# Single and Multiple Intrabeam Scattering in Hadron Colliders

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***Talk outline***

1. Introduction
2. Growth rates for Gaussian beams
3. IBS in non-linear longitudinal well
4. Comparison with experiment

   Conclusions

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# 1. Introduction

## Objective

- Steady growth of the luminosity for last 3 years
  - Present record luminosity $10^{32}$ cm$^{-2}$s$^{-1}$
  - Three fold luminosity increase during next 2-3 years
    - Major part should come from increase of p-bar production
    - but the luminosity growth has to be supported by Tevatron
- Luminosity evolution in Tevatron is driven by
  - Elastic and non-elastic scattering on the residual gas
  - Elastic and non-elastic scattering on counter-rotating beam
  - RF noise & transverse noise (magnetic field fluctuations, quad motion, etc.)
  - Intrabeam scattering
  - Beam-beam effects
- The major aim for the effort is to understand better the beam-beam effects and other possible limitations of the luminosity
- To make any practical conclusions accurate measurements are not less important than good theory

# IBS

- Growth rates for Gaussian beams (Bjorken, Mtingwa, 1982)
  - X-Y coupling needs to be taken into account in the case of Tevatron
  - This approximation is good for many applications but is not self-consistent
    - the beam does not stay gaussian in the coarse of evolution
  - The evolution of beam parameters is determined by system of ordinary differential equations
- Longitudinal degree of freedom is different from transverse ones
  - Non-linear focusing
  - Finite RF bucket size
  - The distribution function evolution is determined by partial diff. equation
- Simultaneous treatment of single and multiple scattering are required in many practical applications
  - Contributions from single and multiple scattering cannot be separated at the store beginning in Tevatron
  - Beam-life time in Recycler with cooling cannot be accurately computed if only single scattering is taken into account
  - The distribution function evolution is determined by integro-differential equation (Boltzmann type)

# 2. Growth rates for Gaussian beams

- Landau collision integral

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial p_i}\left(F_i f\right)=\frac{1}{2}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right)$$

$$F_i(p)=-\frac{4pne^4L_c}{m}\int f(p')\frac{u_i}{|\mathbf{u}|^3}d^3p' \quad , \quad \mathbf{u}=\mathbf{v}-\mathbf{v}'$$

$$D_{ij}=4pne^4L_c\int f(p')\frac{u^2d_{ij}-u_iu_j}{|\mathbf{u}|^3}d^3p'$$

- Integration with Gaussian distrib. in all 3 degrees of freedom yields

$$\frac{d}{dt}\overline{\mathrm{v}_x{}^2}=\frac{2p^{3/2}ne^4L}{m\sqrt{\overline{\mathrm{v}_x{}^2}+\overline{\mathrm{v}_y{}^2}+\overline{\mathrm{v}_z{}^2}}}\Psi\left(\sqrt{\overline{\mathrm{v}_x{}^2}},\sqrt{\overline{\mathrm{v}_y{}^2}},\sqrt{\overline{\mathrm{v}_z{}^2}}\right)$$

  - Equations for other degrees of freedom are obtained by cyclic substitution
    Here:

$$\Psi(x,y,z)=\frac{\sqrt{x^2+y^2+z^2}}{p}\int_0^\infty\frac{\sqrt{2}\,t^3dt}{\left(x^2+t^2\right)^{3/2}\left(y^2+t^2\right)^{1/2}\left(z^2+t^2\right)^{1/2}}\left(\frac{y^2-x^2}{y^2+t^2}+\frac{z^2-x^2}{z^2+t^2}\right)$$

- Energy conservation requires: $\Psi(x,y,z)+\Psi(y,z,x)+\Psi(z,x,y)=0$

- The function $\Psi(x,y,z)$ does not depend on $x^2+y^2+z^2$ and therefore for further analysis we choose $x^2+y^2+z^2=1$
- The function $\Psi(x,y,z)$ is determined so that

$$\Psi(0,x,x)=1 \qquad \Psi(x,x,0)=\Psi(x,0,x)=-1/2 \qquad \Psi(x,x,x)=0 \qquad (1)$$

- When two parameters coincide the integral can be computed

$$\Psi\left(x,\sqrt{\frac{1-x^2}{2}},\sqrt{\frac{1-x^2}{2}}\right)=2\hat{\Psi}(x)$$

$$\Psi\left(\sqrt{\frac{1-z^2}{2}},\sqrt{\frac{1-z^2}{2}},z\right)=-\hat{\Psi}(z)$$

$$\hat{\Psi}(x)=\frac{1}{\sqrt{2p}}\frac{1}{3x^2-1}\left(\frac{1}{\sqrt{2}}\frac{3x^2+1}{\sqrt{3x^2-1}}\ln\left(\frac{\sqrt{2}x-\sqrt{3x^2-1}}{\sqrt{2}x+\sqrt{3x^2-1}}\right)+6x\right)$$

- For practical applications the function $\Psi(x,y,z)$ can be approximated as

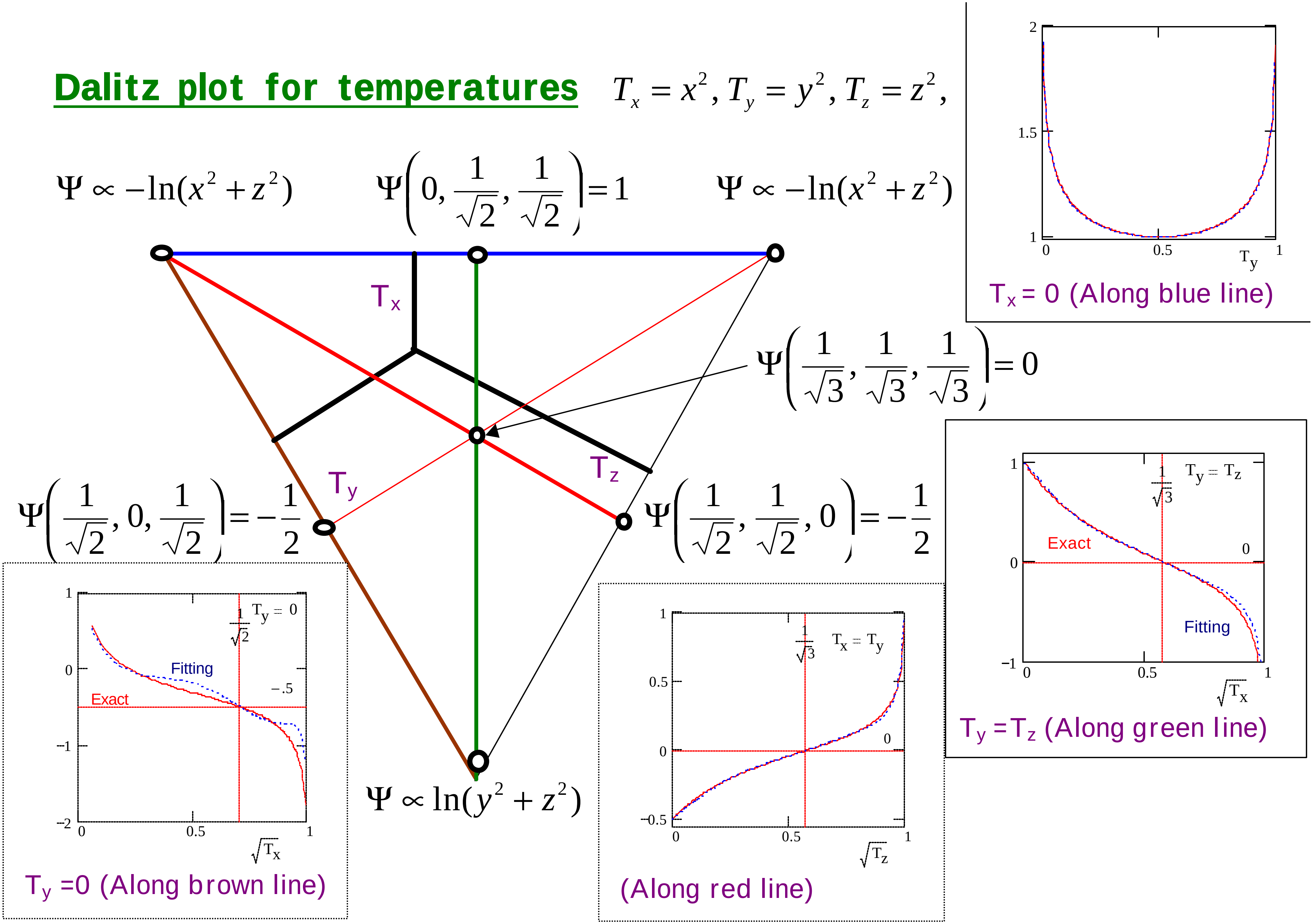
$$\Psi(x,y,z)\approx\frac{\sqrt{2}}{p}\ln\left(\frac{y^2+z^2}{\sqrt{x^2+y^2}\sqrt{x^2+z^2}}\right)+0.688\frac{(x-y)(x-z)}{\sqrt{x^2+y^2}\sqrt{x^2+z^2}}-0.055\left(y^2-z^2\right)^2-$$

$$7.47x^6|y-z|^3-0.8x\left(1-3x^3\right)?\left(1-3x^3\right) \quad , \qquad ?\left(1-3x^3\right)=\begin{cases}1, & x\geq 0\\ 1, & x<0\end{cases}$$

  - This function has correct asymptotics, satisfy conditions (1), and coincide with exact expression within
  - ~1% for x=0, ~10% for entire range of parameters

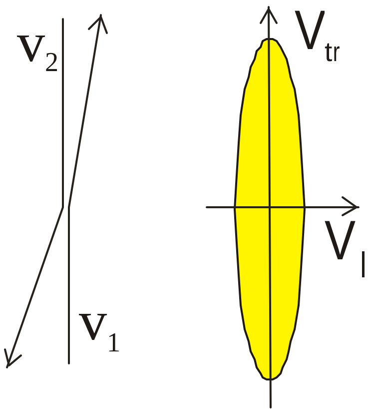
# Dalitz plot for temperatures

$T_x = x^2, T_y = y^2, T_z = z^2,$



# Accelerator specific corrections

- **General Recipe**
  - Find local density and velocity spreads and compute average temperature growth across the beam cross-section. Then average along beam orbit.
    - Take into account that axis of 3D ellipsoid of velocities not necessarily coincide with local coordinate frame axis
    - Take into account additional excitation of transverse degrees of freedom due to non-zero dispersions




- **Details**
  - For Tevatron significant simplifications are due to
    - $v_\parallel << v_x, v_y$ in the beam frame
    - $g >> Q_x, Q_y$
  - After averaging over the bunch length and the cross-section

$$\frac{d}{dt}\left(s_p^{\ 2}\right)\equiv\left\langle\frac{d}{dt}\left(\frac{\overline{p_\parallel^{\ 2}}}{p}\right)\right\rangle_s=\frac{1}{4\sqrt{2}}\frac{e^4}{m_p^{\ 2}c^3 g_i^{\ 3} b_i^{\ 3}}\left\langle\frac{N_i}{s_1 s_2 s_s}\frac{\Psi\left(s_p/g, q_1, q_2\right)}{\sqrt{q_1^{\ 2}+q_2^{\ 2}+\left(s_p/g\right)^2}}L_C\right\rangle_s$$

  - Here $q_1$ and $q_2$ are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and $s_1$ and $s_2$ are ellipse semi-axis in the x-y plane

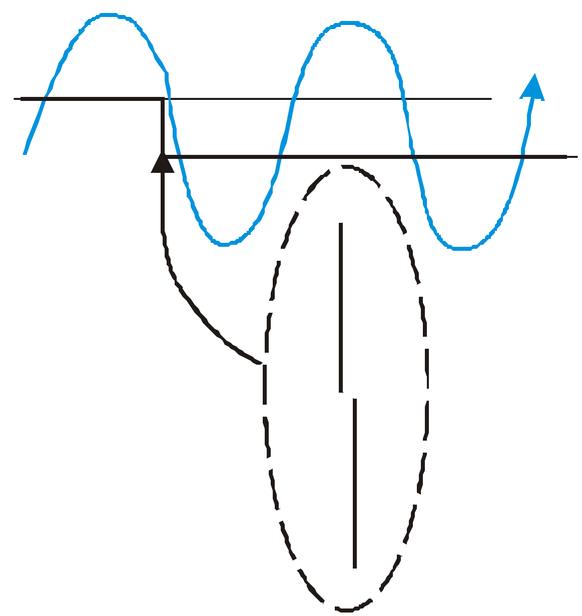
## ➢ Uncoupled motion

$$s_1 \equiv s_x = \sqrt{e_x b_y + D^2 s_p{}^2}\,, \qquad s_2 \equiv s_y = \sqrt{e_y b_y}\,,$$

$$q_1 \equiv q_x = \sqrt{\frac{e_x}{b_x}\left(1 + \frac{(D' b_x + a_x D)^2 s_p{}^2}{e_x b_x + D^2 s_p{}^2}\right)}\,, \qquad q_2 \equiv q_y = \sqrt{e_y / b_y}$$

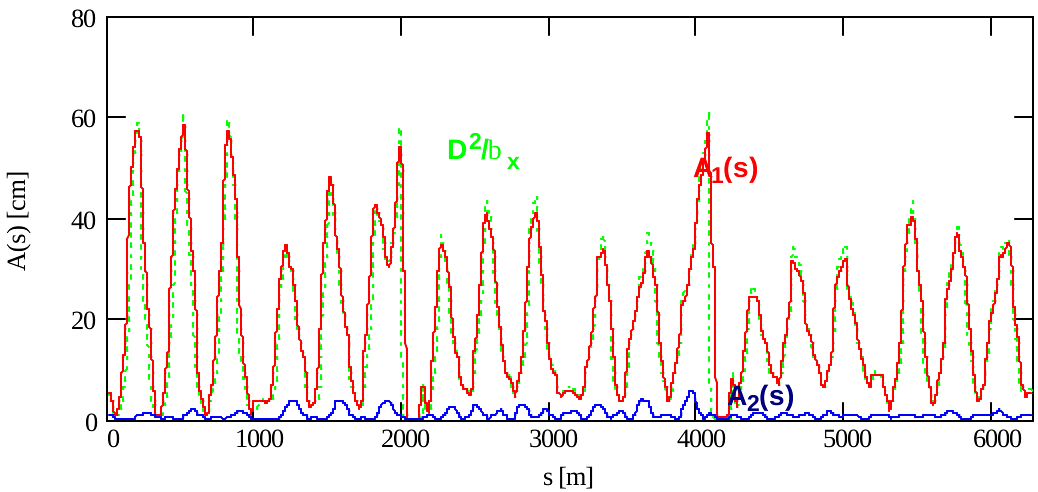
- Additional transverse emittance growth due to finite dispersion dominates emittance change due to “direct” scattering

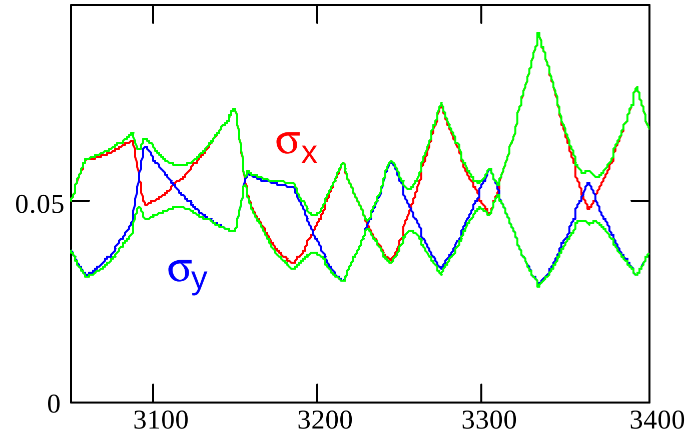
$$\frac{de_x}{dt} = \left\langle \frac{D^2 + (D' b_x + a_x D)^2}{b_x} \frac{d\overline{q_{\parallel}{}^2}}{dt} \right\rangle_s$$

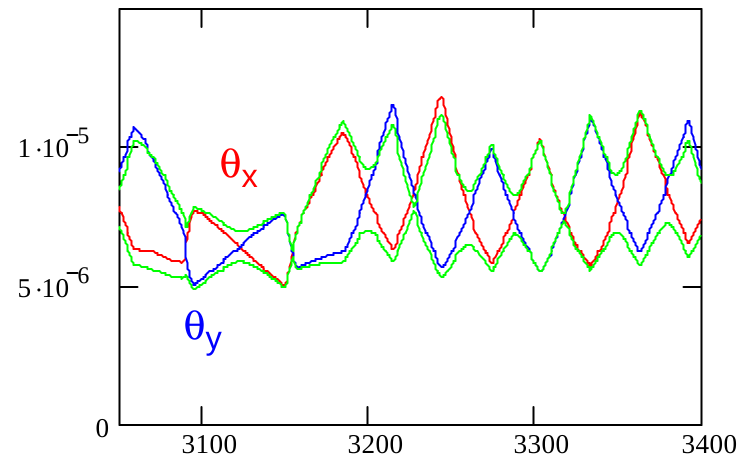


- X-Y coupled motion
  - Measured Tevatron optics with coupling has been used in calculations!!!
  - Coupling effects are sufficiently small
    - little corrections for density and angular spread
  - Emittance growth related to mode 2 (y mode) is about 5% of mode 1 (x-mode)

$$\frac{de_1}{dt} = \left\langle A_1 \frac{d}{dt} \overline{\left(\frac{\Delta p}{p}\right)^2} \right\rangle_s , \quad \frac{de_2}{dt} = \left\langle A_2 \frac{d}{dt} \overline{\left(\frac{\Delta p}{p}\right)^2} \right\rangle_s$$




Top - Beam size projections and ellipsoid semi-axis
Bottom – projections for angular spreads and ellipsoid semi-axis

- In measurements both modes contribute to the beam sizes

$$s_x^{\,2} = b_{1x}e_1 + b_{2x}e_2$$

$$s_y^{\,2} = b_{1y}e_1 + b_{2y}e_2$$

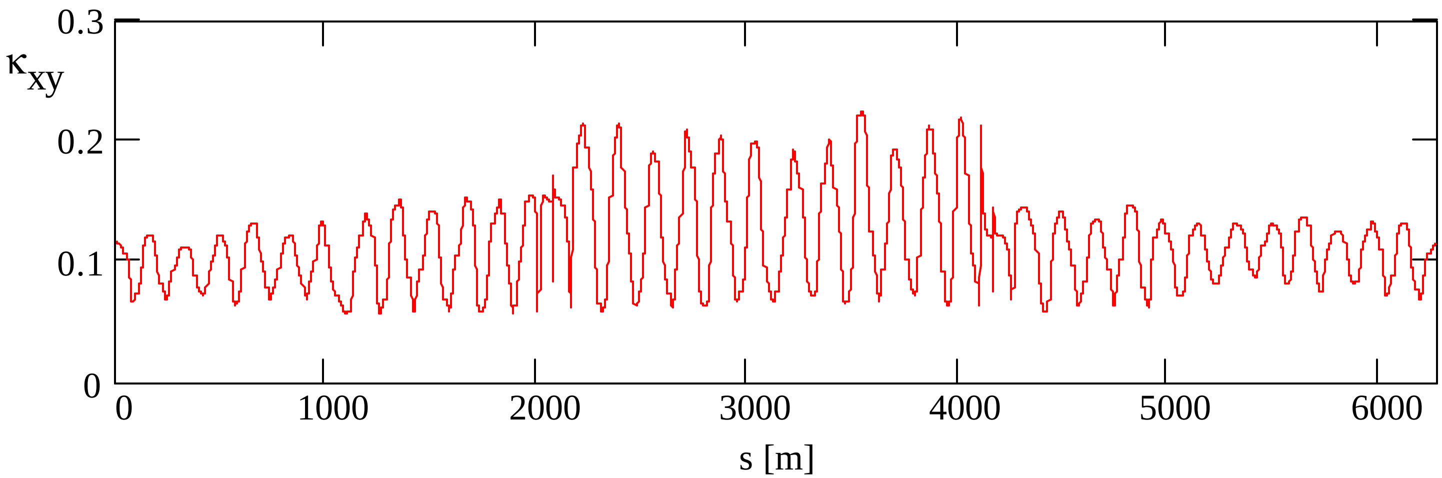
- That yields for observed emittance growth

$$\frac{de_x}{dt} = \frac{de_1}{dt} + \frac{b_{2x}}{b_{1x}}\frac{de_2}{dt}$$

$$\frac{de_y}{dt} = \frac{b_{1y}}{b_{2y}}\frac{de_1}{dt} + \frac{de_2}{dt}$$

- That leads to an increase of observed coupling
  - $k_{xy} \approx 0.1$ at Synchrotron light emittance monitor

$$k_{xy} \equiv \frac{de_y/dt}{de_x/dt + de_y/dt}$$



# 3. IBS in Non-linear Longitudinal Well

## Diffusion equation

- In the case $v_{\|} \ll v_x, v_y$ the friction in Landau collision integral can be neglected
- Diffusion equation

$$D(I) = \langle D(p) \rangle_{period}$$

1D: $$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial p}\left( D(p)\frac{\partial f}{\partial p} \right)$$ $\Rightarrow$ 2D: $$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I}\left( I\frac{D(I)}{w(I)}\frac{\partial f}{\partial I} \right)$$

  - $I$ is the action and $w$ is the frequency for dimensionless Hamiltonian of synchrotron motion: $H = \frac{p^2}{2} + 2\left(\sin\frac{j}{2}\right)^2$
  - Diffusion coefficient depends on distribution, ($I$)

$$D(I) = 4L_c\tilde{A}\langle n(j)\rangle_{period} \qquad \tilde{A} = p^2\sqrt{\frac{p}{2}}\frac{\left(a - 1/g_i^2\right)e^4 q^2}{eV_0 m_p c b_i g_i^2 C}\left\langle \frac{N_i}{s_1 s_2}\frac{\Psi\left(s_p/g, q_1, q_2\right)}{\sqrt{q_1^2 + q_2^2 + \left(s_p/g\right)^2}} \right\rangle_s$$

Here: $n(j) = \int f\left(I(p,j)\right)dp$ , $\int_{-p}^{p} n(j)\,dj = 1$

$a$ - momentum compaction, $q$ - harmonic number

$V_0$ - RF voltage, $C$ - ring circumference

# Simultaneous treatment of single and multiple scattering

- Boltzmann type equation
  - In the case $v_{\parallel} << v_{\perp}$ one can write for Coulomb scattering in long. direction

$$\frac{\partial f}{\partial t} = \left\langle \tilde{A} \int n(j) \frac{f(p+q) - f(p)}{|q|^3} \mathrm{d}q \right\rangle_{period} = \left\langle \tilde{A} \int n(j) \frac{f(I') - f(I)}{|p-p'|^3} d(j - j') \mathrm{d}I' dy\, dy' \right\rangle_{period}$$

- After simplification we obtain

$$\frac{\partial f(I,t)}{\partial t} = \tilde{A} \int_0^{\infty} W(I,I') \left( f(I',t) - f(I,t) \right) \mathrm{d}I'$$

$$W(I,I') = \frac{2ww'}{p} \int_0^{\min(a,a')} \frac{dj}{pp'} n(j) \left[ \frac{1}{|p-p'|^3} + \frac{1}{|p+p'|^3} \right] \xrightarrow{E' \geq E}$$

$$\frac{ww'}{p(E-E')^3} \left[ (E-E') \int_0^a n(j) \frac{dx}{p} + 2 \int_0^a n(x)\, p\, dx \right] .$$

$a \equiv a(I)$ is the motion amplitude

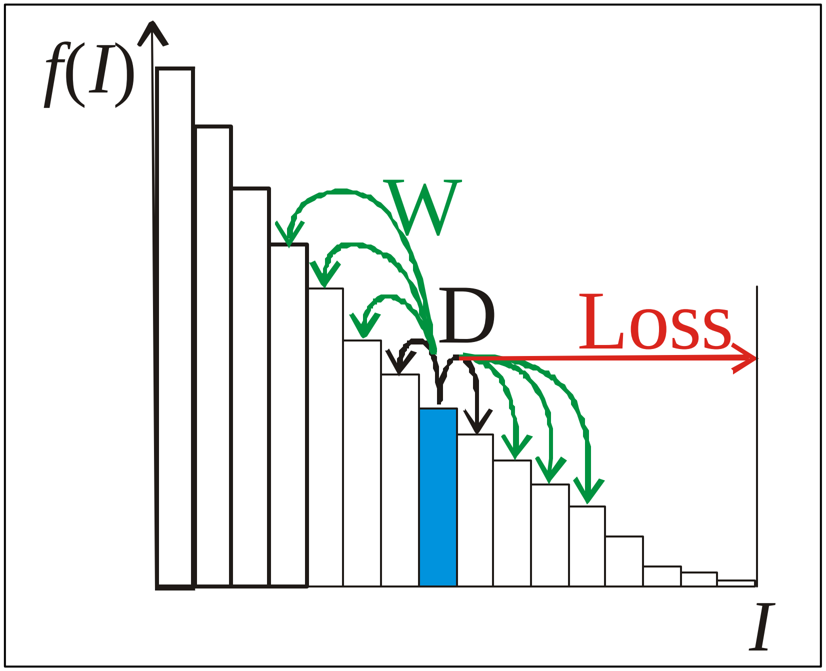
  - The kernel is symmetric: $W(I,I') = W(I',I)$,
  - The kernel divergence needs to be limited at the minimum action change corresponding to the maximum impact parameter

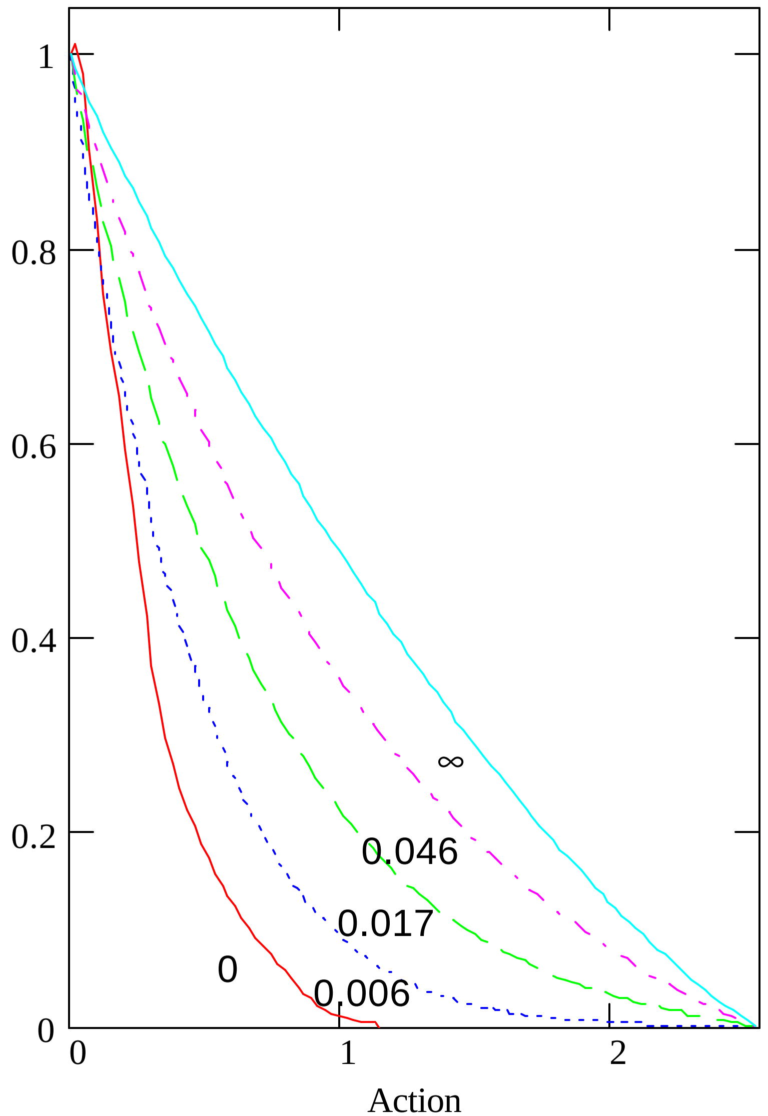
## Numerical model

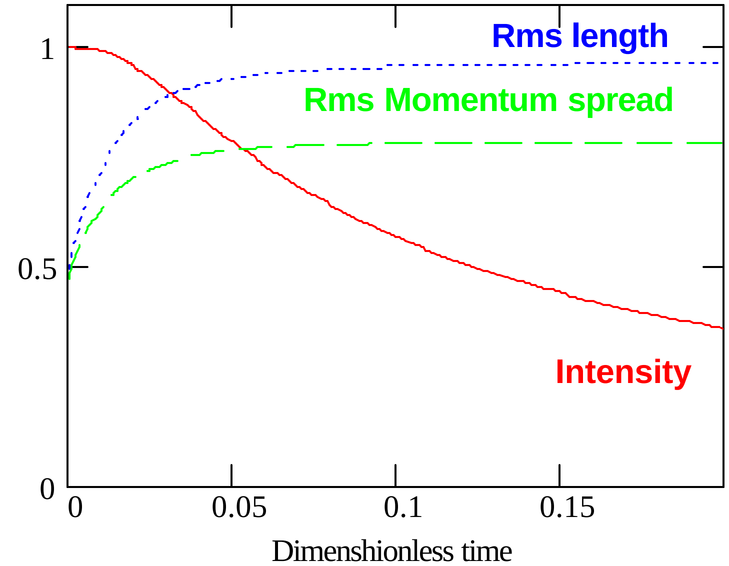
- Set of bins
  - Transition probabilities
    - Nearby bins – diffusion equation to resolve divergence of $W(I, I')$
    - Far away bins – transition probabilities are described by $W(I, I')$
    - Particle loss outside bucket need to be added
- In matrix form

$$\mathbf{f}_{n+1} = \mathbf{f}_n + \mathbf{W}\mathbf{f}_n \Delta t$$

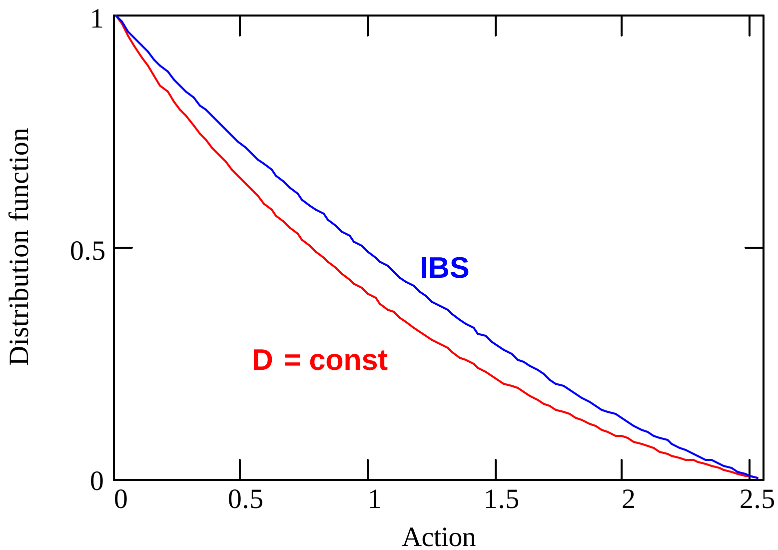
W – is matrix of transition probabilities. It is a symmetric matrix





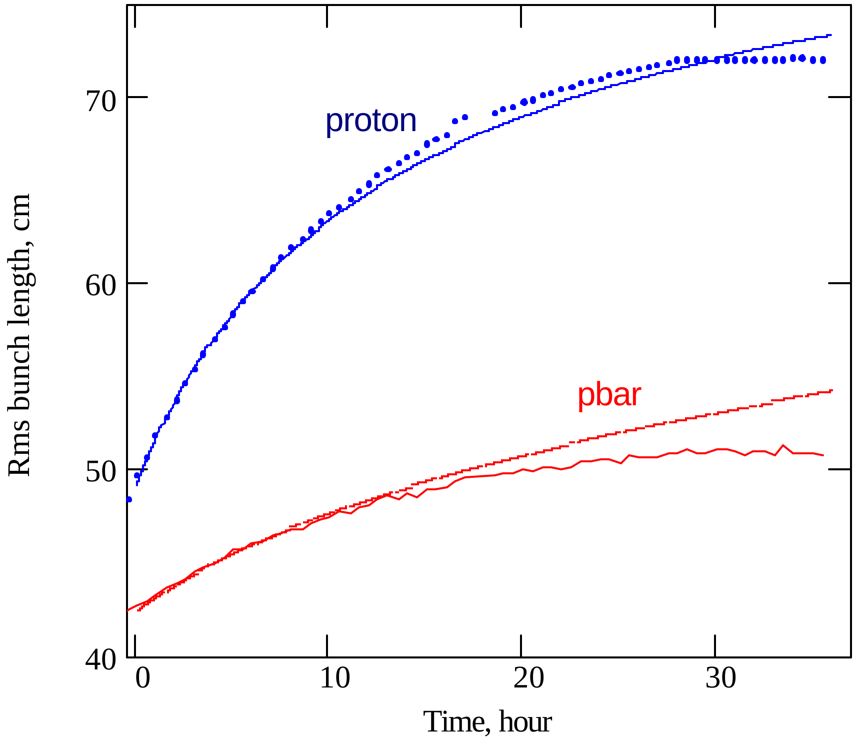

Dependencies: $N(t)/N_0, \sqrt{\overline{j^2(t)}}, \sqrt{\overline{p^2(t)}}$



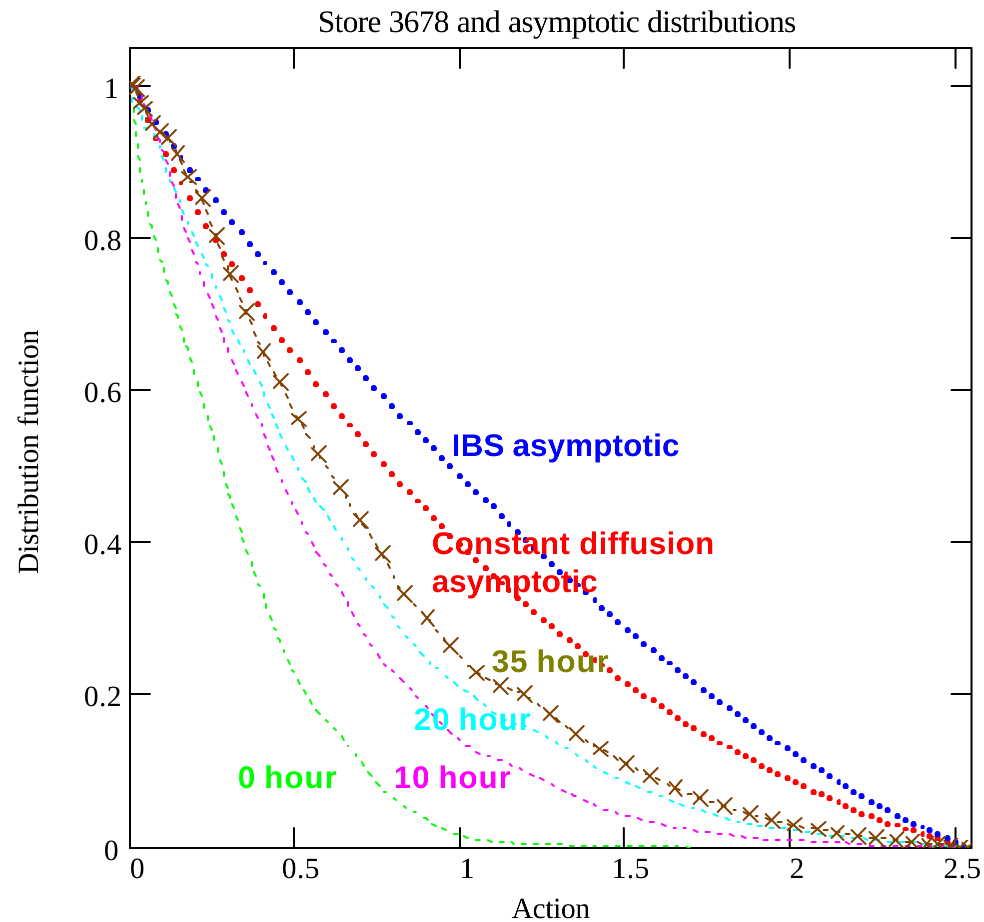

Dependence of longitudinal distribution on time for IBS. Measured initial distribution is used

Asymptotic distributions

# 5. Comparison with experiment

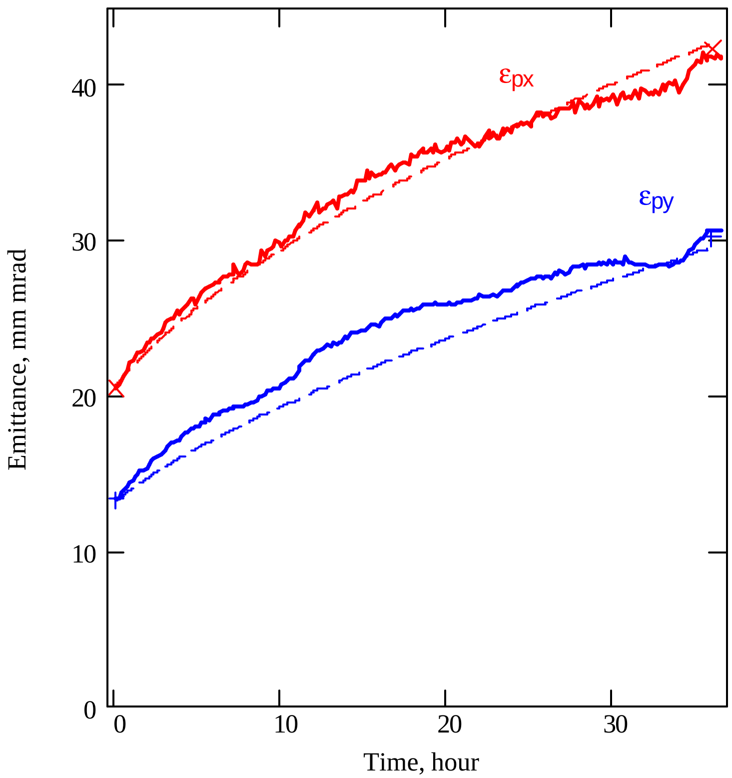


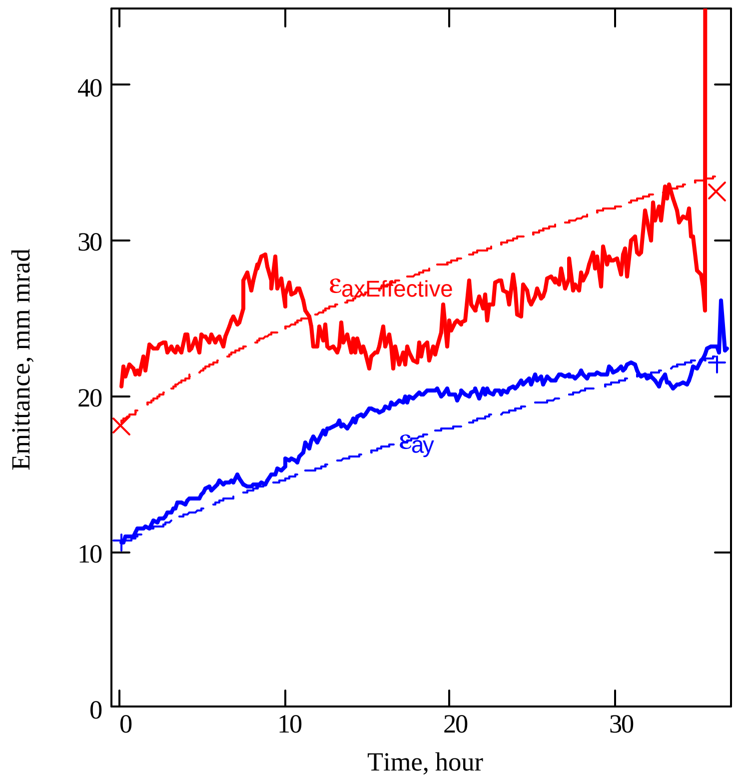

Dependence of computed and measured bunch length on time for Store 3678



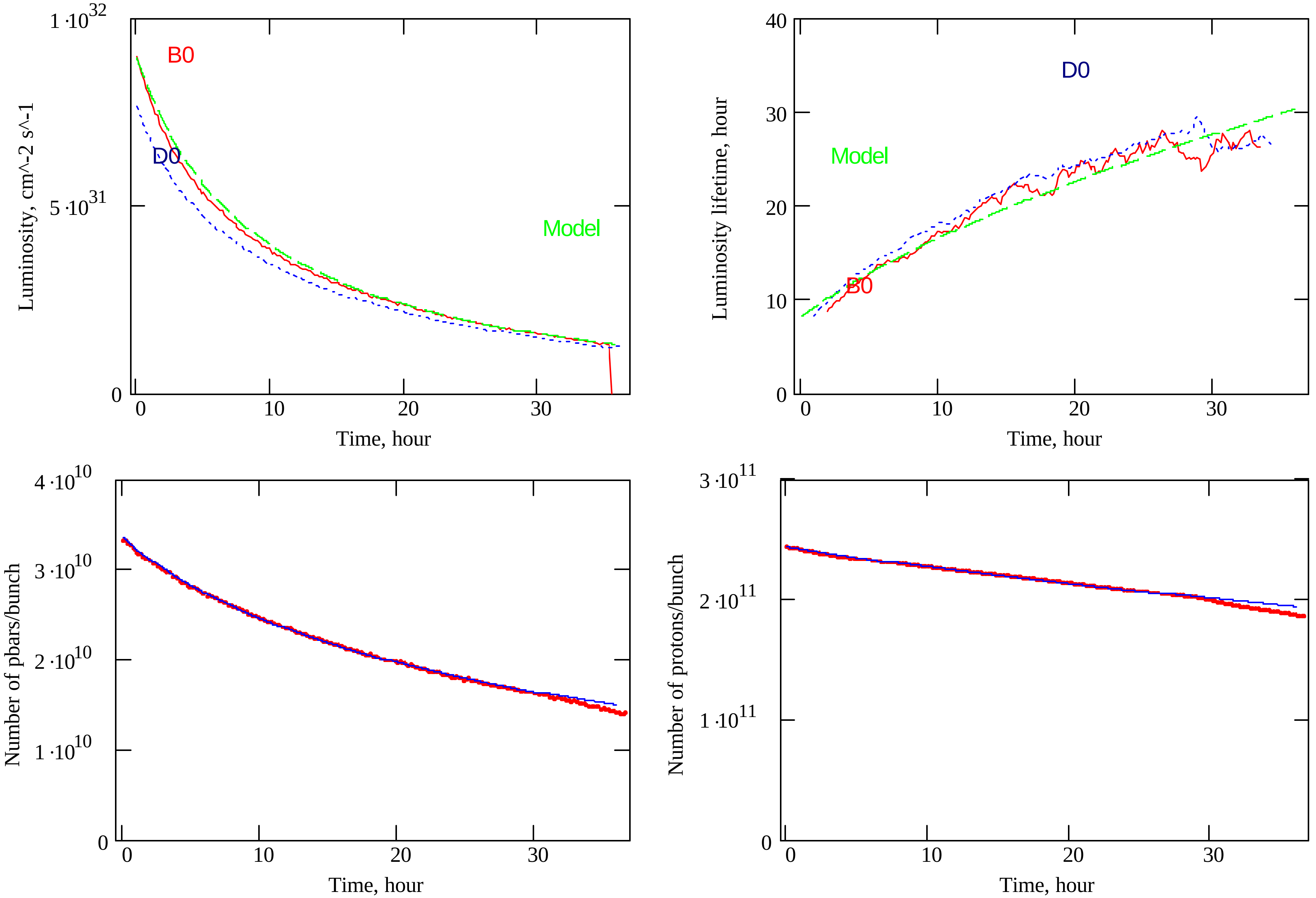

Evolution of proton distribution for Store 3678

- Good coincidence for proton bunch lengthening.. It is not affected by choice of free parameters
- $\bar{p}$ bunch lengthening is affected by RF noise (~20% of IBS): $S_f$=3·10$^{11}$ rad$^2$s, $d\sigma_s/dt|_{t=0}$= 0.12cm/hour
    - Sextupole power supply was lost at t=28 hour => beam-beam effects




- Both Proton and pbar emittance growths have contributions from scattering at the residual gas. At the store beginning it gives:
  - 2*14% of IBS for protons
  - 2*64% of IBS for pbars
  - Gas pressure was set by matching particle loss due to nuclear scattering
- For pbars there is unaccounted emittance growth (2.4 times of IBS)
  - Beam-beam effects
  - Noise in magnets
  - Preliminary measurements of bunch motion at $\omega_b$ yields $\sigma$ = 0.1 $\mu$m
    - That is consistent with the measurements




Luminosity and beam parameters evolution for Store 3678

# Conclusions

- Theory describes well observed evolution of parameters for proton beam
- Observed discrepancy for antiprotons is related to other effects which are presently not taken into account
  - Noise in magnets and beam-beam effects are the most probable reasons
- To get such good agreement the improvements in theory as well as in experiment have been required

## Backup transparencies

## 1. Emittance Growth Rates for X-Y coupled motion in the case of pancake distribution

- **Growth rates for the momentum spread in the bunched beam**
  - After averaging over the bunch length and the cross-section

$$\frac{d}{dt}\left(\sigma_p^{\;2}\right)\equiv\left\langle\frac{d}{dt}\left(\frac{\overline{p_{\parallel}^{\;2}}}{p}\right)\right\rangle_s=\frac{1}{4\sqrt{2}}\frac{e^4}{m_p^{\;2}c^3\gamma_i^{\;3}\beta_i^{\;3}}\left\langle\frac{N_i}{\sigma_1\sigma_2\sigma_s}\frac{\Psi\left(\sigma_p/\gamma,\theta_1,\theta_2\right)}{\sqrt{\theta_1^{\;2}+\theta_2^{\;2}+\left(\sigma_p/\gamma\right)^2}}L_C\right\rangle_s$$

- Here $\theta_1$ and $\theta_2$ are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and $\sigma_1$ and $\sigma_2$ are ellipse semi-axis in the x-y plane
- This equation is still valid for arbitrary gaussian distribution (pancake distr. is not required)
- For uncoupled beam

$$\sigma_1\equiv\sigma_x=\sqrt{\varepsilon_x\beta_y+D^2\sigma_p^{\;2}},\qquad\sigma_2\equiv\sigma_y=\sqrt{\varepsilon_y\beta_y},$$

$$\theta_1\equiv\theta_x=\sqrt{\frac{\varepsilon_x}{\beta_x}\left(1+\frac{(D'\beta_x+\alpha_x D)^2\sigma_p^{\;2}}{\varepsilon_x\beta_x+D^2\sigma_p^{\;2}}\right)},\qquad\theta_2\equiv\theta_y=\sqrt{\varepsilon_y/\beta_y}$$

## ➢ Optics is described with Mais-Ripken beta-functions

- For coupled motion the eigen-vectors can be parameterized as

$$\mathbf{x}(s) = \mathrm{Re}\left(\tilde{e}_1 \mathbf{v}_1(s) e^{-im_1(s)} + \tilde{e}_2 \mathbf{v}_2(s) e^{-im_2(s)}\right),$$

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{b_{1x}} \\ -\dfrac{i(1-u)+a_{1x}}{\sqrt{b_{1x}}} \\ \sqrt{b_{1y}} e^{in_1} \\ -\dfrac{iu + a_{1y}}{\sqrt{b_{1y}}} e^{in_1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \sqrt{b_{2x}} e^{in_2} \\ -\dfrac{iu + a_{2x}}{\sqrt{b_{2x}}} e^{in_2} \\ \sqrt{b_{2y}} \\ -\dfrac{i(1-u)+a_{2y}}{\sqrt{b_{2y}}} \end{bmatrix}$$

- To find beam sizes and local angular spreads First introduce bilinear form describing the beam ellipse in 4D space

$$\hat{\mathbf{x}}^T \hat{?} \hat{\mathbf{x}} = 1$$

$$\hat{\Xi}_{11} = \frac{(1-u)^2 + a_{1x}{}^2}{e_1 b_{1x}} + \frac{u^2 + a_{2x}{}^2}{e_2 b_{2x}}, \qquad \hat{\Xi}_{22} = \frac{b_{1x}}{e_1} + \frac{b_{2x}}{e_2},$$

$$\hat{\Xi}_{33} = \frac{u^2 + a_{1y}{}^2}{e_1 b_{1y}} + \frac{(1-u)^2 + a_{2y}{}^2}{e_2 b_{2y}}, \qquad \hat{\Xi}_{44} = \frac{b_{1y}}{e_1} + \frac{b_{2y}}{e_2},$$

$$\hat{\Xi}_{12} = \hat{\Xi}_{21} = \frac{a_{1x}}{e_1} + \frac{a_{2x}}{e_2}, \qquad \hat{\Xi}_{34} = \hat{\Xi}_{43} = \frac{a_{1y}}{e_1} + \frac{a_{2y}}{e_2},$$

$$\hat{\Xi}_{13} = \hat{\Xi}_{31} = \frac{\left[a_{1x}a_{1y} + u(1-u)\right]\cos n_1 + \left[a_{1y}(1-u) - a_{1x}u\right]\sin n_1}{e_1\sqrt{b_{1x}b_{1y}}} + \frac{\left[a_{2x}a_{2y} + u(1-u)\right]\cos n_2 + \left[a_{2x}(1-u) - a_{2y}u\right]\sin n_2}{e_2\sqrt{b_{2x}b_{2y}}} ,$$

$$\hat{\Xi}_{14} = \hat{\Xi}_{41} = \sqrt{\frac{b_{1y}}{b_{1x}}}\frac{a_{1x}\cos n_1 + (1-u)\sin n_1}{e_1} + \sqrt{\frac{b_{2y}}{b_{2x}}}\frac{a_{2x}\cos n_2 - u\sin n_2}{e_2} ,$$

$$\hat{\Xi}_{23} = \hat{\Xi}_{32} = \sqrt{\frac{b_{1x}}{b_{1y}}}\frac{a_{1y}\cos n_1 - u\sin n_1}{e_1} + \sqrt{\frac{b_{2x}}{b_{2y}}}\frac{a_{2y}\cos n_2 + (1-u)\sin n_2}{e_2} ,$$

$$\hat{\Xi}_{24} = \hat{\Xi}_{42} = \frac{\sqrt{b_{1x}b_{1y}}\cos n_1}{e_1} + \frac{\sqrt{b_{2x}b_{2y}}\cos n_2}{e_2} .$$

- **Then we can write the distribution function in the following form**

$$f(\hat{\mathbf{x}},q_{\parallel}) = C_1 \exp\left(-\frac{1}{2}\left(\hat{\mathbf{x}} - \mathbf{D}q_{\parallel}\right)^T \mathbf{?} \left(\hat{\mathbf{x}} - \mathbf{D}q_{\parallel}\right)\right)\exp\left(-\frac{q_{\parallel}^{2}}{2s_{p}^{2}}\right)$$

where $\mathbf{D} = \left[D_x \quad D'_x \quad D_y \quad D'_y\right]^T$

- After integration over momentum spread we obtain

$$f(\hat{\mathbf{x}}) = C_2 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T \mathbf{?}'\hat{\mathbf{x}}\right), \quad \mathbf{?}' = \mathbf{?} - \frac{\mathbf{?DD}^T\mathbf{?}}{s_p^{-2} + \mathbf{D}^T\mathbf{?}\,\mathbf{D}}$$

- **Beam sizes**
  - Size projections

$$s_x \equiv \sqrt{\overline{x^2}} = \sqrt{e_1 b_{1x} + e_2 b_{2x} + D_x^{\,2} q_{\parallel}^{\,2}}$$

$$s_y \equiv \sqrt{\overline{y^2}} = \sqrt{e_1 b_{1y} + e_2 b_{2y} + D_y^{\,2} q_{\parallel}^{\,2}}$$

$$a_{xy} \equiv \frac{\overline{xy}}{s_x s_y} = \frac{e_1\sqrt{b_{1x}b_{1y}}\cos n_1 + e_2\sqrt{b_{2x}b_{2y}}\cos n_2 + D_x D_y q_{\parallel}^{\,2}}{s_x s_y}$$

    - Ellipse semi-axis

$$s_{1,2} = \sqrt{\frac{2\left(1 - a_{xy}^{\,2}\right)}{s_x^{\,2} + s_y^{\,2} \pm \sqrt{\left(s_x^{\,2} - s_y^{\,2}\right)^2 + 4a_{xy}^{\,2} s_x^{\,2} s_y^{\,2}}}}$$

- **Local transverse velocity spreads**

- Bilinear form for angular spreads

$$[0 \quad q_x \quad 0 \quad q_x]\,\mathbf{?}' \begin{bmatrix} 0 \\ q_x \\ 0 \\ q_y \end{bmatrix} = q_x{}^2 \Xi_{22} + 2 q_x q_y \Xi_{24} + q_y{}^2 \Xi_{44} = 1$$

- Ellipse semi-axis in the plane of local angular spreads (x’-y’ plane)

$$q_{1,2} = \sqrt{\frac{2}{\Xi_{22} + \Xi_{44} \pm \sqrt{\left(\Xi_{22} - \Xi_{44}\right)^2 + 4\Xi_{24}{}^2}}}$$

## ➢ Additional transverce emittance growth due to finite dispersion

- For uncoupled motion

$$\frac{d\varepsilon_x}{dt} = \left\langle A_x \frac{d}{dt}\left(\overline{\frac{p_{\parallel}^{\,2}}{p}}\right)\right\rangle_s \quad , \qquad A_x = \frac{D^2 + \left(D'\beta_x + \alpha_x D\right)^2}{\beta_x}$$

- **Coupled motion:** momentum change excites both hor. and vert. motions

$$\begin{bmatrix} D_x \\ D'_x \\ D_y \\ D'_y \end{bmatrix}\frac{\Delta p}{p} \equiv \mathbf{D}\frac{\Delta p}{p} = \mathrm{Re}\left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2\right) = \mathbf{VA} \quad , \quad V = [\mathbf{v}'_1, -\mathbf{v}''_1, \mathbf{v}'_2, -\mathbf{v}''_2] \quad , \qquad \mathbf{a} = \begin{bmatrix} a'_1 \\ a''_1 \\ a'_2 \\ a''_2 \end{bmatrix}$$

$$\mathbf{a} = \frac{\Delta p}{p}\mathbf{V}^{-1}\mathbf{D}$$

  - Then the emittance growth is

$$\frac{d\varepsilon_1}{dt} = \left\langle A_1 \frac{d}{dt}\overline{\left(\frac{\Delta p}{p}\right)^2}\right\rangle_s \; , \qquad \frac{d\varepsilon_2}{dt} = \left\langle A_2 \frac{d}{dt}\overline{\left(\frac{\Delta p}{p}\right)^2}\right\rangle_s$$

Here: $A_{1,2} = \mathbf{D}^T\mathbf{B}_{1,2}\mathbf{D}$ , $\mathbf{B}_1 = \left(\mathbf{V}^{-1}\right)^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^{-1}$, $\mathbf{B}_2 = \left(\mathbf{V}^{-1}\right)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{V}^{-1}$

- Expressing matrix V through beta-functions we finally obtain

$$\mathbf{B}_1 = \begin{bmatrix} \dfrac{(1-u)^2 + a_{1x}{}^2}{b_{1x}} & a_{1x} & B_{1_{13}} & B_{1_{14}} \\ a_{1x} & b_{1x} & B_{1_{23}} & \sqrt{b_{1x} b_{1y}} \cos n_1 \\ B_{1_{13}} & B_{1_{23}} & \dfrac{u^2 + a_{1y}{}^2}{b_{1y}} & a_{1y} \\ B_{1_{14}} & \sqrt{b_{1x} b_{1y}} \cos n_1 & a_{1y} & b_{1y} \end{bmatrix}$$

$$B_{1_{13}} = \frac{\left(u(1-u) + a_{1x} a_{1y}\right)\cos n_1 + \left(a_{1y}(1-u) - a_{1x} u\right)\sin n_1}{\sqrt{b_{1x} b_{1y}}}$$

$$B_{1_{14}} = \sqrt{\frac{b_{1y}}{b_{1x}}}\left(a_{1x}\cos n_1 + (1-u)\sin n_1\right)$$

$$B_{1_{23}} = \sqrt{\frac{b_{1x}}{b_{1y}}}\left(a_{1y}\cos n_1 - u\sin n_1\right)$$

$$\mathbf{B}_2 = \begin{bmatrix} \dfrac{u^2 + a_{2x}{}^2}{b_{2x}} & a_{2x} & B_{2_{13}} & B_{2_{14}} \\ a_{2x} & b_{2x} & B_{2_{23}} & \sqrt{b_{2x} b_{2y}} \cos n_2 \\ B_{2_{13}} & B_{2_{23}} & \dfrac{(1-u)^2 + a_{2y}{}^2}{b_{2y}} & a_{2y} \\ B_{2_{14}} & \sqrt{b_{2x} b_{2y}} \cos n_2 & a_{2y} & b_{2y} \end{bmatrix}$$

$$B_{2_{13}} = \frac{\left(u(1-u) + a_{2x} a_{2y}\right)\cos n_2 + \left(a_{2x}(1-u) - a_{2y} u\right)\sin n_2}{\sqrt{b_{2x} b_{2y}}}$$

$$B_{2_{14}} = \sqrt{\frac{b_{2y}}{b_{2x}}}\left(a_{2x} \cos n_2 - u \sin n_2\right)$$

$$B_{2_{23}} = \sqrt{\frac{b_{2x}}{b_{2y}}}\left(a_{2y} \cos n_2 + (1-u) \sin n_2\right)$$

- Finally, for ultra-relativistic beam ($g >> Q_x, Q_y$), we obtain

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ \\ e_2 \end{bmatrix}_s = \frac{1}{4\sqrt{2}}\frac{e^4}{m_p{}^2 c^3 g_i{}^3 b_i{}^3}\left\langle \frac{N_i}{s_1 s_2 s_s}\frac{L_C}{\sqrt{q_1{}^2 + q_2{}^2 + \left(s_p/g\right)^2}}\begin{bmatrix} \hat{A}_1 \Psi\left(s_p/g, q_1, q_2\right) \\ \\ \hat{A}_2 \Psi\left(s_p/g, q_1, q_2\right) \end{bmatrix}\right\rangle_s$$