Chromatic Effects in the Round-to-Flat Beam Transformation

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Abstract

In this note, the chromatic effects in the round-to-flat beam transformation along with the associated impact on the flat beam emittances is studied analytically under the thin lens approximation. We compare our analytical results with simulations performed with various particle tracking programs. It is found that the analytical results are in good agreement with simulations.

1 Introduction

The theory of generating a beam with high transverse emittance ratio, i.e., a flat beam, from an incoming angular momentum dominated beam is treated in several papers [1, 2, 3]. The experimental demonstration of such a round-to-flat transformation at Fermilab/NICADD Photo-injector Lab (FNPL) by using a three skew quadrupole channel is reported in [4]. In this note, we use the results in [5] to compute the quadrupole strengths under the thin lens approximation, but with thermal emittance included in the calculation. We follow the theoretical treatment based on four dimensional beam matrix presented in [6].

The 2×2 transfer matrix for a normal quadrupole in thin lens approximation is given by:

$$M_Q(q) = \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}, \tag{1}$$

where $q = \frac{1}{f}$, f being the focal length of the quadrupole. In practical unit, q is given by:

$$q[1/\mathrm{m}] = \frac{300g[\mathrm{T/m}]}{pc[\mathrm{MeV}]} \tag{2}$$

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where p is the particle momentum, g the transverse magnetostatic field gradient and c the speed of light.

For a drift space of length d, the 2×2 transfer matrix is given by:

$$M_D(d) = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \tag{3}$$

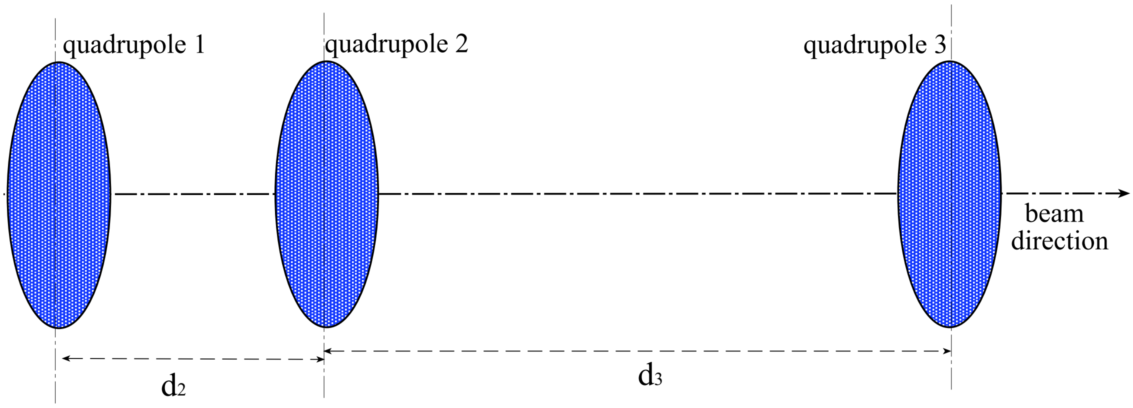

Figure 1: Three quadrupoles separated by drift spaces of lengths d_2 and d_3.

Consider a beam line consisting of three normal quadrupoles[1], with the first two separated by a drift of distance d_2, and the last two by d_3, see Fig. 1. Given the transfer matrix A in (x, x') phase space, an electron with coordinates $X_0 = (x_0, x'_0)^T$ is transformed via $A \cdot X_0$, where A is:

$$A = \begin{bmatrix} 1 & 0 \\ q_3 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix}, \tag{4}$$

and in vertical, $Y_0 = (y_0, y'_0)^T$ in (y, y') trace space is transformed via $B \cdot Y_0$, where B is:

$$B = A(-q_1, -q_2, -q_3, d_2, d_3). \tag{5}$$

In the four dimensional trace space (x, x', y, y'), the 4×4 transfer matrix for the normal quadrupole channel is then:

$$M_{NQ} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \tag{6}$$

Now if the quadrupoles are rotated by $45°$, then the transfer matrix for the skew quadrupole channel is given by:

$$M(q_1, q_2, q_3, d_2, d_3) = R^{-1} M_{NQ} R, \tag{7}$$

where R is the 4×4 matrix describing a rotation of $45°$ in the trace space,given by:

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}, \tag{8}$$

[1]Three is the minimum number of quadrupoles needed to remove the angular momentum.

and I is the 2×2 identity matrix.

From Eq. 7 and Eq. 8,we have the transfer matrix of the skew quadrupole channel as:

$$M(q_1, q_2, q_3, d_2, d_3) = \frac{1}{2} \begin{bmatrix} A+B & A-B \\ A-B & A+B \end{bmatrix}. \tag{9}$$

The general form of a cylindrically symmetric beam matrix [6] at the entrance of the skew quadrupole channel is:

$$\Sigma_0 = \begin{bmatrix} \sigma^2 & 0 & 0 & \kappa\sigma^2 \\ 0 & \kappa^2\sigma^2 + \sigma'^2 & -\kappa\sigma^2 & 0 \\ 0 & -\kappa\sigma^2 & \sigma^2 & 0 \\ \kappa\sigma^2 & 0 & 0 & \kappa^2\sigma^2 + \sigma'^2 \end{bmatrix}, \tag{10}$$

where $\sigma^2 = \langle x^2 \rangle = \langle y^2 \rangle, \sigma'^2 = \langle x'^2 \rangle = \langle y'^2 \rangle$, $\kappa = \frac{eB_z}{2p}$, B_z is the longitudinal magnetic field on the photocathode and p is the particle momentum. The beam matrix at the exit of the skew quadrupole channel is:

$$\Sigma = M\Sigma_0\widetilde{M}, \tag{11}$$

where $\widetilde{M}$ stands for the transpose of M.

In the case of longitudinally cold beam (zero energy spread), the beam matrix Σ could be block diagonalized upon proper choice of transfer matrix M. Generally the beam is not cold, but has an energy spread coming both from stochastic and correlated processes. At FNPL, the skew quadrupole channel is located upstream of a magnetic compressor. If one wishes to compress the beam, a correlated relative energy spread of the order of two percent (rms value) is introduced by running the RF cavity off-crest. Next we address the related chromatic effects in the round-to-flat beam transformation in section 2.

2 Chromatic Effects on the Transfer Matrix

As in Eq. 2, the strength of the quadrupole is related to the beam's momentum. Consider an electron with a small relative momentum deviation $\delta = \frac{p-p_0}{p_0}$ around the average beam momentum p_0. The quadrupole strength for an electron with momentum $p = p_0(1+\delta)$ is given by:

$$q[1/\mathrm{m}] = \frac{300g[\mathrm{T/m}]}{pc[\mathrm{MeV}]} = \frac{300g[\mathrm{T/m}]}{p_0c[\mathrm{MeV}]} \cdot (1 - \delta + \delta^2) + \mathcal{O}(\delta^3) \approx q_0(1 - \delta + \delta^2), \tag{12}$$

where $q_0[1/\mathrm{m}] \doteq \frac{300g[\mathrm{T/m}]}{p_0c[\mathrm{MeV}]}$. Correspondingly, the transfer matrix $M_Q(q)$ may be written as:

$$M_Q(q, \delta) \approx \begin{bmatrix} 1 & 0 \\ q_0 & 1 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ -q_0 & 0 \end{bmatrix} + \delta^2 \begin{bmatrix} 0 & 0 \\ q_0 & 0 \end{bmatrix}, \tag{13}$$

Matrix A and B become:

$$\begin{aligned} A &\approx A_0 + \delta\Delta A_1 + \delta^2\Delta A_2, \\ B &\approx B_0 + \delta\Delta B_1 + \delta^2\Delta B_2, \end{aligned}$$

where the subscript "0" refers to the quadrupole strength for the particle with momentum p_0, and ΔA_i $(i = 1, 2)$and ΔB_i are the modifications to the matrices A_0 and B_0 on the i'th order of δ.

Define the following matrices:

$$\Delta_i^{\pm} \doteq \Delta A_i \pm \Delta B_i,$$
$$\Delta_i \doteq \frac{1}{2}\begin{bmatrix} \Delta_i^+ & \Delta_i^- \\ \Delta_i^- & \Delta_i^+ \end{bmatrix},$$

The transfer matrix of the skew quadrupole channel takes the form of:

$$M(q_1, q_2, q_3, d_2, d_3) \approx M_0 + \delta\Delta_1 + \delta^2\Delta_2, \tag{14}$$

where

$$M_0 \doteq \frac{1}{2}\begin{bmatrix} A_0 + B_0 & A_0 - B_0 \\ A_0 - B_0 & A_0 + B_0 \end{bmatrix}, \tag{15}$$

If the distribution of the relative momentum spread centered on the average energy, then $\langle\delta\rangle$ vanishes. From Eq. 11 and Eq. 14, keeping only the first order modification to the beam matrix, we have:

$$\Sigma \approx M_0\Sigma_0\widetilde{M_0} + \langle\delta^2\rangle(M_0\Sigma_0\widetilde{\Delta_2} + \Delta_1\Sigma_0\widetilde{\Delta_1} + \Delta_2\Sigma_0\widetilde{M_0}) \tag{16}$$

As mentioned in section 1, for beam with zero relative momentum spread, i.e., the first term of Eq. 16, can be block diagonalized given proper transfer matrix M(see, for example, Ref. [6]):

$$M_0\Sigma_0\widetilde{M_0} = \begin{bmatrix} (\varepsilon_{eff} - \mathcal{L})T & 0 \\ 0 & (\varepsilon_{eff} + \mathcal{L})T \end{bmatrix}, \tag{17}$$

where

$$\varepsilon_{eff} = \sigma\sqrt{\sigma'^2 + \kappa^2\sigma^2},$$
$$\mathcal{L} = \kappa\sigma^2,$$
$$T = \tfrac{1}{2}(A_0 + B_0)T_0(\widetilde{A_0} + \widetilde{B_0}),$$
$$T_0 = \begin{bmatrix} \beta & 0 \\ 0 & 1/\beta \end{bmatrix},$$
$$\beta = \frac{\sigma_c}{\sqrt{\sigma'^2 + \kappa\sigma_c^2}}$$

Notice that the determinant of 2×2 matrix T is 2. The two transverse emittances are given by:

$$\varepsilon_{x,y}^0 = \varepsilon_{eff} \mp \mathcal{L}. \tag{18}$$

When there is a relative momentum spread in the beam, the beam matrix varies as a function of it. The two transverse emittances can be calculated as the square roots of the determinants of the top left and bottom right 2×2 sub-matrices of the beam matrix expressed in Eq. 16. Rewrite the second term of Eq.16 as:

$$\langle\delta^2\rangle\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \doteq \langle\delta^2\rangle(M_0\Sigma_0\widetilde{\Delta_2} + \Delta_1\Sigma_0\widetilde{\Delta_1} + \Delta_2\Sigma_0\widetilde{M_0}), \tag{19}$$

Use the fact that for two 2×2 matrices P and Q,

$$|P+Q| = |P| + |Q| + Tr(P^{\dagger}Q), \tag{20}$$

where "||" stands for the determinant, "Tr" for the trace of a matrix, and $P^{\dagger}$ is the symplectic conjugate of P, $P^{\dagger} = J^{-1}\widetilde{P}J$, where J is the 2×2 unit symplectic matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Finally we can write the modified transverse emittances due to chromatic effect as:

$$\varepsilon_{x,y} = \sqrt{(\varepsilon_{eff} \mp \mathcal{L})^2 + \langle \delta^2 \rangle^2 [|\Delta_{11 \text{ or } 22}| + (\varepsilon_{eff} \mp \mathcal{L})^2 \mathrm{Tr}(T\Delta^{\dagger}_{11 \text{ or } 22})]}. \tag{21}$$

In next section, we compare our analytical calculation with simulations.

3 Comparison with simulation

As an example, we chose the following operation parameters for the FNPL flat beam experiment:

$$\begin{aligned} &\gamma = 30, \\ &\sigma_c = 1.00 \text{ mm}, \\ &\kappa = 0.78 \text{ m}^{-1}, \\ &\sigma' = 0.033 \text{ mrad}, \\ &d_2 = 0.35 \text{ m}, \\ &d_3 = 0.85 \text{ m}. \end{aligned}$$

Using the thin lens approximation, and including the thermal emittance, the skew quadrupole strengths are calculated from Ref.[5] to be:

$$\begin{aligned} &q_1 = 1.729 \text{ m}^{-1}, \\ &q_2 = -1.339 \text{ m}^{-1}, \\ &q_1 = 0.628 \text{ m}^{-1}. \end{aligned}$$

The flat beam emittance at zero relative momentum spread is calculated to be:

$$\begin{aligned} &\varepsilon_x^n = 0.021 \text{ mm mrad}, \\ &\varepsilon_y^n = 46.82 \text{ mm mrad}. \end{aligned}$$

The analytical calculation of two transverse emittances and their ratio, as a function of relative momentum spread, is first compared with simulation results from ASTRA[7], ELEGANT[8] and SYNERGIA[9]. These results agree well in general, see Fig. 2 to 4.

A few words about the simulation codes: ASTRA integrates the equation of motion using a Runge-Kutta algorithm; ELEGANT is a six dimensional tracking program and

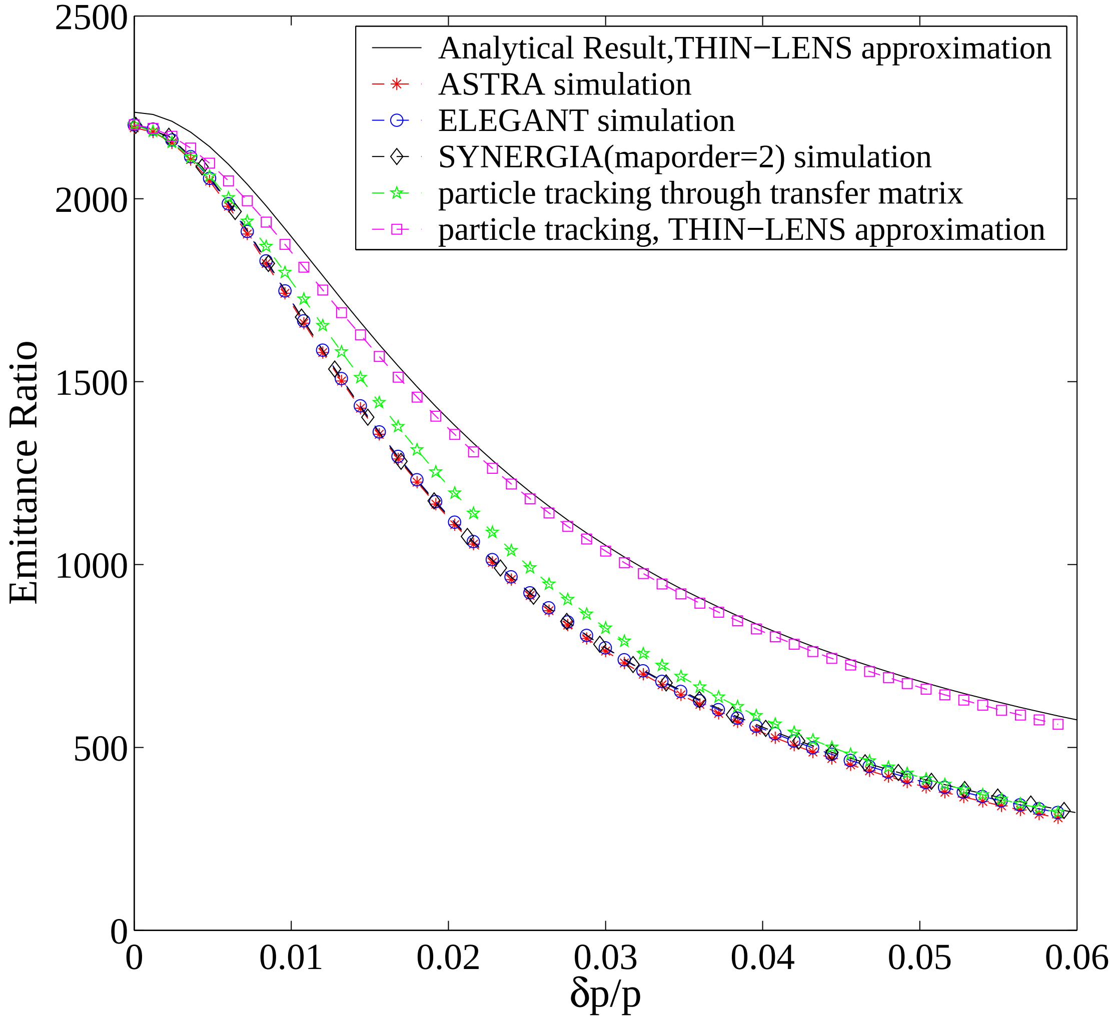

Figure 2: Chromatic effects on emittance ratio. Solid line is obtained from Eq.21. Dashed lines with markers are computed using numerical methods.

we tracked using matrix up to the second order; SYNERGIA is based on Lie algebra and in the simulation done in this note, the "maporder" is set to the second order. Good agreement between second order codes (ELEGANT and SYNERGIA) with the integration of the equations of motion (ASTRA) suggest that higher order effects in beam dynamics are indeed dominated by second order. One other purpose of comparing these simulation codes results is to benchmark SYNERGIA. The latter simulation program has a fully three-dimensional space charge model and it runs fast, though in current simulation the space charge effect is not included since our primary purpose is to investigate the chromatic effects.

On the other hand, we can see that the agreement between the analytical result and simulations is better for lower relative momentum spread values. To further explore the difference, each particle used in the simulation is tracked through the transfer matrix for both the cases when the quadrupole are thick and thin lenses, using the transfer matrix as shown in Eq. 13. We found that in the thick lens case, the tracking results almost overlap

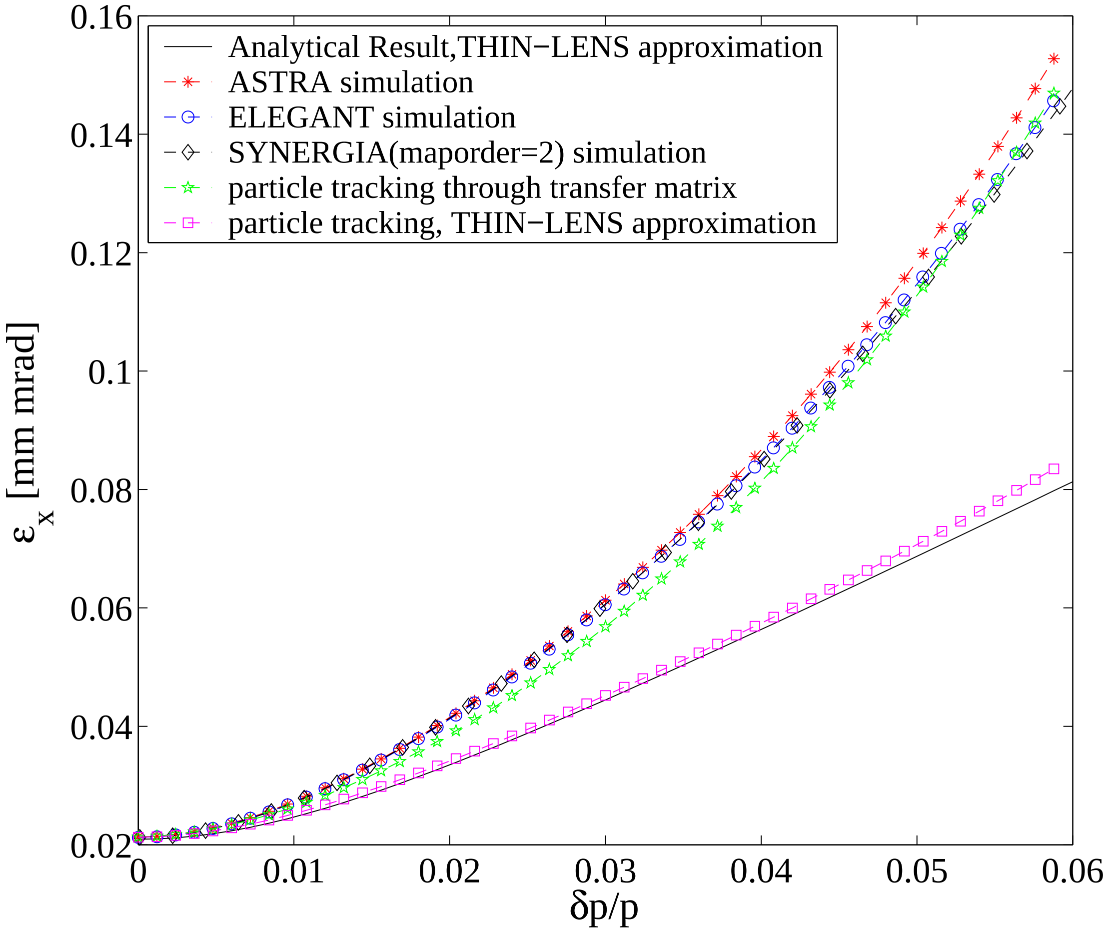

Figure 3: Chromatic effects on horizontal emittance. Solid line is obtained from Eq.21. Dashed lines with markers are computed using numerical methods.

with the simulation results, while the thin lens approximation tracking results agree quite well with the analytical predictions, see Fig. 2, 3, and 4. So the difference between analytical model and simulation results could be explained by the fact that thin lens approximation is used in analytical model in order to computer the skew quadrupole strengths.

Next we study the effect of different spacing between skew quadrupoles by changing d_3 from 85 cm to 35 cm, which is another possibility in the beam line at FNPL. Chromatic effects are studied in these two different cases using analytical mode and ELEGANT simulation. The results are show in Fig. 5 to 7. The analytical model agrees well with ELEGANT simulation. We see that in the case of the spacing between skew quadrupoles are $d_2 = 0.35$ m, $d_3 = 0.85$ m, the two transverse emittance values, especially the smaller one, increases slower as a function of relative momentum spread, and emittance ratio decreases slower as well. So in consideration of chromatic effect, the first choice of skew quadrupole spacing is preferred.

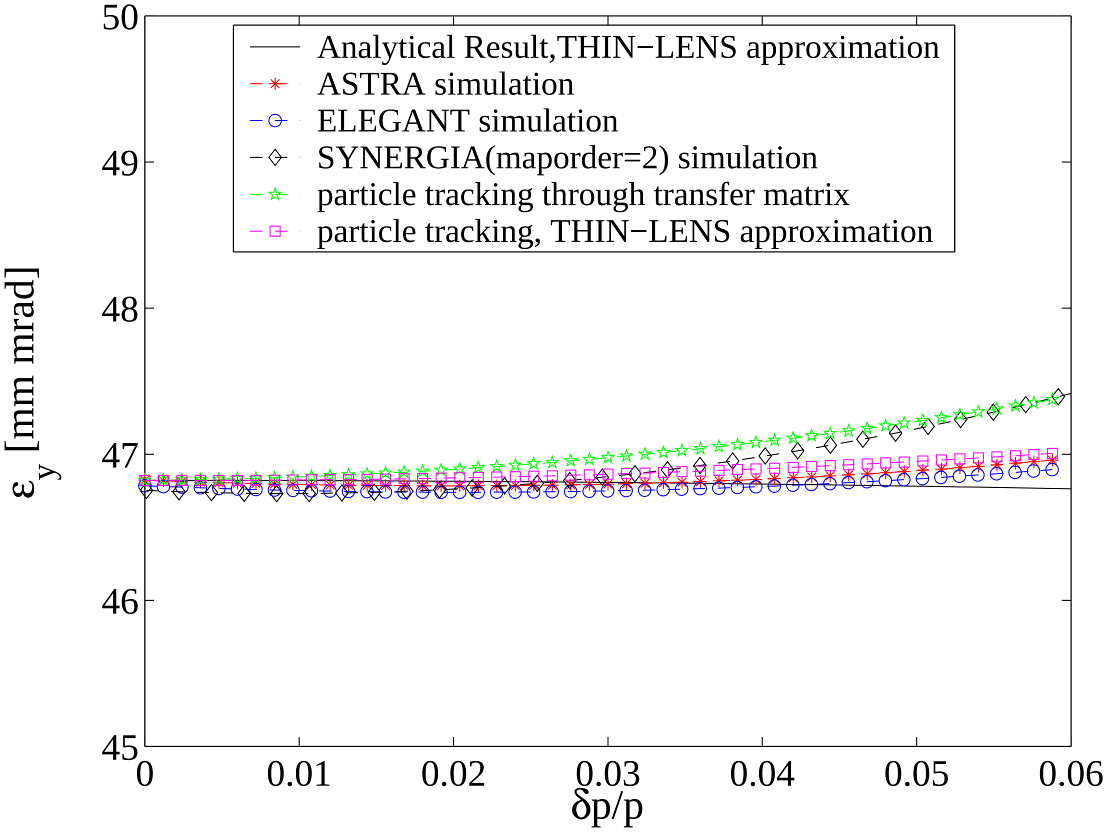

Figure 4: Chromatic effects on vertical emittance. Solid line is obtained from Eq.21. Dashed lines with markers are computed using numerical methods.

4 conclusion

We presented an analytical model of chromatic effects in the round-to-flat beam transformation. Despite our simplified assumptions and the use of the thin lens approximation for the quadrupoles, we find that our analytical model is in decent agreement with numerical simulations.

References

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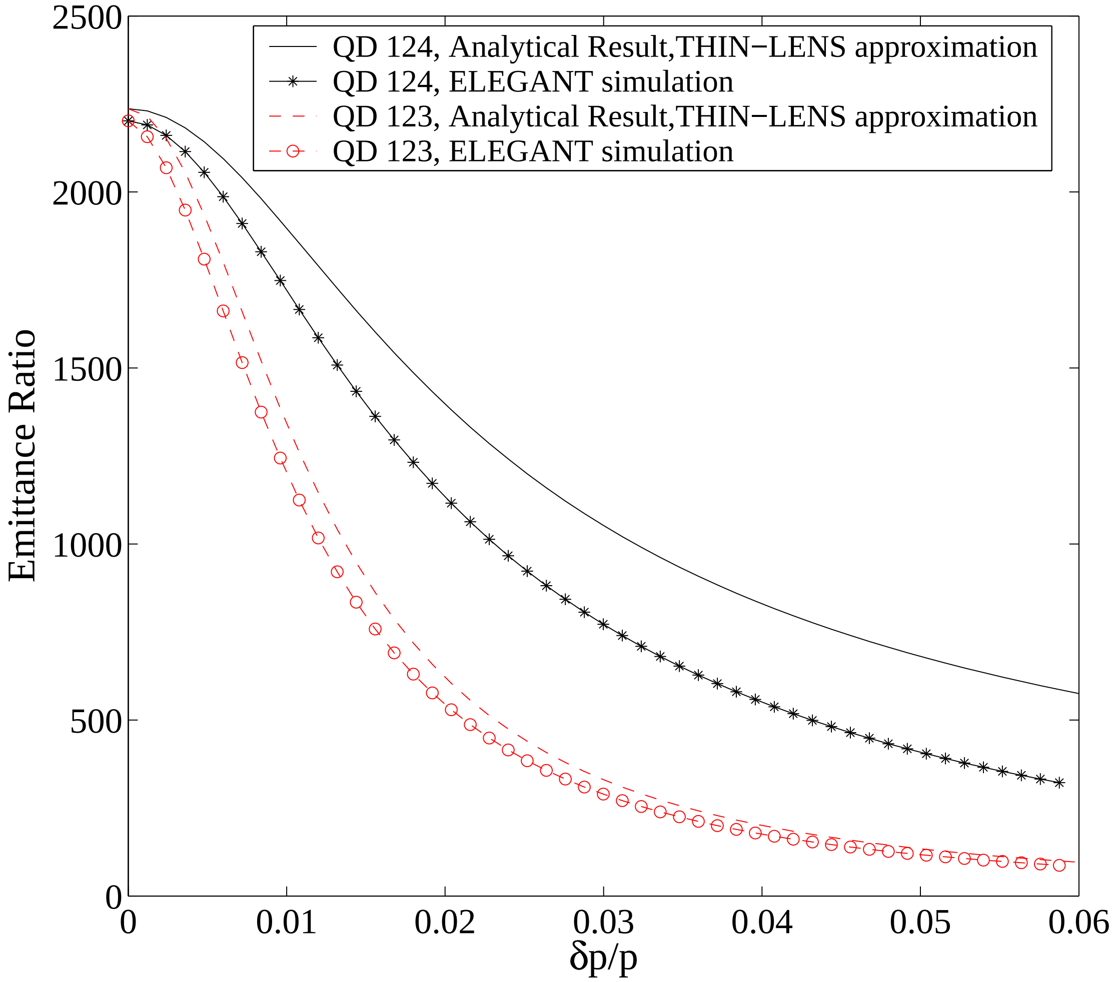

Figure 5: Chromatic effects on emittance ratio for two different spacing between skew quadrupoles: case of QD 124: $d_2 = 0.35$ m, $d_3 = 0.85$ m; case of QD 123: $d_2 = d_3 = 0.35$ m.

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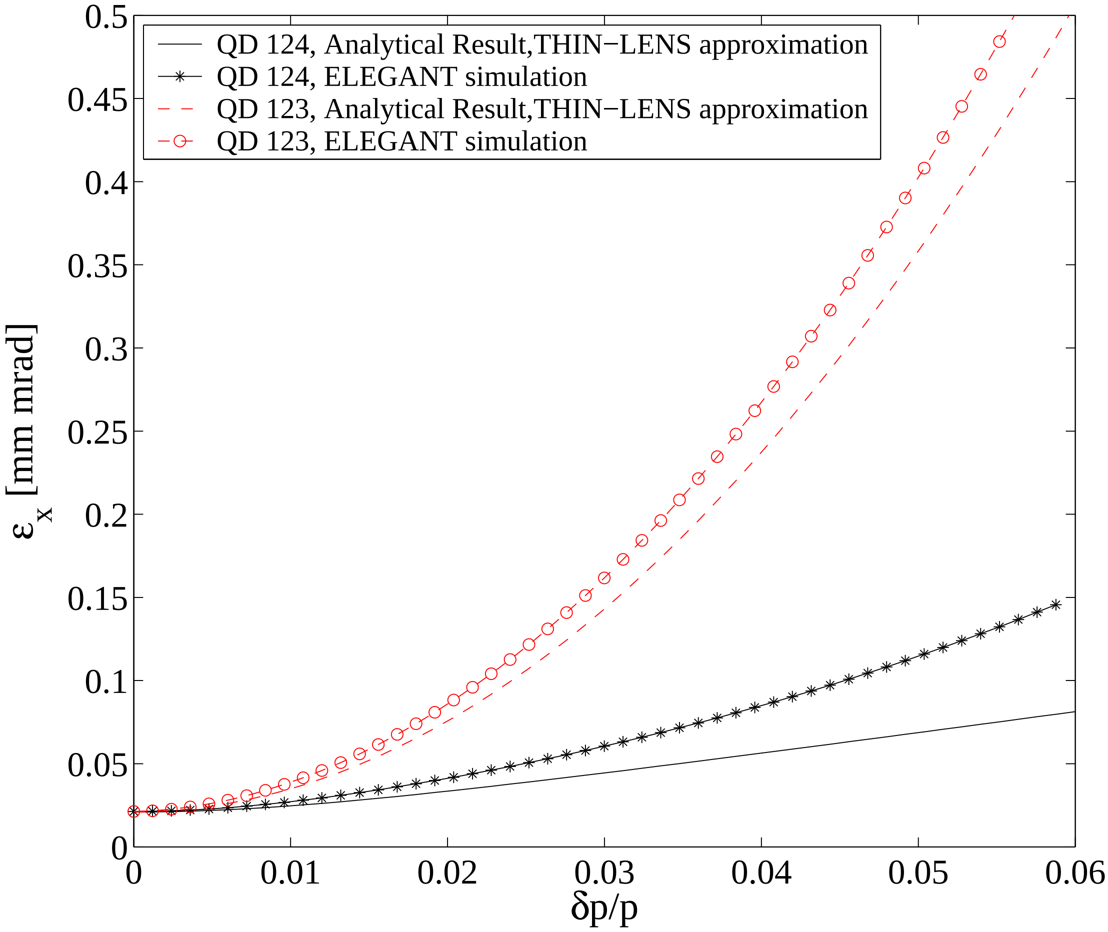

Figure 6: Chromatic effects on horizontal emittance for two different spacing between skew quadrupoles: case of QD 124: $d_2 = 0.35$ m, $d_3 = 0.85$ m; case of QD 123: $d_2 = d_3 = 0.35$ m.

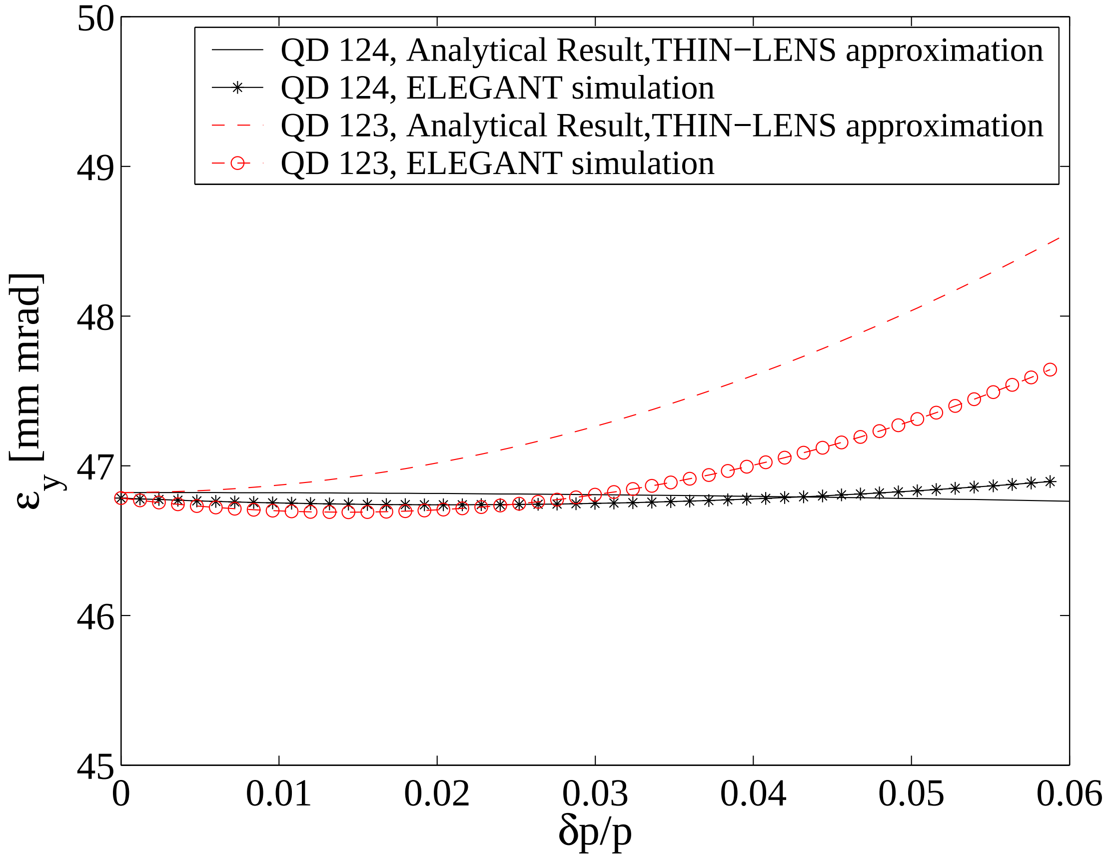

Figure 7: Chromatic effects on vertical emittance for two different spacing between skew quadrupoles: case of QD 124: $d_2 = 0.35$ m, $d_3 = 0.85$ m; case of QD 123: $d_2 = d_3 = 0.35$ m.