Effect of Beam-Beam Head-on Interaction on Beta Star

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July 29, 2005

We wish to estimate the magnitude of the modification to the amplitude function by the two head-on beam-beam interactions in the Tevatron. To begin, we consider the effect from a single interaction point (IP). We will treat the interaction as a thin lens kick centered at the IP. Let q be the equivalent quadrupole strength due to the beam-beam interaction, $q = 1/f_{bb} = 4\pi\xi/\beta^*$, where $\xi = 3r_0N/2\epsilon_N$ is the "beam-beam parameter," r_0 is the classical radius of the proton, N the number of protons per bunch, and ϵ_N the 95% normalized emittance. And let M_0 be the one-turn matrix of the accelerator in the absence of beam-beam,

$$M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu_0 & \beta_0^* \sin\mu_0 \\ -\frac{1}{\beta_0^*}\sin\mu_0 & \cos\mu_0 \end{pmatrix},$$

beginning and ending at the interaction point. We now insert the linearized beam-beam interaction and recompute the matrix at the midpoint of the equivalent lens:

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -q/2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q/2 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq/2 & d - bq/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q/2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a - bq/2 & b \\ c - (a+d)q/2 + bq^2/4 & d - bq/2 \end{pmatrix} = \begin{pmatrix} \cos\mu & \beta^*\sin\mu \\ -\frac{1}{\beta^*}\sin\mu & \cos\mu \end{pmatrix}. \end{aligned}$$

In the last step we have made use of the fact that here $\alpha^* = \alpha_0^* = 0$. Thus, we see that

$$\frac{\beta^*}{\beta_0^*} = \frac{\sin\mu_0}{\sin\mu}.$$

Now,

$$\begin{aligned} \mathrm{Tr}M = 2\cos\mu &= a + d - bq = 2\cos\mu_0 - bq \\ \longrightarrow \cos^2\mu &= \cos^2\mu_0 - bq\cos\mu_0 + \frac{1}{4}b^2q^2 \\ \text{or,} \qquad \sin^2\mu = 1 - \cos^2\mu &= \sin^2\mu_0 + bq\cos\mu_0 - \frac{1}{4}b^2q^2. \end{aligned}$$

Since $b = \beta^*\sin\mu$, then $bq = 4\pi\xi\sin\mu$, and so

$$\sin^2\mu = \sin^2\mu_0 + 4\pi\xi\cos\mu_0\sin\mu - \frac{1}{4}(4\pi\xi)^2\sin^2\mu$$

from which, upon solving for $\sin\mu$, we get

$$\sin\mu = \frac{2\pi\xi\cos\mu_0 \pm \sqrt{\sin^2\mu_0 + (2\pi\xi)^2}}{1 + (2\pi\xi)^2}$$

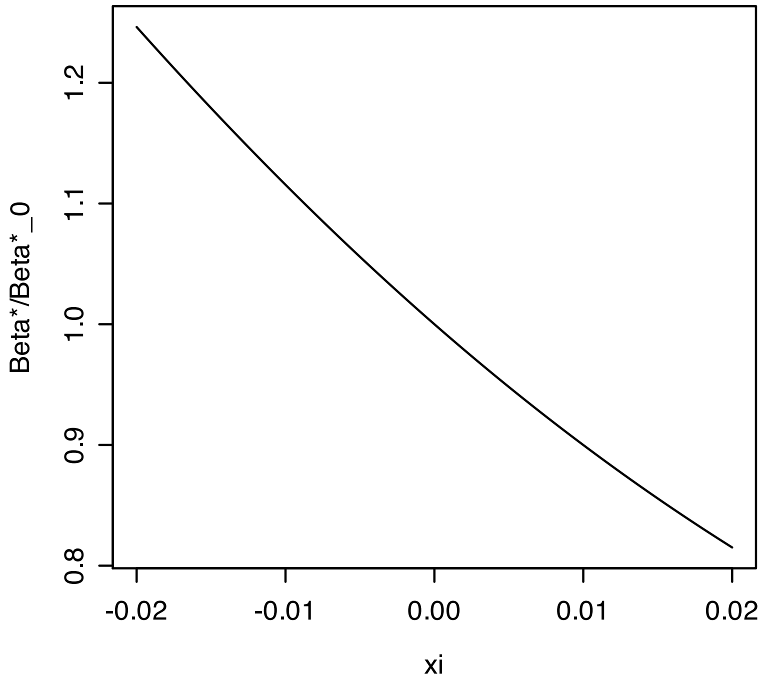

Figure 1: Ratio of β^*/β^*_0 *vs.* ξ.

where the sign before the radical is chosen to be the same as the sign of $\sin\mu_0$.

Thus, the value of the new amplitude function at the center of the IP is

$$\beta^* = \frac{\beta^*_0 \sin\mu_0 \left[1 + (2\pi\xi)^2\right]}{2\pi\xi\cos\mu_0 \pm \sqrt{\sin^2\mu_0 + (2\pi\xi)^2}}. \tag{1}$$

Written in terms of the relative change in beta, for small ξ,

$$\frac{\Delta\beta^*}{\beta^*_0} = \frac{\beta^*}{\beta^*_0} - 1 \approx -2\pi\xi\cot\mu_0. \tag{2}$$

A typical value for ξ in the Tevatron might be $\xi = 3(1.5\times10^{-18})(2.7\times10^{11})/(2\cdot 20\pi\times10^{-6}) \approx$ 0.01. Note: that for $\xi = 0.01$ and $\mu_0 = 2\pi\cdot 20.584$, Equation 1 yields $\beta^*/\beta^*_0 - 1$ = -10.0%, while Equation 2 gives $\Delta\beta^*/\beta^*_0$ = -10.8%. The beam-beam interaction enhances the collision rate with our choice of tune. In Figure 1 is a plot of Equation 1 for a relevant range of interest for the Tevatron conditions and for the tune value given above. If the tune were just below the half-integer, or if the beam-beam interaction were defocusing rather than focusing, then the effect would be to increase, rather than decrease, the amplitude function over this range.

Next, we consider the condition where there are two interaction regions, separated by an integer number of betatron wavelengths in the absence of beam-beam interactions. For the Tevatron, the two interaction points have identical values of β_0^* and by design are separated in phase by $\Delta\psi = 7 \cdot (2\pi)$. In the absence of beam-beam the single turn matrix corresponding to either IP is numerically identical to the matrix M_0 from our earlier discussion. Let M_n be the transport matrix between the two detectors (*i.e.*, from CDF to D0) and $M_{\nu-n}$ for the remainder of the ring. Then the one-turn transport matrix corresponding to the first IP, including beam-beam, would be

$$M_1 = M_{q_1/2}\, M_{\nu-n}\, M_{q_2}\, M_n\, M_{q_1/2}$$

where M_{q_i} represents the thin lens kick due to a beam-beam interaction at the i-th IP. Likewise, the matrix for the second IP would be

$$M_2 = M_{q_2/2}\, M_n\, M_{q_1}\, M_{\nu-n}\, M_{q_2/2}.$$

Note that the matrix $M_n = I$, the identity matrix, since $\Delta\psi/2\pi = integer$. Additionally,

$$M_{\nu-n} = \begin{pmatrix} \cos 2\pi(\nu-n) & \beta_0^* \sin 2\pi(\nu-n) \\ -\frac{1}{\beta_0^*}\sin 2\pi(\nu-n) & \cos 2\pi(\nu-n) \end{pmatrix} = \begin{pmatrix} \cos\mu_0 & \beta_0^*\sin\mu_0 \\ -\frac{1}{\beta_0^*}\sin\mu_0 & \cos\mu_0 \end{pmatrix},$$

i.e., numerically identical to M_0. Following through with the matrix multiplications yields

$$M_1 = \begin{pmatrix} a - b(q_2 + q_1/2) & b \\ c - aq_1/2 - (d - bq_1/2)(q_2 + q_1/2) & d - bq_1/2 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} a - bq_2/2 & b \\ c - a(q_1 + q_2/2) - (q_2/2)(d - b(q_1 + q_2/2) & d - b(q_1 + q_2/2) \end{pmatrix}.$$

The *1-2* elements show that $\beta_2^* = \beta_1^* \equiv \beta^* = \beta_0^*(\sin\mu_0/\sin\mu)$. Additionally, since $\beta_1^* q_1 = 4\pi\xi$ and $\beta_2^* q_2 = 4\pi\xi$, then $q_1 + q_2 = 2q = 8\pi\xi$ and we find, similar to our earlier result, that

$$\beta^* = \frac{\beta_0^* \sin\mu_0 \left[1 + (4\pi\xi)^2\right]}{4\pi\xi\cos\mu_0 \pm \sqrt{\sin^2\mu_0 + (4\pi\xi)^2}}. \tag{3}$$

One major difference here, however, is that while α^* was zero previously, this is no longer the case. We see from M_1 that $2\alpha_1^*\sin\mu = a - d - bq_2 = -bq_2 = -\beta^*\sin\mu\; q_2 = -4\pi\xi\sin\mu$, or

$$\alpha_1^* = -2\pi\xi$$

and similarly, for the second IP,

$$\alpha_2^* = +2\pi\xi.$$

In each case, the change in α through the thin lens interaction must be $\Delta\alpha = q\beta = 4\pi\xi$. Thus, at the first IP the incoming value of α is $-4\pi\xi$ and the outgoing value is zero. At the second IP the incoming value of α is zero and the outgoing value is $4\pi\xi$. (Remember that β' = slope of $\beta = -2\alpha$.) So the effect of the two beam-beam interactions is to (approximately) double the change in β^* at each IP, and to produce a "kink" in the amplitude function across the interaction region, as depicted in Figure 2.

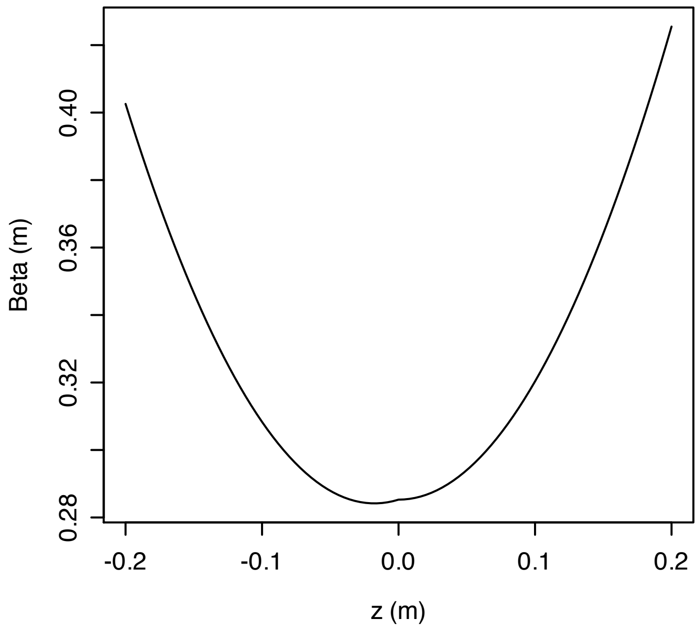

Figure 2: Amplitude Function distortion at one IP due to head-on "thin lens" collisions at two (in-phase) IPs. In the absence of beam-beam the minimum value would be $\beta_0^* = 35$ cm.

The lensing effect of the two beam-beam interactions produces a distortion of the amplitude function around the ring. The amplitude of this distortion is given by[1]

$$\left|\frac{\Delta\beta}{\beta}\right| = \frac{1}{2}|\det \Delta J| + \sqrt{|\det \Delta J| + \left(\frac{\det \Delta J}{2}\right)^2}$$

$\approx \sqrt{|\det \Delta J|}$ for small mismatches. Here,

$$\Delta J = \begin{pmatrix} \Delta\alpha & \Delta\beta \\ -\Delta\gamma & -\Delta\alpha \end{pmatrix}.$$

The two mismatch conditions tend to add coherently around the ring, though some cancellation occurs between the IP's. For $\xi = 0.01$ the amplitude of the mismatch would be ∼23% between the IP's and ∼28% in the "long arc."

While the value of the amplitude function is reduced at the IP, the luminosity of the collider does not go up proportionally, due to the finite bunch length. The integral normally performed to calculate the "hour glass factor," for round colliding beams with long Gaussian shaped bunches, namely,

$$\frac{1}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty}\frac{e^{-z^2/\sigma_z^2}dz}{1+(z/\beta_0^*)^2}$$

now becomes

$$\frac{1}{\sqrt{\pi}\sigma_z}\int_{-\infty}^{\infty}\frac{e^{-z^2/\sigma_z^2}dz}{f(z)}$$

where

$$f(z) = \begin{cases} b(\xi)\left[1 + \left(\frac{z}{b(\xi)\beta_0^*}\right)^2\right] & \text{for} \quad z \leq 0 \\ b(\xi)\left[1 - \Delta \cdot \left(\frac{z}{b(\xi)\beta_0^*}\right) + (1+\Delta^2)\left(\frac{z}{b(\xi)\beta_0^*}\right)^2\right] & \text{for} \quad z > 0 \end{cases},$$

$\Delta \equiv 4\pi\xi$, and $b(\xi) \equiv \beta^*(\xi)/\beta_0^*$ as given by Equation 3.

Figure 3 shows the Hour Glass integral as a function of bunch length for the cases where $\xi = 0$ and for $\xi = 0.01$, with $\beta_0^* = 35$ cm. While the beam-beam lowers the value of β^* by over 20%, the

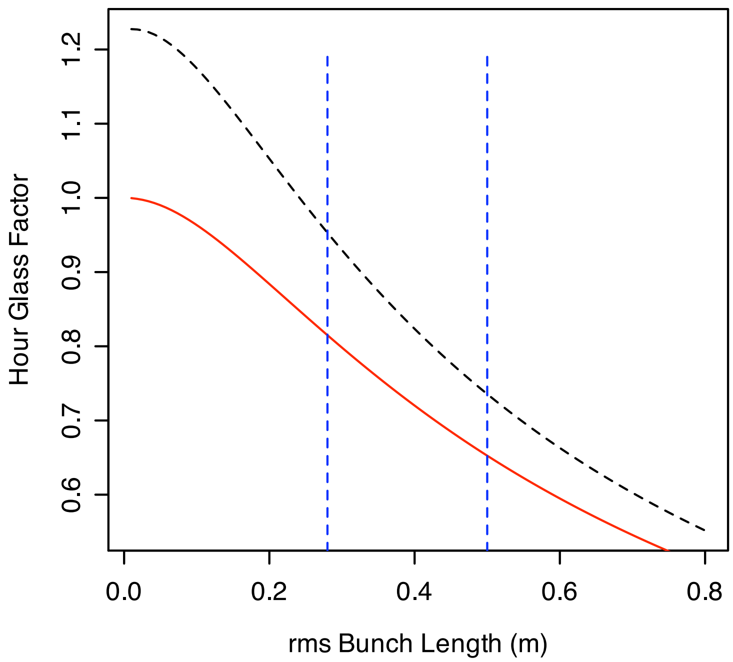

Figure 3: Hour Glass integral *vs.* bunch length. The solid curve is for $\xi = 0$ while the dashed curve is for $\xi = 0.01$ in the Tevatron with two beam-beam head-on interactions, and with $\beta_0^* = 35$ cm. The two vertical lines correspond to bunch lengths of 50 cm and 28 cm.

increase in luminosity would only be about 12% due to the rapid increase of β over the extent of the luminous region.

In applying the above to the Tevatron one must remember that (a) the interaction is clearly not a thin lens kick, and (b) the distortion of the proton optics due to the less intense antiproton bunches is smaller than the distortion of the antiproton optics due to the protons. However, it should be expected that the lattice distortions about the ring due to the two head-on interactions will be significant, and there should be a certain degree of asymmetry across a luminous region due to the phasing of the two interaction points. Also, in practice, the phase advance between the interaction regions is re-tuned to provide proper phases at beam collimators, and thus is no longer an integral number of wavelengths. However, the beta-wave in the long arc will still be of the same approximate magnitude calculated above.

Another point to make is that the hour glass factor masks many benefits which can be reaped from lowering the amplitude function, either through beam-beam or by optical manipulations. Shortening the bunch length would clearly provide an improvement in luminosity. For instance, the present day goal is to obtain a value of β_0^* of 28 cm. This would naively increase the luminosity by 25%, but due to the hour glass effect only about 12% would be realized. The beam-beam pinch brings this potential increase to 14%. However, shortening the bunch length from today's value of 50 cm to 28 cm would further increase the luminosity by another 35%. This would require, for instance, a new RF system in the Tevatron with frequency 212 MHz (4 $\times$ present) and a total RF voltage of 2.5 MV (2.5 times present). A similar approach has been in use at other proton rings, such as HERA and RHIC.

References

[1] M. J. Syphers, D. A. Edwards, T. Sen, Amplitude Function Mismatch, Proc. 1993 Part. Accel. Conf., IEEE 93CH3279-7, 134-136 (1993).