

Single Particle Trajectories -- Analytical Methods

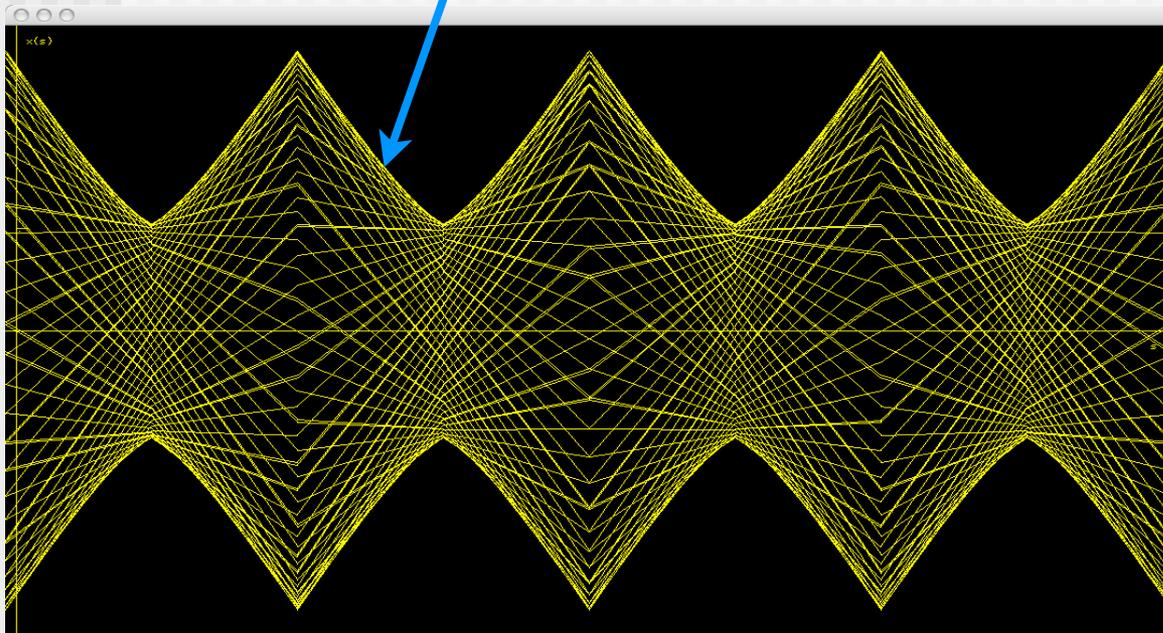
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Pushing the “Envelope”

Envelope described by an
“amplitude function”



- We saw, for a FODO system, that the motion of a single particle is contained within an “envelope”
-
- Wish to determine its functional form, and the rate at which the phase of the oscillatory motion develops
-
- Decouple motion of individual particle from intrinsic properties of the accelerator design

Hill's Equation -- Analytical Solution

- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: “similar” to simple harmonic oscillator equation, but “spring constant” is not *constant* -- depends upon longitudinal position, s .

- So, assume solution is sinusoidal, with a phase which advances as a function of location s ; also assume amplitude is modulated by a function which also depends upon s :

$$x(s) = A\sqrt{\beta(s)} \sin[\psi(s) + \delta]$$

- Plug into Hill's Equation...

Analytical Solution (cont'd)

$$x(s) = A\sqrt{\beta(s)} \sin[\psi(s) + \delta]$$

$$x' = \frac{1}{2}A\beta^{-\frac{1}{2}}\beta' \sin[\psi(s) + \delta] + A\sqrt{\beta} \cos[\psi(s) + \delta]\psi'$$

$$x'' = \dots$$

Plug into Hill's Equation, and collect terms...

$$\begin{aligned} x'' + K(s)x &= A\sqrt{\beta} \left[\psi'' + \frac{\beta'}{\beta}\psi' \right] \cos[\psi(s) + \delta] \\ &+ A\sqrt{\beta} \left[-\frac{1}{4} \frac{(\beta')^2}{\beta^2} + \frac{1}{2} \frac{\beta''}{\beta} - (\psi')^2 + K \right] \sin[\psi(s) + \delta] = 0 \end{aligned}$$

A and δ are constants of integration, defined by the initial conditions (x_0, x'_0) of the particle. For arbitrary A, δ , must have contents of each $[] = 0$ simultaneously.

Analytical Solution (cont'd)

- Thus, we must have ...

$$\psi'' + \frac{\beta'}{\beta} \psi' = 0$$

and

$$-\frac{1}{4} \frac{(\beta')^2}{\beta^2} + \frac{1}{2} \frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

$$\beta \psi'' + \beta' \psi' = 0$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$(\beta\psi')' = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

$$\beta\psi' = \text{const}$$

$$\psi' = 1/\beta$$

Note: the phase advance is an observable quantity. So, while could choose different value of *const*, then β would just be scale accordingly; so, we can choose *const* = 1.

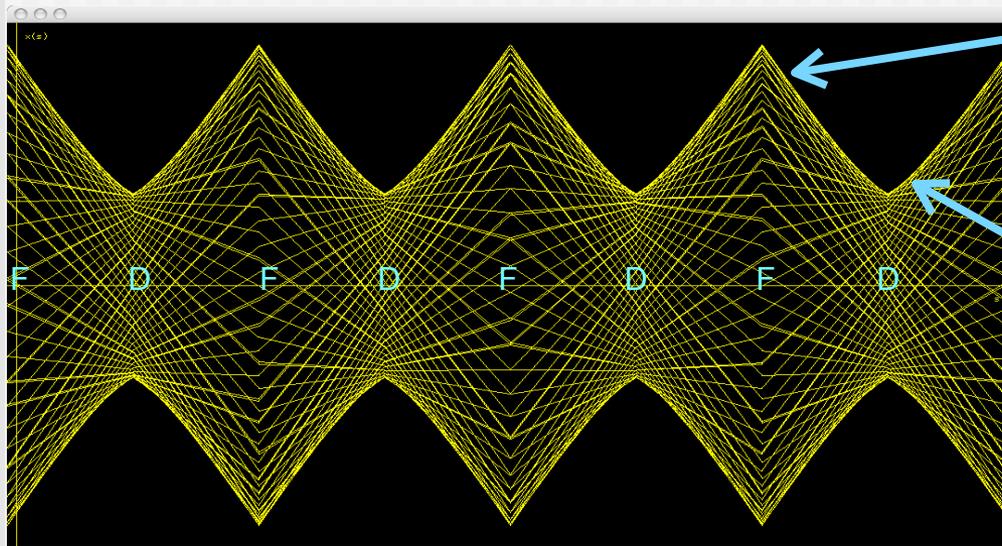
The function $\beta(s)$ is the local wavelength ($\lambda/2\pi$) of the oscillatory motion.

Differential equation that the amplitude function must obey

Some Comments

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But it also is a local wavelength of the motion.
- This seems strange at first, but ...
 - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
 - Thus, the spacing and/or strengths (i.e., $K(s)$) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

The Amplitude Function, β



Higher β --
smaller phase advance
larger beam size

Lower β --
greater phase advance
smaller beam size

- Since the amplitude function is a wavelength, it will have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant \bar{A} must have units of $m^{1/2}$, and it must be numerically small. More on this subject next time...

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\beta'' + 4K\beta = \text{const.}$$

is the general equation of motion for the amplitude function, β .
(in regions where K is either zero or constant)

Piecewise Solutions

■ $K = 0$:

$$\beta'' = \text{const} \longrightarrow \beta(s) = \beta_0 + \beta'_0 s + \frac{1}{2} \beta''_0 s^2 \quad \text{Parabola!}$$

- since $\beta > 0$, then from original diff. eq.: $2\beta\beta'' - (\beta')^2 = 4$
- Therefore, parabola is always concave up $\beta'' > 0$

■ $K > 0, K < 0$:

$$\beta(s) \sim \sin / \cos \quad \text{or} \quad \sinh / \cosh + \text{const}$$

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,
$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1 + \alpha^2}{\beta}$$
- Collectively, β, α, γ are called the Courant-Snyder Parameters (sometimes called “Twiss” or “lattice” parameters)

The Transport Matrix

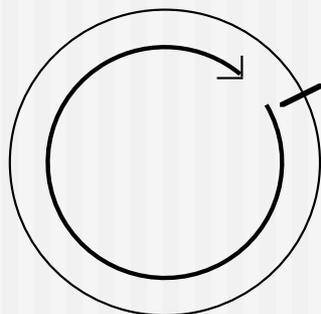
- We can write: $x(s) = a\sqrt{\beta} \sin \psi + b\sqrt{\beta} \cos \psi$
- Solve for a and b in terms of initial conditions and write in matrix form
 - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos \Delta\psi + \alpha_0 \sin \Delta\psi) & \sqrt{\beta_0\beta} \sin \Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}} \sin \Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}} \cos \Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2} (\cos \Delta\psi - \alpha \sin \Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Periodic Solutions

- Within a system made up of periodic sections it is natural to want the beam envelope to have the same periodicity.
- Taking the previous matrix to be that of a periodic section, and demanding the C-S parameters be periodic yields...

$$M_{periodic} = \begin{pmatrix} \cos \Delta\psi + \alpha \sin \Delta\psi & \beta \sin \Delta\psi \\ -\gamma \sin \Delta\psi & \cos \Delta\psi - \alpha \sin \Delta\psi \end{pmatrix}$$



$M_{periodic}$

values of β , α above correspond to one particular point in the accelerator

Periodicity and the “Tune”

- We see from above that matrix of a periodic section (which, for example, could be an entire synchrotron!) has a Trace which is

$$\text{trace}(M_{\text{periodic}}) = 2 \cos \Delta\psi$$

- If the matrix *does* represent an entire synchrotron, then the total phase advance is just 2π x the tune:

$$\Delta\psi = 2\pi\nu = \oint \frac{ds}{\beta(s)}$$

Propagation of Courant-Snyder Parameters

- We note that can write periodic matrix corresponding to location s as:

$$M_0 = \begin{pmatrix} \cos \Delta\psi + \alpha \sin \Delta\psi & \beta \sin \Delta\psi \\ -\gamma \sin \Delta\psi & \cos \Delta\psi - \alpha \sin \Delta\psi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin \Delta\psi$$

■

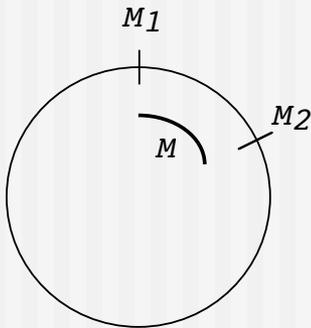
$$= I \cos \Delta\psi + J \sin \Delta\psi = e^{J\Delta\psi}$$

■

$$\text{■ where } J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \det J = 1, \quad \text{trace}(J) = 0; \quad J^2 = -I$$

Tracking β , α , γ ...

- Let M_1 and M_2 be the “periodic” matrices at two points, and M propagates the motion between them. Then,



$$M_2 = M M_1 M^{-1}$$

(M_1, M_2 are “once around”)

- Or, equivalently,

$$J_2 = M J_1 M^{-1}$$

- So, if know parameters (*i.e.*, J) at one point, can find them at another point if given the matrix for motion in between

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1 \rightarrow 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1 \rightarrow 2}$$

- So, from knowledge of matrices, can “transport” phase and C-S parameters along a beam line

Simple Examples

- Propagation through a Drift

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
$$\implies \Delta\psi = \tan^{-1} \left(\frac{L}{\beta_1 - L\alpha_1} \right)$$
$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$
$$\alpha = \alpha_0 - \gamma_0 L$$
$$\gamma = \gamma_0$$

- Propagation through a Thin Lens

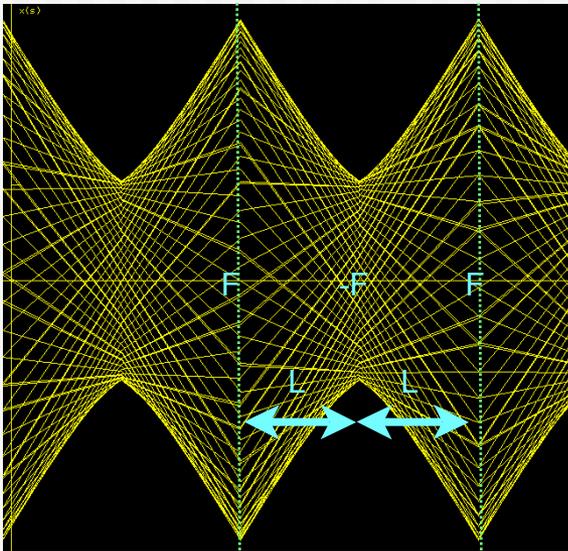
$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$
$$\implies \Delta\psi = 0$$
$$\beta = \beta_0$$
$$\alpha = \alpha_0 + \beta_0/F$$
$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

Choice of Initial Conditions

-
- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a “ring,” then natural to choose the periodic solution for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- Will discuss optical “mismatches” and their implications in future talks

Computation of Courant-Snyder Parameters

- As an example, consider a FODO system



$$\begin{aligned}
 M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
 &= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
 \end{aligned}$$

- Thus, use above matrix to compute functions at exit of the F quad..

FODO Cell

- From the matrix:

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Here, μ is phase advance through one periodic cell

$$\text{trace} M = a + d = 2 - L^2/F^2 = 2 \cos \mu \quad \rightarrow \quad \sin \frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin \mu} = 2F \sqrt{\frac{1 + \sin \mu/2}{1 - \sin \mu/2}} \quad \alpha = \frac{a - d}{2 \sin \mu} = \sqrt{\frac{1 + \sin \mu/2}{1 - \sin \mu/2}}$$

- If go from D quad to D quad, get

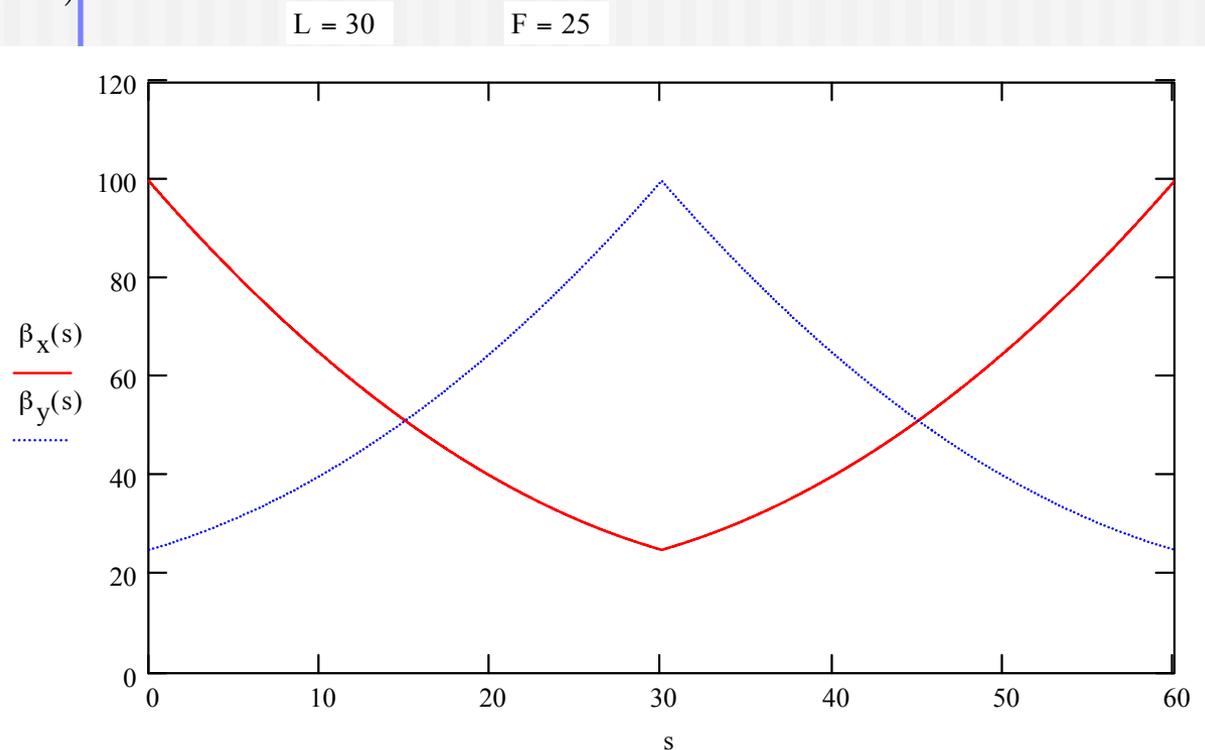
- at exit:

$$\beta = 2F \sqrt{\frac{1 - \sin \mu/2}{1 + \sin \mu/2}}, \quad \alpha = \sqrt{\frac{1 - \sin \mu/2}{1 + \sin \mu/2}}$$

Periodic FODO Cell Functions

■ Tevatron Cell

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25 \text{ m})\sqrt{1.6/0.4} = 100 \text{ m}$$
$$\beta_{min} = 2(25 \text{ m})\sqrt{0.4/1.6} = 25 \text{ m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$



Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
 - TRANSPORT
 - SYNCH
 - MAD
 - CHEF
 - many more ...

An Example -- NuMI Beam Line

Using CHEF

(Michelotti, Ostiguy)

The screenshot displays the CHEF (Coupled Hardware Emulation Framework) software interface. The main window is titled "CHEF: Beamline Browser" and shows a list of beamline components. The components are organized into a tree structure, with the NuMI beamline expanded to show its sub-components. The components are listed in a table with columns for Name, Type, and Azimuth.

Name	Type	Azimuth
NUMI	beamline	0-365.179
EXTRACT	beamline	0-60.0581
FODO	beamline	60.0581-117.653
HT104	beamline	60.0581-61.6186
HBPM105	beamline	61.6186-62.307
Q105	beamline	62.307-65.789
Q105UP:	drift	62.466
HQF105:Q120	quadrupole	63.99
MQF105:	marker	63.99
HQF105:Q120	quadrupole	65.514
Q105DN:	drift	65.789
COR105	beamline	65.789-66.3799
VAC105	beamline	66.3799-79.0292
VBPM106	beamline	79.0292-79.7014
Q106	beamline	79.7014-83.0732
COR106	beamline	83.0732-83.5815
VAC106	beamline	83.5815-95.964
HBPM107	beamline	95.964-96.9374
Q107	beamline	96.9374-100.276
COR107	beamline	100.276-100.785
VAC107	beamline	100.785-112.827
COR108	beamline	112.827-113.35
VBPM108	beamline	113.35-114.199
Q108	beamline	114.199-117.653
VDOWN	beamline	117.653-175.598
CARRIER:	drift	238.485
VUP	beamline	238.485-365.179

The right-hand window shows the "C:/Documents and Settings/Mike Syphers/Desktop/numi.lat" file, which contains the lattice configuration for the NuMI beam line. The configuration is a list of elements and their parameters, such as quadrupoles, drifts, and markers. The elements are listed in a table with columns for element name and parameters.

Element Name	Parameters
!Quads with F in the name are focusing, with a D are defocusing	
HQD101	: QUADRUPOLE, TYPE = Q120, L = 0.5*LQ120, K1 = -115.64654/BRHO
IP100	: DRIFT, L = L6IN lion pump
Q101UP	: DRIFT, L = .174625
Q101DN	: DRIFT, L = 2.16815
IP101A	: DRIFT, L = L6IN
Q101	: LINE = (IP100,Q101UP,HQD101,MQD101,HQD101,Q101DN,IP101A)
HQF102	: QUADRUPOLE, TYPE = Q120, L = 0.5*LQ120, K1 = 132.87501/BRHO
HQ02DN	: DRIFT, L = .2139953
Q102	: LINE = (HQF102,MQF102,HQF102,HQ02DN)
HQD103	: QUADRUPOLE, TYPE = Q120, L = 0.5*LQ120, K1 = -130.68561/BRHO
HQ03DN	: DRIFT, L = .2139957
Q103	: LINE = (HQD103,MQD103,HQD103,HQ03DN)
HQF104	: QUADRUPOLE, TYPE = Q60, L = 0.5*LQ60, K1 = 44.92258/BRHO
HQ04DN	: DRIFT, L = .26229

The bottom window shows the "CHEF: Uncoupled Lattice Functions: NUMI" plot. The plot displays the horizontal and vertical beta and alpha functions as a function of arc length. The x-axis is "Arc Length [m]" ranging from 0 to 350. The left y-axis is "Beta [m]" ranging from 0 to 120. The right y-axis is "Alpha" ranging from -10 to 10. The plot shows four curves: Horizontal Beta (black), Vertical Beta (red), Horizontal Alpha (black), and Vertical Alpha (red). The beta functions show oscillatory behavior, while the alpha functions show a more complex, non-oscillatory behavior.

NuMI Beam Line using CHEF

