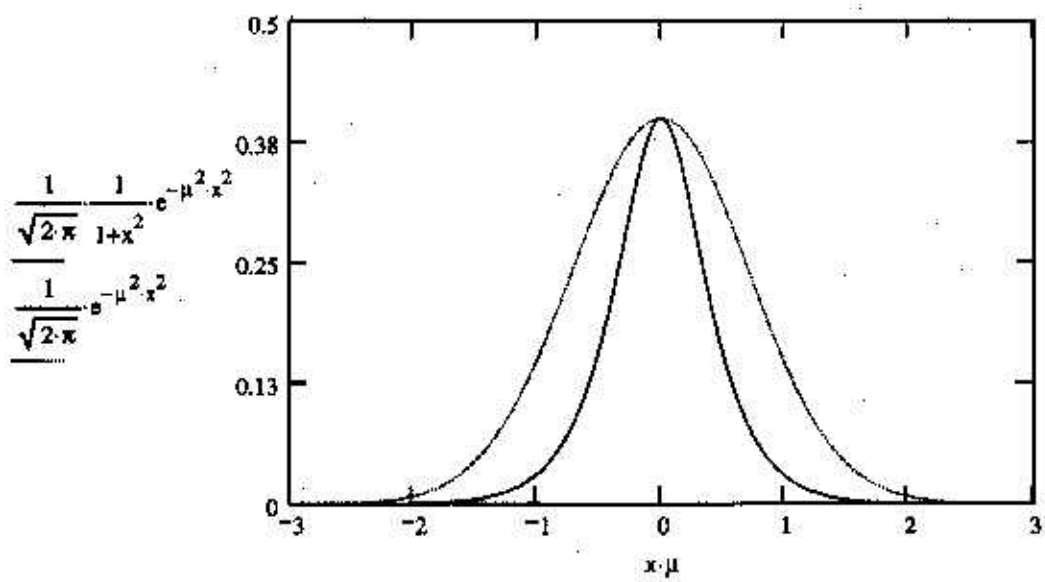
Here, x = ratio of beta_star to sigma_z:

$$u_{rms}(x) := \left[\frac{\frac{\sqrt{\pi}}{2\cdot x} - \frac{\pi}{2}\cdot e^{x^2}\cdot(1-\Phi(x))}{\frac{\pi}{2}\cdot e^{x^2}\cdot(1-\Phi(x))} \right]^{\frac{1}{2}}$$



$\beta_{star} := 35\cdot cm$ $\quad \sigma_z := 70\cdot cm$ $\quad \mu := \frac{\beta_{star}}{\sigma_z}$ $\quad z_{rms}(x) := \beta_{star}\cdot u_{rms}(x)$

$\frac{\sigma_z}{\sqrt{2}} = 49.497\,cm$ $\quad u_{rms}(\mu) = 0.913$ $\quad z_{rms}(\mu) = 31.938\,cm$



$$R = f \cdot \iiiint \frac{\Sigma}{dA} \left[\rho_1 \, dA \, (v_1 - v_2) \, dt \right] \rho_2 \, dA \, dz$$

$$\rho_1 = \frac{N_1}{(2\pi)^{3/2} \sigma_z \, \sigma_x(z) \, \sigma_y(z)} \, e^{-x^2/2\sigma_x(z)^2} \, e^{-y^2/2\sigma_y^2(z)} \, e^{-(z-vt)^2/2\sigma_z^2}$$

$$\rho_2 = \frac{N_2}{(2\pi)^{3/2} \sigma_z \, \sigma_x(z) \, \sigma_y(z)} \, e^{-x^2/2\sigma_x(z)^2} \, e^{-y^2/2\sigma_y(z)^2} \, e^{-(z+vt)^2/2\sigma_z^2}$$

$$\mathcal{L} = \frac{f N_1 N_2}{(2\pi)^{3/2} \sigma_z} \cdot \sqrt{2} \, \frac{\sqrt{2}\, v}{\sqrt{2\pi}\, \sigma_z} \int_{-\infty}^{\infty} e^{-v^2 t^2/\sigma_z^2} \, dt$$

$$\cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sigma_x \sigma_y} \iiint_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, \sigma_x(z)} \, e^{-\frac{x^2}{2\sigma_x(z)^2}} \, \frac{1}{\sqrt{2\pi}\, \sigma_y(z)} \, e^{-\frac{y^2}{2\sigma_y(z)^2}} \, e^{-z^2/\sigma_z^2} \, dx\, dy$$

$$= \frac{f N_1 N_2}{(2\pi)^{3/2} \sigma_z} \cdot \sqrt{2} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sigma_x(z) \sigma_y(z)} \, e^{-z^2/\sigma_z^2} \, dz$$

let $\sigma_x(z) = \sqrt{\epsilon \beta(z)} = \sqrt{\epsilon \beta^* \left[1 + \left(\frac{z - z_0}{\beta^*} \right)^2 \right]} = \sigma^* \sqrt{1 + \left(\frac{z - z_0}{\beta^*} \right)^2}$

and let $\sigma_y(z) = \sigma^* \sqrt{1 + \left(\frac{z + z_0}{\beta^*} \right)^2}$

then,

$$\mathcal{L} = \frac{f N_1 N_2}{4\pi \sigma^{*2}} \cdot \frac{\sqrt{2}}{\sqrt{2\pi}\, \sigma_z} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1 + \left(\frac{z - z_0}{\beta^*} \right)^2} \sqrt{1 + \left(\frac{z + z_0}{\beta^*} \right)^2}} \, e^{-z^2/\sigma_z^2} \, dz$$

$$\left[1+\left(\frac{z-z_0}{\beta^*}\right)^2\right]\left[1+\left(\frac{z+z_0}{\beta^*}\right)^2\right]$$

$$= 1 + \frac{z^2}{\beta^{*2}} - \cancel{2\frac{z_0 z}{\beta^{*2}}} + \frac{z_0^2}{\beta^{*2}} + \frac{z^2}{\beta^{*2}} + \cancel{2\frac{z_0 z}{\beta^{*2}}} + \frac{z_0^2}{\beta^{*2}} + \left(\frac{z^2-z_0^2}{\beta^{*2}}\right)^2$$

$$= 1 + 2\left(\frac{z}{\beta^*}\right)^2 + 2\left(\frac{z_0}{\beta^*}\right)^2 + \left(\frac{z}{\beta^*}\right)^4 - 2\left(\frac{z_0}{\beta^*}\right)^2\left(\frac{z}{\beta^*}\right)^2 + \left(\frac{z_0}{\beta^*}\right)^4$$

$$= 1+2\left(\frac{z_0}{\beta^*}\right)^2 + \left(\frac{z_0}{\beta^*}\right)^4 + 2\left[1-\left(\frac{z_0}{\beta^*}\right)^2\right]\left(\frac{z}{\beta^*}\right)^2 + \left(\frac{z}{\beta^*}\right)^4$$

$$= \left[1+\left(\frac{z_0}{\beta^*}\right)^2\right]^2 + 2\left[1-\left(\frac{z_0}{\beta^*}\right)^2\right]\left(\frac{z}{\beta^*}\right)^2 + \left(\frac{z}{\beta^*}\right)^4$$

$u = z/\beta^*$

$$\Rightarrow \mathcal{L}/\mathcal{L}_0 = 2\cdot\frac{\sqrt{2}}{\sqrt{2\pi}}\left(\beta^*/\sigma_z\right)\cdot\int_0^\infty \frac{1}{\sqrt{a+bu^2+cu^4}}\, e^{-\mu^2 u^2}\, du$$

$$a = \left[1+\left(\frac{z_0}{\beta^*}\right)^2\right]^2 \qquad b = 2\left[1-\left(\frac{z_0}{\beta^*}\right)^2\right] \qquad c = 1$$

$$\mu = \beta^*/\sigma_z$$

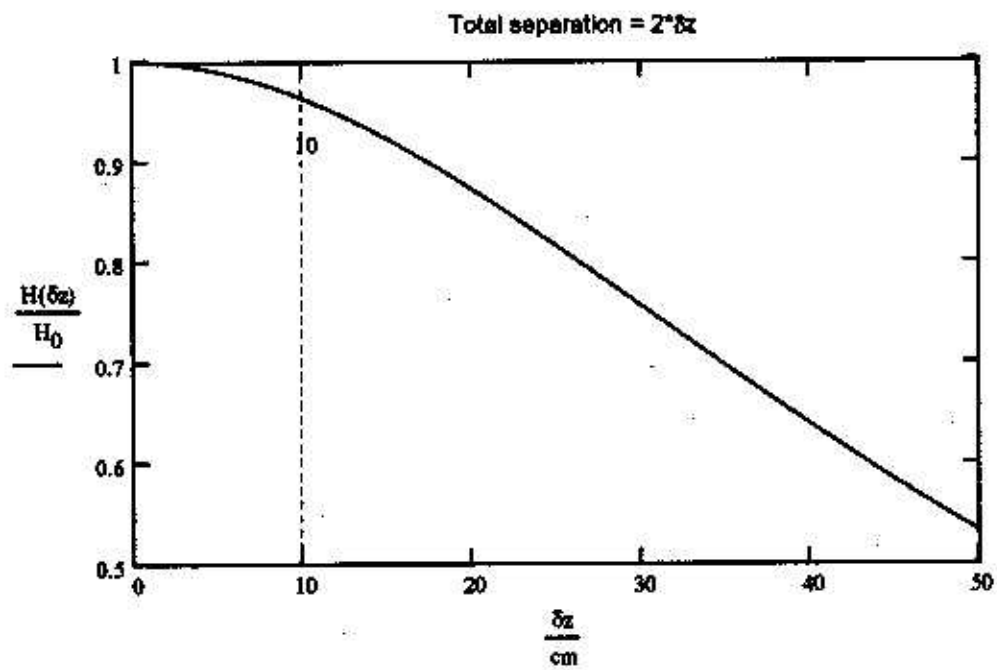
$z_0 := 0 \cdot cm$ $\qquad \beta_{star} := 35 \cdot cm$ $\qquad \sigma_z := 60 \cdot cm$

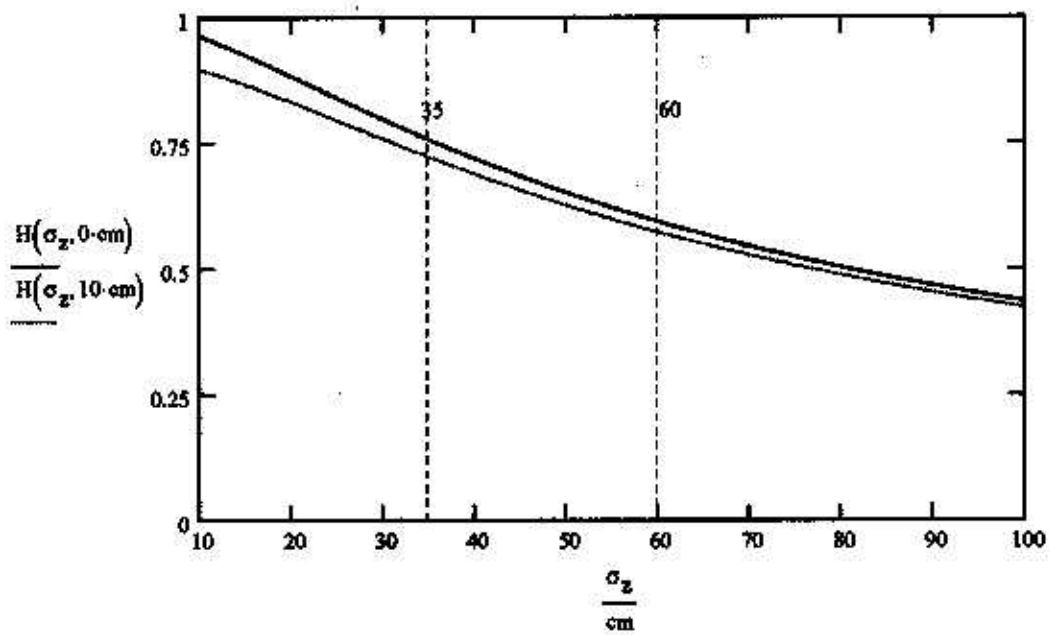
$$a(x) := \left[1 + \left(\frac{x}{\beta_{star}}\right)^2\right]^2 \qquad b(x) := 2 \cdot \left[1 - \left(\frac{x}{\beta_{star}}\right)^2\right] \qquad \mu := \frac{\beta_{star}}{\sigma_z}$$

$$H(z_0) := \frac{2 \cdot \sqrt{2}}{\sqrt{2 \cdot \pi}} \cdot (\mu) \cdot \int_0^8 \frac{1}{\sqrt{a(z_0) + b(z_0) \cdot u^2 + u^4}} \cdot e^{-(\mu \cdot u)^2} \, du$$

$H(z_0) = 0.595$ $\qquad H_0 := H(0 \cdot cm)$

$\delta z := 0 \cdot cm, 0.1 \cdot cm .. 50 \cdot cm$ $\qquad 1 - \frac{H(10 \cdot cm)}{H_0} = 3.572\%$



$$H(\sigma_z, z_0) := \frac{2\cdot\sqrt{2}}{\sqrt{2\cdot\pi}}\cdot\left(\frac{\beta_{star}}{\sigma_z}\right)\cdot\int_0^8 \frac{1}{\sqrt{a(z_0) + b(z_0)\cdot u^2 + u^4}}\cdot e^{-\left(\frac{\beta_{star}}{\sigma_z}\cdot u\right)^2}\,du$$

$\sigma_z := 10\cdot cm, 11\cdot cm .. 100\cdot cm$