LONGITUDINAL PHASE SPACE TRAJECTORIES

MJS

$$\left.\begin{aligned} \frac{d\Delta E}{dn} &= eV(\sin\phi - \sin\phi_s) \\ \frac{d\phi}{dn} &= 2\pi h\eta \frac{\Delta E}{\beta^2 E_s} \end{aligned}\right\} \quad \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta eV}{\beta^2 E_s}(\sin\phi - \sin\phi_s)$$

Find 1st integral of the motion...

$$\int \underbrace{\frac{d^2\phi}{dn^2}}\frac{d\phi}{dn}\,dn = \int \underbrace{\frac{2\pi h\eta eV}{\beta^2 E_s}(\sin\phi - \sin\phi_s)}\frac{d\phi}{dn}\,dn$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 = -\frac{2\pi h\eta eV}{\beta^2 E_s}(\cos\phi + \phi\sin\phi_s) + \text{const.}$$

$$\Rightarrow \frac{1}{2}\left(\frac{2\pi h\eta}{\beta^2}\right)^2\left(\frac{\Delta E}{E_s}\right)^2 + \frac{2\pi h\eta eV}{\beta^2 E_s}(\cos\phi + \phi\sin\phi_s) = \text{const.}$$

$$\Rightarrow \boxed{\left(\frac{\Delta E}{E_s}\right)^2 + \frac{\beta^2}{\pi h\eta}\left(\frac{eV}{E_s}\right)(\cos\phi + \phi\sin\phi_s) = \text{constant}}$$

Suppose $\phi = \phi_s + \Delta\phi$, $\Delta\phi$ is small

$$\Rightarrow \left(\frac{\Delta E}{E_s}\right)^2 + \frac{\beta^2}{\pi h\eta}\left(\frac{eV}{E_s}\right)\left(\cos\phi_s\cos\Delta\phi - \sin\phi_s\sin\Delta\phi + (\phi_s + \Delta\phi)\sin\phi_s\right) = \text{const.}$$

$$\Rightarrow \left(\frac{\Delta E}{E_s}\right)^2 + \frac{\beta^2}{\pi h\eta}\left(\frac{eV}{E_s}\right)\left[\cos\phi_s\left(1 - \tfrac{1}{2}\Delta\phi^2\right) + \phi_s\sin\phi_s\right] = \text{const.}$$

or, $$\boxed{\left(\frac{\Delta E}{E_s}\right)^2 - \frac{\beta^2\cos\phi_s}{2\pi h\eta}\left(\frac{eV}{E_s}\right)\Delta\phi^2 = \text{constant}}$$ for small $\Delta\phi$

suppose $\Delta E = 0$ @ $\Delta\phi = \Delta\hat{\phi}$ $\Rightarrow$ $\text{constant} = -\frac{\beta^2\cos\phi_s}{2\pi h\eta}\left(\frac{eV}{E_s}\right)\Delta\hat{\phi}^2$

but, $\Delta\phi = 0$ @ $\Delta E = \Delta\hat{E}$ $\Rightarrow$ $\text{constant} = \left(\frac{\Delta\hat{E}}{E_s}\right)^2$

$$\beta^2\frac{\Delta\hat{p}}{p} = \boxed{\frac{\Delta\hat{E}}{E_s} = \sqrt{-\frac{\beta^2\cos\phi_s}{2\pi h\eta}\cdot\frac{eV}{E_s}}\;\Delta\hat{\phi}}$$