General Solutions for a Transverse to Longitudinal Emittance Exchange Beamline

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Abstract

The goal of this paper is to expand upon the development of an emittance echange beamline as developed in Cornacchia and Emma [1]. In particular, to clarify what types of beamlines are needed before and after a deflecting mode cavity to effect a perfect emittance exchange. General properties of the transfer matricies of these beamlines are listed and some specific cases are shown. The hope is that this will aid those designing such beamlines.

1 Introduction

A scheme to exchange the longitudinal emittance with one of the transverse emittances was proposed by Cornacchia and Emma in 2002 [1]. In this paper the authors show many of the properties of a beamline to generate such an exchange. Furthermore, a beamline is proposed based on a chicane with a deflecting mode cavity in the dispersive region. Such a beamline does not exchange the emittances without coupling between the two planes. A subsequent paper by Emma, *et. al.* shows the design of a beamline based on two identical doglegs with a deflecting mode cavity after the first dogleg [2]. This beamline will exchange the emittances without residual coupling.

It has been proposed to perform such an experiment at the A0 photinjector prior to it moving to NML. This will be the topic of Tim Koeth's thesis. In the design of a beamline for the experiment, it was decided that a standard chicane will not be used and some design will be chosen to completely exchange the emittances without coupling. The purpose of this paper is to expand upon the derivation of beamline properties in Ref. [1] and derive what are the the necessary properties of the subsections of the beamline to effect such an exchange.

2 General Properties

The matrix of an emittance exchange beamline takes the form of

$$M = M_{ac} M_{cav} M_{bc} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{1}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are 2×2 blocks of the matrix, M_{cav} is the cavity matrix, and M_{bc} and M_{ac} symplectic matricies for the before cavity and after cavity sections ofthe beamline respectively. It is assumed that there is no RF in either the before or after cavity sections of the emittance exchange beamline. The requirement of symplecticity means that M_{bc} and M_{ac} have the form of

$$M_{bc} = \begin{pmatrix} a & b & 0 & \eta \\ c & d & 0 & \eta' \\ c\eta - a\eta' & d\eta - b\eta' & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$$M_{ac} = \begin{pmatrix} e & f & 0 & D \\ g & h & 0 & D' \\ gD - eD' & hD - fD' & 1 & \chi \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

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The cavity matrix takes the form

$$M_{cav} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

as given in reference [1]. For completeness, we note that the phase space variables are x, x', z, δ which are the transverse position and angle, the distance from the bunch center ($z > 0$ refers to the front of the bunch), and fractional momentum offset.

Using these matricies the matrix for the total emittance exchange is

$$M = \begin{pmatrix} cf(1+k\eta) + ae + ak(D - f\eta') & df(1+k\eta) + be + bk(D - f\eta') & fk & D + e\eta + f\eta' + k(D\eta + f\xi) \\ ch(1+k\eta) + ag + ak(D' - h\eta') & dh(1+k\eta) + bg + bk(D' - h\eta') & hk & D' + g\eta + h\eta' + k(D'\eta + h\xi) \\ \left[\begin{array}{c} c(hD - fD') + a(gD - eD' + k\chi) \\ +(c\eta - a\eta')\left[1 + k(hD - fD')\right] \end{array}\right] & \left[\begin{array}{c} d(hD - fD') + b(gD - eD' + k\chi) \\ +(d\eta - b\eta')\left[1 + k(hD - fD')\right] \end{array}\right] & 1 + k(hD - fD') & \left[\begin{array}{c} D(g\eta + h\eta') + D'(e\eta + f\eta') \\ +k\xi(hD - fD') + \xi + \chi(1 + k\eta) \end{array}\right] \\ ak & bk & 0 & 1 + k\eta \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{5}$$

This matrix automatically fulfills all of the relations outlined in Appendix A of Reference [1] since it is a symplectic matrix. According to Equations 28 of Reference [1] read

$$\epsilon_x^2 = |\mathbf{A}|^2 \epsilon_{x0}^2 + (1 - |\mathbf{A}|)^2 \epsilon_{z0}^2 + \lambda^2 \epsilon_{x0} \epsilon_{z0} \tag{6}$$

$$\epsilon_z^2 = (1 - |\mathbf{A}|)^2 \epsilon_{x0}^2 + |\mathbf{A}|^2 \epsilon_{z0}^2 + \lambda^2 \epsilon_{x0} \epsilon_{z0} \tag{7}$$

$$\tag{8}$$

where $\epsilon_{x,z}$ are the transverse and longitudinal emittances after the exchange, and $\epsilon_{x0,z0}$ are the emittances prior. These equations state that $\det\mathbf{A} = |\mathbf{A}| = 0$ is required for emittance exchange. The $\lambda^2\epsilon_{x0}\epsilon_{z0}$ term is not explicitly written out, but in the text below Equations 28 the authors state that "$\lambda^2 = 0$ if and only if all $\mathbf{A}_{ij} = 0$ or the trivial case of no coupling at all, where all $\mathbf{B}_{ij} = \mathbf{C}_{ij} = 0$." It is clear from Equation 5 that in the case where the cavity is off, $k = 0$, there is no coupling of the emittances. However, it is not clear that if $\mathbf{A}_{ij} = 0$ then $\lambda^2 = 0$.

From Equations 19 of Reference [1]

$$\lambda^2\epsilon_{x0}\epsilon_{z0} = \mathrm{trace}\left\{(\mathbf{A}\sigma_{\mathbf{x}}\mathbf{A}^T)^a\mathbf{B}\sigma_{\mathbf{z}}\mathbf{B}^T\right\} = \mathrm{trace}\left\{(\mathbf{C}\sigma_{\mathbf{x}}\mathbf{C}^T)^a\mathbf{D}\sigma_{\mathbf{z}}\mathbf{D}^T\right\} \tag{9}$$

where $\sigma_{\mathbf{x},\mathbf{z}}$ is the sigma matrix for the transverse and longitudinal phase space prior to the exchange , $\mathbf{X}^T$ denotes the matrix transpose and $\mathbf{X}^a$ denote the adjoint matrix. When written in terms of the $\mathbf{A}$ and $\mathbf{B}$ blocks of the M matrix this term becomes

$$\begin{aligned}
\lambda^2\epsilon_{x0}\epsilon_{z0} \;=\; & (\mathbf{A}_{21}\mathbf{B}_{11} - \mathbf{A}_{11}\mathbf{B}_{21})^2\sigma_x^2\sigma_z^2 + (\mathbf{A}_{21}\mathbf{B}_{12} - \mathbf{A}_{11}\mathbf{B}_{22})^2\sigma_x^2\sigma_{z'}^2 \\
& +(\mathbf{A}_{22}\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21})^2\sigma_{x'}^2\sigma_z^2 + (\mathbf{A}_{22}\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22})^2\sigma_{x'}^2\sigma_{z'}^2 \\
& +2(\mathbf{A}_{21}\mathbf{B}_{11} - \mathbf{A}_{11}\mathbf{B}_{21})(\mathbf{A}_{21}\mathbf{B}_{12} - \mathbf{A}_{11}\mathbf{B}_{22})\sigma_x^2\sigma_{zz'} \\
& +2(\mathbf{A}_{22}\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21})(\mathbf{A}_{22}\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22})\sigma_{x'}^2\sigma_{zz'} \\
& +2(\mathbf{A}_{21}\mathbf{B}_{11} - \mathbf{A}_{11}\mathbf{B}_{21})(\mathbf{A}_{22}\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21})\sigma_{xx'}\sigma_z^2 \\
& +2(\mathbf{A}_{21}\mathbf{B}_{12} - \mathbf{A}_{11}\mathbf{B}_{22})(\mathbf{A}_{22}\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22})\sigma_{xx'}\sigma_{z'}^2 \\
& +4\left[(\mathbf{A}_{21}\mathbf{B}_{12} - \mathbf{A}_{11}\mathbf{B}_{22})(\mathbf{A}_{22}\mathbf{B}_{11} - \mathbf{A}_{12}\mathbf{B}_{21})\right. \\
& \left.(\mathbf{A}_{21}\mathbf{B}_{11} - \mathbf{A}_{11}\mathbf{B}_{21})(\mathbf{A}_{22}\mathbf{B}_{12} - \mathbf{A}_{12}\mathbf{B}_{22})\right]\sigma_{xx'}\sigma_{zz'}
\end{aligned} \tag{10}$$

where it is clear that if all of $\mathbf{A}_{ij} = 0$ then $\lambda^2 = 0$. One can surmise from the form that if the above equation were written in terms of the $\mathbf{C}$ and $\mathbf{D}$ blocks of M then this term would be zero if $\mathbf{D}_{ij} = 0$. We will note at this point that the chicane designed in Reference [1] has only the $\sigma_{x'}^2\sigma_{z'}^2$ term.

$$\lambda^2\epsilon_{x0}\epsilon_{z0} = 4\eta^2\sigma_{x'}^2\sigma_{z'}^2 \tag{11}$$

3 SubBeamline Properties

Now that some of the general properties of the emittance exchange matrix have been fleshed out and the assumptions stated we want to answer the following question,"What is needed of the elements of M_{bc}, M_{cav}, M_{ac}, to effect a perfect emittance exchange?"

There are four equations, one for each element of the $\mathbf{A}$ block. At first glance there are 12 unknowns (a, b, c, d, e, f, g, h, k, η, η', D, D') However, the first 4 are related by the symplectic condition. This is true for the second 4 elements as well. This leaves 10 free parameters. One can solve the system of equations $\mathbf{A}_{ij} = 0$, and arrive at the following relations between the free parameters:

$$k = -\frac{1}{\eta} \tag{12a}$$

$$D = e\eta + f\eta' \tag{12b}$$

$$D' = g\eta + h\eta' \tag{12c}$$

Plugging these into the matrix for the emittance exchange, one gets

$$M = \begin{pmatrix} 0 & 0 & -\frac{f}{\eta} & e\eta + f\eta' - f\frac{\xi}{\eta} \\ 0 & 0 & -\frac{h}{\eta} & g\eta + h\eta' - h\frac{\xi}{\eta} \\ c\eta - a\eta' - a\frac{\chi}{\eta} & d\eta - b\eta' - b\frac{\chi}{\eta} & 0 & 0 \\ -\frac{a}{\eta} & -\frac{b}{\eta} & 0 & 0 \end{pmatrix}. \tag{13}$$

and the equation for the after cavity matrix becomes

$$M_{ac} = \begin{pmatrix} \frac{D - f\eta'}{\eta} & f & 0 & D \\ \frac{D'}{\eta} + \frac{\eta'}{D} + f\frac{D'\eta'}{D\eta} & \frac{fD'+\eta}{D} & 0 & D' \\ -\eta' & \eta & 1 & \chi \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

Now we will note the following interesting points of the solution. Firstly, it is general involving only the properties of the cavity, symplectic transport lines without RF, and perfect emittance exchange. The second is that the solution of the cavity strength is only determined by the dispersion in the cavity. It is not obvious that it should not depend on other things, such as the slope of the dispersion through the cavity. Thirdly, the requirements of the after cavity transport only depend on the dispersion and its slope in the cavity. It does not depend on the other elements of the transport matrix prior to the cavity. Fourth there are no requirements for the elements of M_{bc} other than it generate the required η, η'.

4 Consequences of the solution

At first glance this result looks like it may be nothing more than an existance proof, i.e. solutions exist. However, the form of the matrix for after the beamline places some restrictions on the types of beamlines to be used. For example, the chicane type solution is particularly attractive for its simplicity. However, for any beamline that will return the incoming dispersion and its slope to zero with the cavity off, like the chicane, perfect emittance exchange cannot occur. To zero the dispersion and slope without the cavity, assuming no dispersion prior to the beamline, the following conditions must be met:

$$\begin{aligned} D &= -(e\eta + f\eta') \\ D' &= -(g\eta + h\eta') \end{aligned}$$

These differ from Equations 12 by a negative sign, so the only solution is if $\eta = \eta' = 0$. This also requires an infinite strength deflecting mode cavity.

Most of the designs with which we are familar are based on a dogleg which generates dispersion with zero slope prior to the cavity.

$$M_{bc} = \begin{pmatrix} 1 & L & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Pursuing this design leads to three types of solutions. The first is a beamline after the cavity which generates further dispersion without slope.

$$M_{ac} = \begin{pmatrix} \frac{D}{\eta} & f & 0 & D \\ 0 & \frac{\eta}{D} & 0 & 0 \\ 0 & \eta & 1 & \chi \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Setting $D = \eta$ and $f = L$ gives the solution that is proposed in Reference [2]. That is, a double dogleg with a deflecting cavity between the doglegs.

Another solution is to have a beamline which generates zero dispersion but with a slope.

$$M_{ac} = \begin{pmatrix} 0 & \frac{-\eta}{D'} & 0 & 0 \\ \frac{D'}{\eta} & h & 0 & D' \\ 0 & \eta & 1 & \chi \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

A beamline composed of a drift-quad-drift-dipole can satisfy the requirements of this matrix if the focal length of the quad is equal to the length of the following drift. This solution is similar to the design that Helen has arrived at [3].

Other beamline designs are also possible which are a composition of the two beamlines where neither the dispersion or its slope are zeroed as some point.

5 Conclusion

The design requirements of a transverse to longitudinal emittance exchange beamline have been expanded from the initial description given in Reference [1]. Using only the assumptions of symplecticity and no accelerating RF we have been able to show the properties needed for the beamlines before and after the deflecting mode cavity. The beamline prior to the cavity only has to generate the desired dispersion at the cavity. The cavity strength has to be matched to the generated dispersion. The beamline after the cavity needs to satisfy Equations 12 to effect a perfect emittance exchange.

If the beamline prior to the cavity generates dispersion with no slope, three types of solutions appear for after the cavity. The first is a solution which generates further dispersion with no slope, such as the dogleg proposed by Reference [2]. The second causes the dispersion to go through zero at some point, such as the solution proposed by Reference [3]. The last is an admixture of the two.

A final consequence is that standard chicane type solutions will not produce a perfect emittance exchange.

This paper does not take into account second order type effects. These need to be investigated for any proposed solution for the A0 experiment. However, it simplifies the process of making the first pass in desiging a beamline for A0.

References

[1] M. Cornacchia and P.Emma. Phys. Rev. ST Accel. Beams 5, 084001 (2002).

[2] P. Emma, Z. Huang, K.-J. Kim,and P. Piot. Phys. Rev. ST Accel. Beams 9, 100702 (2006).

[3] Helen Edwards. Private Communication.