Single Particle Trajectories -- Analytical Methods

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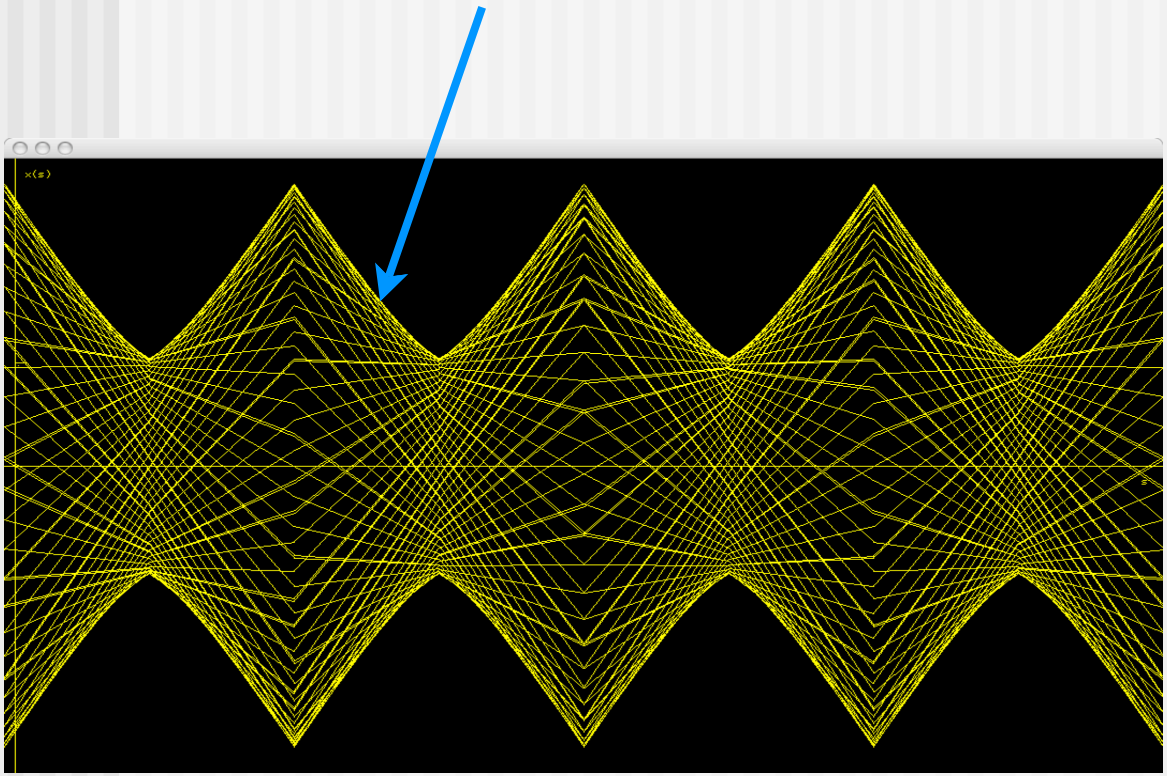
Pushing the “Envelope”

Envelope described by an
“amplitude function”



- We saw, for a FODO system, that the motion of a single particle is contained within an “envelope”
- Wish to determine its functional form, and the rate at which the phase of the oscillatory motion develops
- Decouple motion of individual particle from intrinsic properties of the accelerator design

Hill's Equation -- Analytical Solution

- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

-
- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, *s*.
- So, assume solution is sinusoidal, with a phase which advances as a function of location *s*; also assume amplitude is modulated by a function which also depends upon *s*:

$$\boxed{x(s) = A\sqrt{\beta(s)}\,\sin[\psi(s) + \delta]}$$

- Plug into Hill's Equation...

Analytical Solution (cont'd)

$$
\begin{aligned}
x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\
x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\
x'' &= \ldots
\end{aligned}
$$

Plug into Hill's Equation, and collect terms...

$$
\begin{aligned}
x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\
&\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0
\end{aligned}
$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously.

Analytical Solution (cont'd)

- Thus, we must have ...

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

and

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

Differential equation that the amplitude function must obey

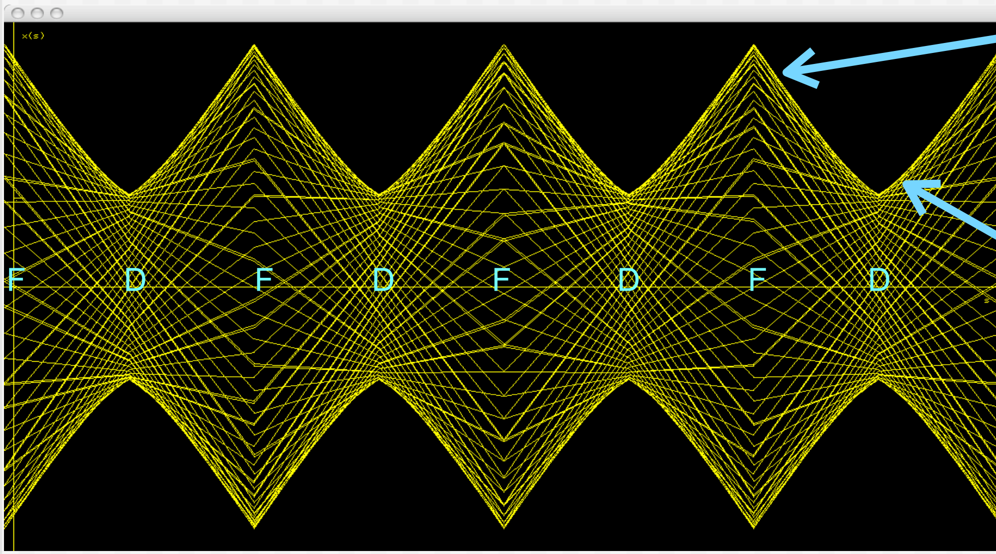
The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of $const$, then β would just be scale accordingly; so, we can choose $const$ = 1.

Some Comments

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But it also is a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., $K(s)$) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

The Amplitude Function, β



Higher β --
smaller phase advance
larger beam size

Lower β --
greater phase advance
smaller beam size

- Since the amplitude function is a wavelength, it will have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant A must have units of $m^{1/2}$, and it must be numerically small. More on this subject next time...

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.
(in regions where K is either zero or constant)

Piecewise Solutions

- $K = 0$:

$$\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

Parabola!

  - since $\beta > 0$, then from original diff. eq.: $2\beta\beta'' - (\beta')^2 = 4$
  - Therefore, parabola is always concave up $\beta'' > 0$

- $K > 0$, $K < 0$:

$$\beta(s) \sim \sin / \cos \quad \text{or} \quad \sinh / \cosh \ + \ const$$

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,
-

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}$$

- Collectively, β, α, γ are called the Courant-Snyder Parameters (sometimes called “Twiss” or “lattice” parameters)

The Transport Matrix

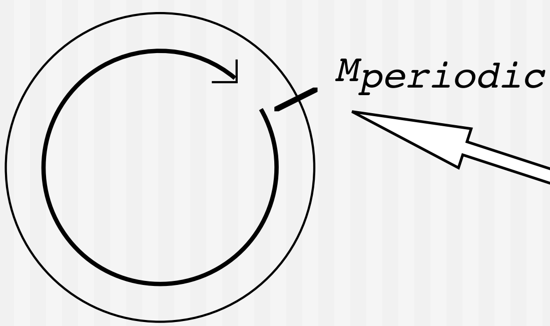
- We can write: $x(s) = a\sqrt{\beta}\sin\psi + b\sqrt{\beta}\cos\psi$
- Solve for a and b in terms of initial conditions and write in matrix form
  - we get:

$$\left(\begin{array}{c} x \\ x' \end{array}\right) = \left(\begin{array}{cc} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{array}\right) \left(\begin{array}{c} x_0 \\ x'_0 \end{array}\right)$$

Periodic Solutions

- Within a system made up of periodic sections it is natural to want the beam envelope to have the same periodicity.
- Taking the previous matrix to be that of a periodic section, and demanding the C-S parameters be periodic yields...

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$



values of β, α above correspond to one particular point in the accelerator

Periodicity and the "Tune"

- We see from above that matrix of a periodic section (which, for example, could be an entire synchrotron!) has a Trace which is

$$trace(M_{periodic}) = 2\cos\Delta\psi$$

- If the matrix *does* represent an entire synchrotron, then the total phase advance is just 2**π** x the tune:

$$\Delta\psi = 2\pi\nu = \oint \frac{ds}{\beta(s)}$$

Propagation of Courant-Snyder Parameters

- We note that can write periodic matrix corresponding to location *s* as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi
\end{aligned}
$$

-

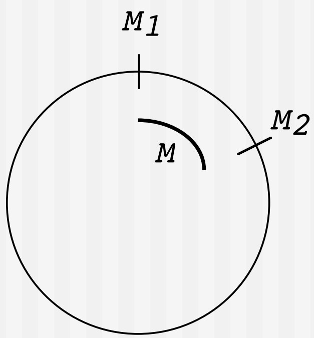
- $$= \quad I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}$$

-

- where $J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$ $\qquad detJ = 1, \quad trace(J) = 0; \quad J^2 = -I$

Tracking β, α, γ ...

- Let M_1 and M_2 be the "periodic" matrices at two points, and M propagates the motion between them. Then,



$$M_2 = M\ M_1\ M^{-1}$$

$(M_1, M_2 \text{ are "once around"})$

- Or, equivalently, $$\boxed{J_2 = M\ J_1\ M^{-1}}$$
  - So, if know parameters (*i.e.*, J) at one point, can find them at another point if given the matrix for motion in between

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan\Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase and C-S parameters along a beam line

Simple Examples

- Propagation through a Drift

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

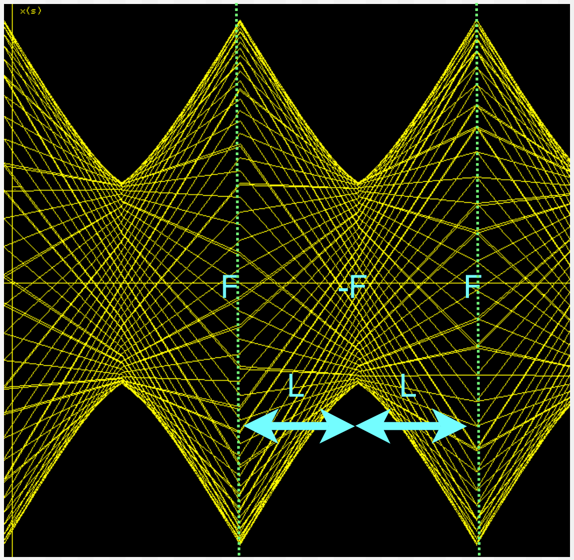
$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

Choice of Initial Conditions

-
- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the periodic solution for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- Will discuss optical "mismatches" and their implications in future talks

Computation of Courant-Snyder Parameters

- As an example, consider a FODO system



$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Thus, use above matrix to compute functions at exit of the F quad..

FODO Cell

- From the matrix:

$$M = \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Here, μ is phase advance through one periodic cell

$$traceM = a+d = 2-L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1+\sin\mu/2}{1-\sin\mu/2}} \qquad \alpha = \frac{a-d}{2\sin\mu} = \sqrt{\frac{1+\sin\mu/2}{1-\sin\mu/2}}$$

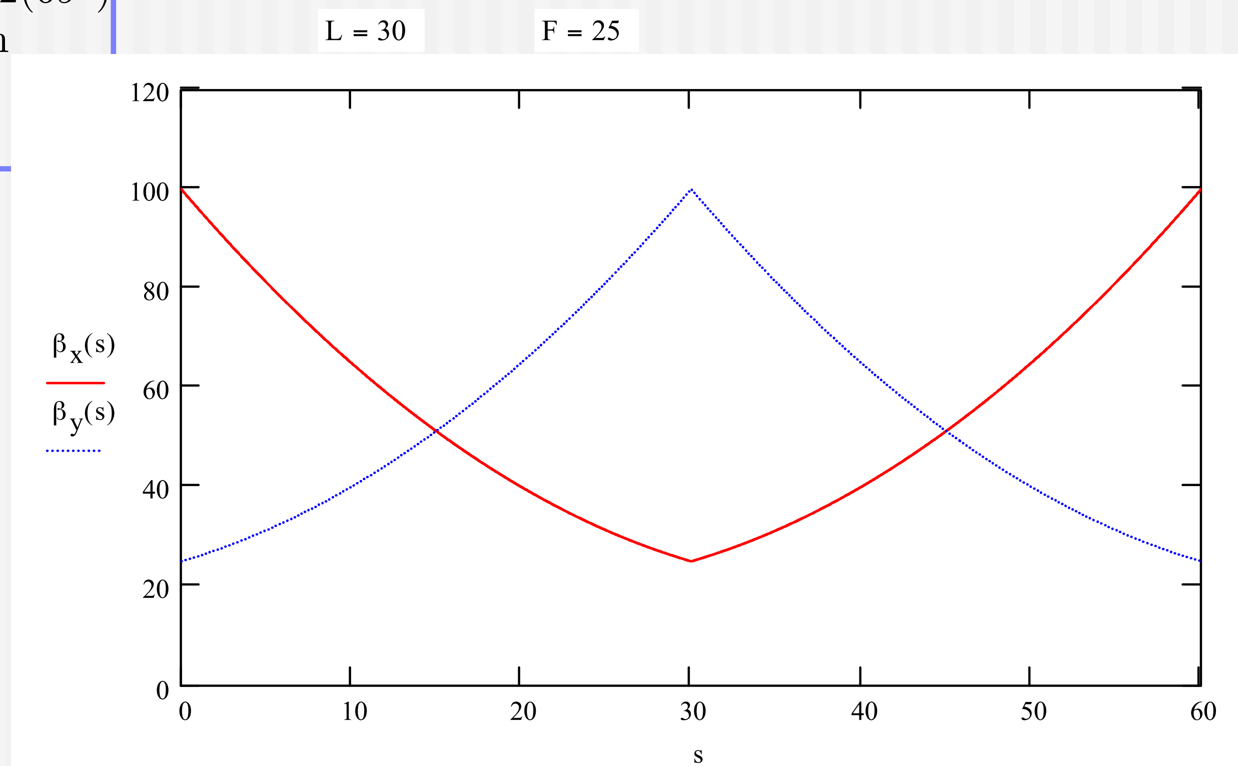
- If go from D quad to D quad, get
  - at exit:

$$\beta = 2F\sqrt{\frac{1-\sin\mu/2}{1+\sin\mu/2}}, \qquad \alpha = \sqrt{\frac{1-\sin\mu/2}{1+\sin\mu/2}}$$

Periodic FODO Cell Functions

- Tevatron Cell

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25\ \mathrm{m})\sqrt{1.6/0.4} = 100\ \mathrm{m}$$
$$\beta_{min} = 2(25\ \mathrm{m})\sqrt{0.4/1.6} = 25\ \mathrm{m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$



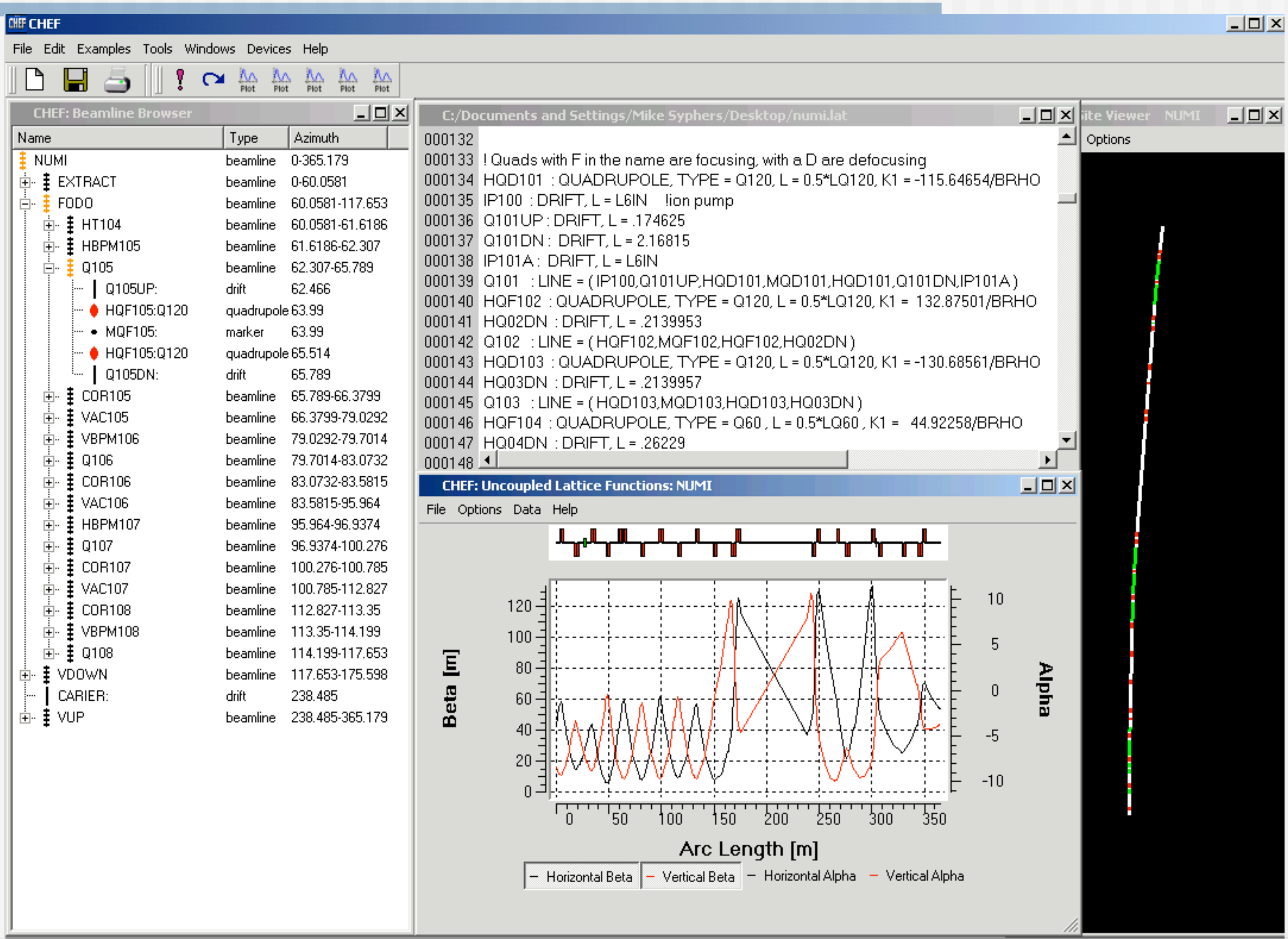
Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT
  - SYNCH
  - MAD
  - CHEF
  - many more ...

An Example -- NuMI Beam Line

- Using CHEF

(Michelotti, Ostiguy)



NuMI Beam Line using CHEF