# Strategies For Measuring Resolution of new BPM Electronics

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January 7, 2004

**Abstract**

A statement in the requirements for the new BPM electronics is that the differential resolution not exceed 7 microns (one $\sigma$). This note discusses how such a measurement can be done, and how accurately $\sigma$ might be measured, given the availability of only one prototype of the new BPM design. The result is that $\sigma$ can be measured to 15% accuracy using a sequence of 35 3-bump measurements, or if 30% accuracy is adequate, a sequence of 14 3-bump measurements.

## Contents



## 1 Overview

A statement in the requirements for the new BPM electronics is that the differential resolution not exceed 7 microns (one $\sigma$). Before proceeding with production of the new BPMs, it is prudent to assess whether the proposed design will meet this requirement. This note discusses how such a measurement can be done, and how accurately $\sigma$ might be measured, given the availability of only one prototype of the new BPM design. It also discusses the extent to which a second prototype might (or might not) improve such measurements.

(There is some debate about whether the physics of beam tuning is such that this requirement is overly stringent, and thus could be relaxed. This note does not address that issue; at any rate, if one knows a meaningful requirement for resolution, one still faces the question of how to verify that this requirement will be met, using only the prototype electronics.)

"The differential resolution must not exceed 7 microns (one $\sigma$)." This can be a confusing statement: Let me clarify what I will consider it to mean (this definition is subject to dispute by the experts if it is inappropriate):

Any given change in circumstances (a stray field, shift is some position, or what have you) will result in an acutal shift in beam position at location $P$, of $x$ microns. Consider an experiment of measuring the position at $P$, using a BPM located at $P$, twice–once before and once after after the change in position. This experiment will likely indicated some change in position, even if the actual change is quite small (since the *precision* of the new BPM devices is at the level of about 2 microns). However, for sufficiently small change $x$, it is not certain that the direction of the measured change will correctly match the direction of the actual change. There is some probability $F(x)$ that the change will be resolved, that is, that a change will be observed *and that it will have the correct sign.* For $x \to 0$, $F(x) \to .5$. Assuming that the BPM is not worthless, for large $x$, $F(x) \to 1$.

Assuming that the sum of various uncertainties in the BPM measurement takes the shape of a Gaussian, $F(x)$ will have the form of an error fucntion. More precisely, the Gaussian assumption implies that

$$F(x) = \frac{1}{2}\left[1 + \mathrm{erf}(\frac{x}{\sqrt{2}\sigma})\right] \tag{1}$$

. And $\sigma$ is the standard deviation of the Gaussian and the resolution of the device.

Given this definition, the question is how best to measure $\sigma$ (or to verify that $\sigma$ does not exceed 7 microns), given just one prototype of the new BPM electronics at one fixed location. There are four challenges in determining $\sigma$:

1. How to set up $x$ for a given trial.
2. How to determine an approximation to $F(x)$.
3. How to estimate $\sigma$, either from that approximation to $F(x)$ or from other considerations involving measurements.
4. How to optimize the process, in the sense that we want to answer the question of whether $\sigma$ exceeds 7 microns without planning more measurements than necessary.

In answering the above challenges, we can assume the following resources:

1. Beam time for some small number of trials of setting DFG values and taking measurements.
2. Availability of a reasonably centered beam for a starting point.
3. We assume that the $\beta$-functions at BPM points are known to some relative accuacy $\epsilon$. Hopefully, as long as $\epsilon$ is small, our conclusions will not depend strongly on its exact value.

4. The DFG currents can be set with a precision and accuracy which is adequate to reproducibly induce small changes (small compared to the 7 micron goal) in beam position. And we can assume these currents do not drift significantly in the time spans during which we will do our measurements to determine $\sigma$.

5. We are not assuming perfect knowledge of the relationship between DFG current and position movement in response, but we do assume linearity in the region near our central measurement point.

6. In addition to the measurements taken at both the new BPM, we have at our disposable the measurements taken at the existing BPM's. These have associated disretization errors on the scale of 150 microns, and other uncertainties which have not been measured but appear (from looking at fluctuations) to also be on the order of 100 microns. *(The procedure recommended will not use these other BPM measurements, unless they are needed to pre-calibrate the scale of displacement versus DFG currents.)*

It will be sufficient to measure the resolution at the single available BPM at point $P$, and when the beam is nearly centered on its ideal orbit. It is acceptable that the error on the estimate of resolution be as large as 15%, since if we could say that the resolution were going to be 6-8 microns, that would satisfy the requirement for all practical purposes. And, in designing our measurement, if we have a scheme which would have greater errors if the resolution is far from 7 microns, that is fine–the answers 14-28 microns or 1-4 microns both answer our question as to whether the specified requirement of 7 is met.

However, there is a soft but important cost constraint: Each experiment of setting DFG currents and measuring positions takes some time, and beam study time is a very precious commodity. The total number of measurements must be minimized, and in fact if we were to find that you can't answer the 7 micron question using a reasonable number of measurements, we would have to re-assess whether this requirement is worth verifying at all.

## 2 Measurements Using a Single New BPM

The idea, of course, will be to set up a series of 3-bump trials, noting the displacement measured at $P$, and to deduce from that data the value of $\sigma$. We take aedvantage of the fact that by doing $N$ displacement measurements, we effectively can extract $N(N-1)$ 2-measurement comparisons.

The measurement program logically consists of three steps:

1. Calibrate the beam movement per unit DFG current by doing a series of 3-bumps with displacements up to several times the current BPM precision, and fitting to find the slope.

2. By making small adjustments in the DFG currents, take a series of measurements in the new BPM which can be plotted against the "known" displacements.

3. Analyze the data from that series of measurements to evaluate the best estimate of $\sigma$. The statistics of this step are greatly simplified if current adjustments in the previous step were taken to have equal step sizes.

A series of calcualtions and perhaps simulations should allow us to evaluate in advance the step sizes and ranges to use in (1) and (2) to minimize the number of trials, while determining $\sigma$ to decent accuracy.

In fact, step (1), the calibration of beam movement per unit DFG current, is probably not necessary, because this is already known to better than the 15% accuracy needed to address the issue of whether $\sigma$ is smaller than 7 microns.

## 2.1 Determination of $\sigma$

The idea behind determining $\sigma$ for the measurement given by the new BPM at $P$ is to do a sequence of trial measurements, and use these to form an ensemble of two-measurement trials. In each two-measurement trial, we ask the question *"Did the BPM measure a change in the correct direction?"* Since the separation between the two known displacements is about $x$, where $x$ varies depending on which two measurements form the trial, this allows us to form a estimate of $F(x)$, and from that, we can derive $\sigma$.

In detail, one workable scheme is:

- Choose a number of trials $N_t$. For example, one could decide to do 21 trials.
- Choose a step size $\rho$ (in displacement, whch via our calibration becomes a step size in current) between each trial. $\rho$ must be small enough that we would not worry that a displacement of $N_t\rho/2$ would put us in a nonlinear region. But more crucially, $\rho$ must be small compared to the 7 micron scale we wish to investigate, and $N_t\rho$ must be large compared to that scale. For example, with $N_t = 21$, one might choose $\rho = 1\mu$ and thus go out to separations of up to 21 microns.
- Starting at a current expected to produce a displacement of $-\rho(N_t - 1)/2$, create a sequence of 3-bumps using DFG's near $P$. For each 3-bump, increase the current to move the expect displacement by the step size $\rho$. Note the measurements on the new BPM. This gives a sequence of $N_t$ measurements $y_i$. (Use the type of measurement for which you are intersted in determining $\sigma$. For example, do not average multiple readings if you would not be doing so in actual BPM usage.)
- Form an ensemble of $N_t(N_t - 1)/2$ pair-trials by pairing each of the $N_t$ measurements with another one. Each pair-trial represents measuring two points, separate by distance $x$ which some multiple of $\rho$. The ensemble will contain $N_t - j$ pairs where the distance is $j\rho$.
- For each value of $j$ from 1 to $N_t - 1$, gather together the $N_t - j$ pairs of measurements, and count how many $C_j$ have the correct relation among the $y$ values (that is, for how many pairs do the $y$ values differ in the correct direction). Let the fraction $C_j/(N_t - j)$ be called $f_j$. For example, $f_1$ will assumedly be near to $1/2$ (because at a separation of one step the resolution is worthless) while $f_{N_t-1}$ will be near to 1 because at that distance the resolution is very good.
- Assign to each $f_j$ an error according to the binomial distribution, of $\sqrt{(N_t - j)f_j(1 - f_j)}$.
- Fit the function $F(j) = 2f_j - 1$ versus $j$, with the given errors in $f_j$, to the form $F(j) = \mathrm{erf}(j/\mu)$.

- The best estimate for $\sigma$ is then $\rho\mu/\sqrt{2}$.

The only thing to decide becomes the optimal choices of step-size $\delta$, and number of trials $N_t$. These are dictated by the need for accuracy in $\sigma$ which, remember, is about 15% when $\sigma$ comes out to 7 microns.

### 2.1.1 Modification of Current Steps

A small refinement in the selection of the sequence of currents may be advantageous. If one chooses a uniform step size $\rho$ and a number of steps $N$ which is will give an accurate value of $\sigma$ when the actual $\sigma$ is around 7 microns, then it may have no accuracy at all in the case that the BPM resolution is not as good as we thought, with an actual $\sigma$ that is much higher. For example, if we were to choose $N = 21$ and $\rho = 1$, then if the true resolution were to be 25, the measurements would only tell us that the true resolution is somewhat more than or around 20.

To improve the situation, one might wish to use a slowly increasing step size. However, if we wish to retain the idea of grouping several fits all separated by the same distance $x$, then we cannot arbitrarily select step sizes, since then few pairs will share the same separations.

The suggested refinement is to choose, out of the $N$ measurements, some $B$-th measurement. Until that $B$-th measurement, the current increatse by $\rho$ each time; after $B$, the current increases by some small integer multiple of $\rho$ (sy $3\rho$). This will extend the range of meaningful answers, while not significantly affecting the accuracy when the true $\sigma$ is small.

### 2.1.2 Alternative Modification of Current Steps

Naively, one might wish to use a slowly increasing step size, with each step greater than the previous one by some factor which is slightly larger than 1. Unfortunately, that complicates the analysis of results, since you no longer have a set of reasonably precise fractions to wich to fit the erf curve.

The analysis can still be done. Instead of a least-squares fit to find the best erf curve, $\mu$ is found by maximizing the likelihood function for the points of data, against $\mu$ in the probability function $\mathrm{erf}(x/\mu)$. For each pair for which the correct sign of difference is obtained, the likelihood function gets a factor of $.5(1 + \mathrm{erf}(x/\mu))$, while for each pair for which the correct sign of difference is obtained, the likelihood function gets a factor of $1 - .5(1 + \mathrm{erf}(x/\mu)) = .5(1 - \mathrm{erf}(x/\mu))$.

## 2.2 Estimating $\sigma$ – How Accurate Will the Estimate Be?

This estimate of $\sigma$ will be off for three reasons:

- The statistical error in determination of $\sigma$ from the finite number of independent pair-measurements at varying values of $x$.
- A systematic inaccuracy induced by the fact that the pair-measurements are themselves derived from the original sequence, and thus are not truly independant.
- Inaccuracy that stems from inaccuracy in the calibration of current to displacement.

The calibration inaccuracy is unimportant, assuming only that we know the relation between current and displacement to 15%, which is quite a conservative assumption.

The statistical error and the systematic inaccuracy induced by the fact that the pair-measurements re not truly independant can be studied together, using a series of simulations. In each simulation, we make an assumption about the actual value of $\sigma$, and for some given $N$ and $\rho$ generate a series of simulated measurements and do the analysis to find a sample value for $\sigma_{\text{estimated}}$. We repeat this a large number of times $L$, and from this we can determine both the systematic bias (which will manifest itself as an incorrect mean value for the estimate of $\sigma$) and the accuracy in $\sigma$. There are some notorious statistical subtleties involved in estimating the variance of a sample, but here were are not doing that: We will have $L$ independant measurements of a quantity, and it will be valid to discuss the mean and variance of that collection in the usual way.

In the course of doing the simulation, there is choice of how to generate each data point:

- We can simply generate a value for $j\rho$ based on the assumed $\sigma$.
- We can generate that value and then round to the nearest multiple of $\lambda = 150/64 = 2.34$ microns. This would reflect the discretization in the new BPM's, which have an additional six bits of accuracy as compared to the old 150 micron step size.

The latter method is more honest in assessing the resolution power, in that it rolls in the effect of lucky/unlucky discretization and of "ties" counting as non-resolved displacements. Thus it will result in the simulation yielding a $\sigma$ estimate which is just a bit higher than the former. However, there is no doubt that using the discretization in our simulation is the right thing to do, becuase the question we are answering is "how should we measure the displacement resolution of the *actual* system."

That is, when the actual measurements are done to estimate $\sigma$, the result will be a combination of the true gaussian noise fluctations and the discretization error. For discrete bin size $\lambda$, we will find $\sigma = \sqrt{\sigma_{\text{noise}}^2 + \lambda^2/12}$. Since this is the relevant error quantity when using the BPM data to smooth the beam, it is appropriate that this $\sigma$ (rather than just $\sigma_{\text{noise}}$) be used in the simulation as well.

Once we note that what is being approximated is the true $\sigma$ (and not $\sigma_{\text{noise}}$), we now can know the bias inherent in the sequence measurement technique, by noting the mean difference between the estimated $\sigma$ and the actual value.

## 2.3 Results of the Simulation: Bias and Accuracy of $\sigma$

The rules for the simulation become simple: For each value of $N$ and $\rho$ we wish to investigate, perform $L$ trials of the following form: Each trial consists of generating $N$ gaussian random numbers with means ranging from 0 to $(N-1)\rho$.

The deviations of these random numbers should not be the assumed net $\sigma$, because the $\sigma$ appearing in the definition of resolution is greater than the noise in these random numbers for two reasons:

1. The actual $\sigma$ is $\sigma = \sqrt{\sigma_{\text{noise}}^2 + \lambda^2/12}$.
2. The actual $\sigma$ appearing in the definition of the resolution is the difference in the deviations of the two numbers in the pair. That is, ignoring the $\lambda$ effect, if we chose the noise deviation to be $h$ micron, then $\sigma$ would be $\sqrt{2}h$.

Thus the correct value to use for the deviation of the random numbers is $\sigma_{\text{noise}} = \sqrt{(\sigma^2_{\text{assumed}} - \lambda^2)/2}$. Generate $N$ numbers with means ranging from 0 to $(N-1)\rho$ and with this $\sigma_{\text{noise}}$, round the $N$ numbers to the nearest $\lambda$, and then feed them into the analysis engine.

(The actual study will likely go from $-(N-1)\rho/2$ to $+(N-1)\rho/2$ but that is equivalent for the purposes of studying resolution to starting from 0.)

The analysis engine will form the ensemble of $N(N-1)/2$ pairs, and evaluate the results to form the correct resolution fractions $F(x)$, and assign weights based on the binomial distribution to those function values. (In cases where the resolution is perfect, or always wrong, assign a weight based on a hypothetical finding of 1/2 a trial with the other result, rather than the infinite weight the binomial distribution would tell you to assign.) It will then fit to $2f_j - 1$ to the form $\text{erf}(j/\mu)$, which gives a best-fit value for $\mu$. The measured value of $\sigma$ for this trial is then $\sigma = \rho\mu/\sqrt{2}$.

The set of simulations to try to find the optimal beam study for assessing $\sigma$ has four dimensions:

1. The value of $N$. We want to keep $N$ as small as possible without too badly affecting our estimate of $\sigma$, since $N$ represents how many measurements we will really be taking.
2. The value of $\rho$. The test will be sensitive in a range of a few times $\rho$ up to about $\rho N/3$. If $\rho$ is too large or too small we will have no chance of accurately estimating $\sigma$ when $\sigma$ is near 7 microns.
3. The value of $B$, such that the step size past the $B$-th step becomes $3\rho$.
4. The assumed value of $\sigma$. Of course we want to know how accurately we will measure $\sigma$ if it is around 7 microns. But we also want to know how accurate the proposed test will be if $\sigma$ is somewhat off from that.

The suite of assumed $\sigma$ values I use in the set of simulations is $4\mu$, $7\mu$, $10\mu$, $15\mu$, $20\mu$.

The results are a tad disconcerting, at least if one insists on an estimate which is trustworthy to 15% accuracy:

In order to expect to estimate $\sigma$ with a probable error of one micron if the actual value of $\sigma$ is 7 microns, one would have to do about **35 measurements**. The optimum strategy seems to be to do 35 measurements, separated by 2.2 microns each (that is, to do 35 steps ranging from -37.4$\mu$ to +37.4$\mu$ in predicted displacement. This procedure will estimate $\sigma$ to about 15% accuracy whether $\sigma$ is 4, 7, 10, 15, or 20 microns. Due to imperfect independance of the pairs of data, the procedure will deliver a slightly biased estimate, however, the bias is a small fraction of a micron and can thus be corrected for or ignored.

N = 35, rho = 2.2

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias | |
|---|---|---|---|---|---|
| 4 | 2.29395 | 4.07826 | 0.564083 | 0.0782645 | |
| 7 | 4.665 | 7.16711 | 0.986588 | 0.167109 | ***** |
| 10 | 6.87475 | 10.1571 | 1.47837 | 0.157105 | |
| 15 | 10.4767 | 14.8629 | 2.33477 | -0.137089 | |
| 20 | 14.045 | 19.5328 | 3.31009 | -0.467224 | |

What if one is willing to settle for 30Then one can make estimate $\sigma$ measurement using only 14 measurements, with a spacing of 2.7$\mu$. And by increasing the spacing to 8.1$\mu$ for the last four points, one can even get reasonable accuracy if $\sigma$ is out to 15 or 20 microns.

```
N = 14, B = 10, rho = 2.7
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.12501      1.04791     0.125005
   7         4.665        7.14363      1.90382     0.143629  *****
  10       6.87475        10.0565      3.10319     0.0565442
  15       10.4767        15.0069      5.18455     0.00692129
  20        14.045        19.9268      10.6554     -0.0731733
```

#### 2.3.1 Simulation Output

Each simulation used 1000 trials, which seems to give reproducibility of the RMS error result at the scale of the second decimal place. The key number is the RMS error for an actual $\sigma$ of 7 microns.

```
Lambda = 2.34

N = 20, B = 20, rho = 2.3
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.12072     0.756898     0.120716
   7         4.665        7.25442      1.43506     0.254416
  10       6.87475        10.0632      2.22865     0.0632282
  15       10.4767        15.2822       4.3081     0.282193
  20        14.045        20.5305      7.53201      0.53047
[
N = 20, B = 20, rho = 2.0
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.13176     0.775578     0.131756
   7         4.665         7.2064      1.43785     0.206398  *********
  10       6.87475        10.0883      2.31203     0.088268
  15       10.4767        15.1837      5.01361     0.183737
  20        14.045        20.7415      9.14923     0.741475

N = 20, B = 20, rho = 1.6
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.03845      0.74782     0.0384473
   7         4.665        7.28921       1.6335     0.289209
  10       6.87475         10.049      2.61978     0.0489792
  15       10.4767        15.6747      6.68759       0.6747
  20        14.045        24.1372      28.1995       4.1372

N = 20, B = 20, rho = 1.3
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.11997     0.748427     0.119971
   7         4.665         7.3398      1.82566     0.339801
  10       6.87475        10.4123      3.32788     0.412257
  15       10.4767         17.696      20.1824      2.69596
```

```
N = 20, B = 20, rho = 1
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.12879     0.782815     0.128791
   7         4.665        7.57343       2.2545     0.573431
  10       6.87475        11.4029      6.56861      1.40294
  15       10.4767        20.5499      26.6345      5.54991
  20        14.045        30.7018      50.3048      10.7018

N = 20, B = 10, rho = 1
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.11584     0.898101      0.11584
   7         4.665        7.09822      1.82658     0.0982154
  10       6.87475        9.70755       2.6509     -0.292455
  15       10.4767        14.3715      5.10924     -0.628488
  20        14.045        19.6283       10.941     -0.371723

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N = 25, B = 25, rho = 1
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.12863     0.662772     0.128631
   7         4.665        7.25178      1.51272     0.251777
  10       6.87475        10.4734      2.88141     0.473425
  15       10.4767        16.9075      12.5682      1.90752
  20        14.045        23.6257      18.8668      3.62572

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N = 30, B = 30, rho = 2.2
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.06153     0.567861     0.0615253
   7         4.665          7.148      1.11106     0.147999
  10       6.87475        10.1477       1.6548     0.147665
  15       10.4767        14.7701      2.61622     -0.229919
  20        14.045        19.4981      4.01707     -0.501872

N = 30, B = 30, rho = 2.0
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.08322     0.604192     0.0832204
   7         4.665        7.15113      1.08552     0.151128    ****
  10       6.87475        10.0463      1.63488     0.0463032
  15       10.4767        14.7025      2.84273     -0.297457
  20        14.045        19.5242      4.22875     -0.475807

N = 30, B = 30, rho = 1.8
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.07108     0.576913     0.0710804
   7         4.665        7.16681      1.12141     0.166814
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```
  10       6.87475        10.0651       1.67522     0.0650821
  15       10.4767        14.8756       2.88059     -0.124387
  20        14.045        19.6829       4.49619     -0.317066

N = 30, B = 15, rho = 1
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.08468     0.698616     0.0846843
   7         4.665        7.13025      1.31467     0.130253
  10       6.87475        9.77334      2.00232     -0.226659
  15       10.4767        14.0454      3.10638     -0.954601
  20        14.045        18.5547      4.54355     -1.44531

N = 30, B = 30, rho = .5
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.17619     0.718023     0.176191
   7         4.665        7.71693      2.52577     0.716929
  10       6.87475        12.4525      14.2588      2.45254
  15       10.4767        21.4614      25.2098      6.46139
  20        14.045        28.1866       37.333      8.18657

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N = 35, B = 35, rho = 2.3
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.04106     0.547013     0.041056
   7         4.665        7.14681      1.02354      0.14681
  10       6.87475        10.0374       1.4789     0.0374429
  15       10.4767        14.8679      2.40904     -0.132082
  20        14.045        19.6721      3.40111     -0.327948


N = 35, B = 35, rho = 2.2
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.07826     0.564083     0.0782645
   7         4.665        7.16711     0.986588     0.167109   *****
  10       6.87475        10.1571      1.47837     0.157105
  15       10.4767        14.8629      2.33477     -0.137089
  20        14.045        19.5328      3.31009     -0.467224


N = 35, B = 35, rho = 2.1
Sigma    Sigma(Noise)    Estimate     RMS Error     Bias
   4       2.29395        4.08379      0.54589     0.0837896
   7         4.665        7.22475      0.98814      0.22475
  10       6.87475        10.1281      1.47298     0.128147
  15       10.4767        14.8804      2.37916     -0.119581
  20        14.045        19.5986      3.28884     -0.401422
```

N = 35, B = 35, rho = 2.0

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.1003 | 0.56438 | 0.100303 |
| 7 | 4.665 | 7.13677 | 1.01498 | 0.136773 |
| 10 | 6.87475 | 10.1414 | 1.46738 | 0.141444 |
| 15 | 10.4767 | 14.7257 | 2.29323 | -0.274263 |
| 20 | 14.045 | 19.2648 | 3.5122 | -0.735169 |

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N = 40, B = 40, rho = 2.5

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.07242 | 0.529969 | 0.07242 |
| 7 | 4.665 | 7.19897 | 0.950067 | 0.198973 |
| 10 | 6.87475 | 10.1332 | 1.26498 | 0.13317 |
| 15 | 10.4767 | 14.8435 | 2.02634 | -0.156514 |
| 20 | 14.045 | 19.6748 | 2.92626 | -0.325182 |

N = 40, B = 40, rho = 2.0

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.08691 | 0.502204 | 0.0869131 |
| 7 | 4.665 | 7.18831 | 0.929609 | 0.188309 ***** |
| 10 | 6.87475 | 10.092 | 1.37636 | 0.0920412 |
| 15 | 10.4767 | 14.7654 | 2.14373 | -0.234614 |
| 20 | 14.045 | 19.4553 | 3.08455 | -0.544733 |

N = 40, B = 40, rho = 1.5

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.10752 | 0.483877 | 0.10752 |
| 7 | 4.665 | 7.20806 | 0.942227 | 0.208056 |
| 10 | 6.87475 | 10.1264 | 1.36652 | 0.126446 |
| 15 | 10.4767 | 14.9508 | 2.39919 | -0.0492463 |
| 20 | 14.045 | 19.6368 | 3.59537 | -0.363184 |

N = 40, B = 40, rho = 1.0

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.08407 | 0.497232 | 0.08407 |
| 7 | 4.665 | 7.2164 | 1.00879 | 0.216403 |
| 10 | 6.87475 | 10.1819 | 1.57399 | 0.181863 |
| 15 | 10.4767 | 15.1103 | 3.05256 | 0.110318 |
| 20 | 14.045 | 20.231 | 5.43581 | 0.231037 |

N = 40, B = 20, rho = 1

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.07017 | 0.555451 | 0.0701727 |
| 7 | 4.665 | 7.12142 | 1.10394 | 0.12142 |
| 10 | 6.87475 | 9.87747 | 1.58434 | -0.122528 |

| | | | | |
|---|---|---|---|---|
| 15 | 10.4767 | 14.3544 | 2.64673 | -0.645559 |
| 20 | 14.045 | 18.6587 | 3.57477 | -1.34135 |

N = 40, B = 40, rho = .8

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.06335 | 0.489429 | 0.0633478 |
| 7 | 4.665 | 7.20757 | 1.03638 | 0.207567 |
| 10 | 6.87475 | 10.1968 | 1.79374 | 0.196782 |
| 15 | 10.4767 | 15.492 | 3.92596 | 0.492027 |
| 20 | 14.045 | 21.4448 | 8.8844 | 1.44482 |

N = 40, B = 40, rho = .75

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.10686 | 0.506805 | 0.106861 |
| 7 | 4.665 | 7.16496 | 1.06393 | 0.164962 |
| 10 | 6.87475 | 10.2316 | 1.85805 | 0.231617 |
| 15 | 10.4767 | 15.5351 | 4.66116 | 0.535062 |
| 20 | 14.045 | 21.546 | 8.64922 | 1.54598 |

N = 40, B = 40, rho = .5

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.07752 | 0.575696 | 0.077519 |
| 7 | 4.665 | 7.13517 | 1.16759 | 0.135168 |
| 10 | 6.87475 | 9.6472 | 1.66239 | -0.3528 |
| 15 | 10.4767 | 13.9893 | 2.97361 | -1.01074 |
| 20 | 14.045 | 18.2343 | 5.10839 | -1.76569 |

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N = 50, B = 50, rho = 1.5

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.06496 | 0.442941 | 0.0649556 |
| 7 | 4.665 | 7.1995 | 0.826773 | 0.199501 |
| 10 | 6.87475 | 10.0738 | 1.18978 | 0.0738001 |
| 15 | 10.4767 | 14.9383 | 1.81942 | -0.0616957 |
| 20 | 14.045 | 19.857 | 2.54129 | -0.142998 |

N = 50, B = 50, rho = 1.5

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias | |
|---|---|---|---|---|---|
| 4 | 2.29395 | 4.06842 | 0.449385 | 0.068423 | |
| 7 | 4.665 | 7.18546 | 0.825454 | 0.185457 | **** |
| 10 | 6.87475 | 10.1505 | 1.19271 | 0.150462 | |
| 15 | 10.4767 | 14.8953 | 1.81766 | -0.104663 | |
| 20 | 14.045 | 19.7894 | 2.90863 | -0.210592 | |

N = 50, B = 25, rho = 1

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.09588 | 0.496471 | 0.0958808 |

| | | | | |
|---|---|---|---|---|
| 7 | 4.665 | 7.09844 | 0.91457 | 0.0984441 |
| 10 | 6.87475 | 9.95736 | 1.3842 | -0.0426448 |
| 15 | 10.4767 | 14.4448 | 2.21003 | -0.555167 |
| 20 | 14.045 | 18.8127 | 2.9749 | -1.18733 |

N = 50, B = 50, rho = 1

| Sigma | Sigma(Noise) | Estimate | RMS Error | Bias |
|---|---|---|---|---|
| 4 | 2.29395 | 4.06662 | 0.425365 | 0.0666171 |
| 7 | 4.665 | 7.21871 | 0.866704 | 0.218705 |
| 10 | 6.87475 | 10.1595 | 1.27897 | 0.159459 |
| 15 | 10.4767 | 15.0363 | 2.23764 | 0.0363131 |
| 20 | 14.045 | 19.7479 | 3.78972 | -0.252122 |

## 2.4 Calibrating Beam Movement and DFG Current (if Necessary)

Almost certainly, the relation between DFG current and beam displacement is known well enough that it need not be calibrated just for this study. If not, this sub-section discusses how it can be measured, using existing BPMs. Actually, it might be superior to rely on the new BPM, but I'm uncomfortable with the possibility of circular reasoning in doing that.

The calibration step does not have to be very precise, because it will only set the scale of the final resolution answer. For the purposes of addressing the resolution requirement, if we can in the end say that the resolution is 6-8 microns, that will be fine. So the calibration points can be chosen so as to produce just an accuarcy of 15%.

Also, one has to worry in principle about non-linearities in position response to changes in DFG current. But the same series of measurements allows you to get a handle on those non-linearities, and thus to verify that they, too, contribute to the answer at below the 15% level.

The object is to get accurate enough calibration in as few measurements as possible. Assuming we did not know this already (and thus do not have the calibration in hand by doing no trials at all), here is how we could measure it:

The tricky part of planning the calibration step is that the current BPMs have discrete outputs at the level of 150 microns, and that we don't have a handle on the size of their random (non-reproducible) fluctuations. Also, there are three major assets which we might want to utilize make the calibration accurate and quick:

- We can set up the bump to cover some number of BPMs, not necessarily just the two affected by the shortest possible 3-bump.
- We can take advantage of the fact that the $\beta$-functions are slightly different at different BPMs.
- If we can do so without *a priori* knowledge of its resolution, we are free to make use of the more precise measurements from the new BPM.

The strategy will consist of making $2k + 1$ 3-bump measurements, each using currents in the 3 DFGs which are separated by $q_c$, the current (in that DFG) needed for a 3-bump which would be predicted to produce a movement of roughly $\eta$ microns. The central point in this sequence could be zero.

Thus we would measure with $q = mq_c$ where $m$ ranges from $-k$ to $k$. We want $k\eta$ to be large compared to discretization error and fluctuations, but small enough that we don't have to worry about moving the beam too far or about large non-liniiarities. For example, $k\eta$ could be half a millimeter.

Given these measurements, we fit to a linear form, and that fit will have associated error estimates for the slope and origin. The slope gives us our calibration at the remote BPM points; it is trival to use the form for the orbit oscillation plus the $\beta$-functions at the other BPM's and at $P$ to translate that to a calibration of displacement at $P$ versus current. The error estimate for the slope tells us the uncertainty in our calibration.

Actually, given $n$ BPM readings, we would have $n$ such calibrations, and all that remains is to check that they are consistent, within the purported errors in each, and that the overall error is not more than 15%. We could then take the average value to be the calibration needed.

# 3 Measurements Using a Two New BPMs

How much would the situation inprove if there were two of the prototype BPM's at our disposal? The naive answer is that the number of pairs would double, therefore the number of points taken should be reduced by a factor of $\sqrt{2}$. Actually, the situation would be a smidgen better than that, because part of the error in the $\sigma$ estimate comes from the non-independance of the pair measurements, and that factor would be diminished given two independant sequences of measurements.

On the other hand, since there are now two probably incommeasurate $\rho$ values, instead of being able to group many differences toghether to get a more accfurate fraction of correct results, we would get twice as many points with half the data for each. This slightly hurts the estimate accuracy. Also, since $\rho$ needs to be small compared to $\sigma$ to get resolving power, and $j\rho$ needs to be large compared to $\sigma$, there is some worry that we would not be able to reduce the number of measurements taken by the full factor of $\sqrt{2}$.

If we had planned to do 35 measurements to achieve 15% accuracy, then the availability of a second BPM might reduce that to just 25. On the whole, it doesn't seem worth changing any plans for this.

If there were dozens of BPMs available, we would still need on the order of 10 measurements, because of the need to span $\sigma$ by reasonable factors on both ends.

# 4 How Accurately Can Resolution Be Assesed in Full System?

*(Section not done yet.)*