Transverse Emittance and Effects of Optical Mismatches

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Overview

- Found analytical solution to Hill's Equation:

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- So far, discussed amplitude function, β
- What about *A*?
  - Given β(s), how big is the beam at a particular location? mm? cm? m?
  - If perturb the beam's trajectory, how much will it move downstream?
- Single particle behavior *vs*. a "beam"

Betatron Oscillation Amplitude

- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a *betatron*)
- Given

$$\begin{aligned} x &= a\sqrt{\beta}\sin\psi + b\sqrt{\beta}\cos\psi \\ x' &= \frac{1}{\sqrt{\beta}}\left([b - a\alpha]\cos\psi - [a + b\alpha]\sin\psi\right) \\ &\downarrow \\ a &= \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}} \end{aligned}$$

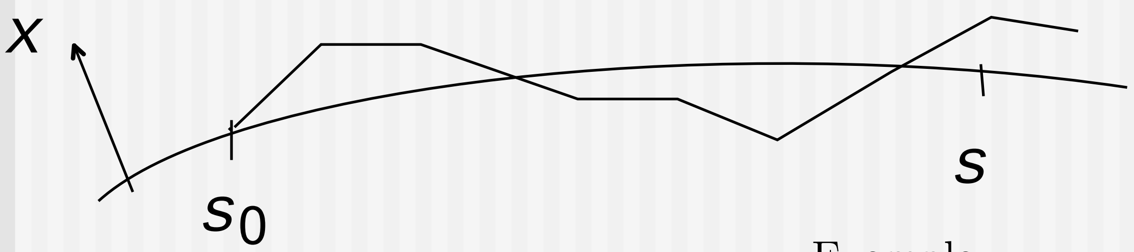
$$\boxed{\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi\right]}$$

$$\text{amplitude: } A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}$$

Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = x'_0 = \Delta\theta$
- Then, downstream, we have

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\,\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.1$ mrad, $\beta_0 = 49$ m, $\beta(s) = 64$ m, and $\Delta\psi = n \times 2\pi + 30^\circ$. Then $x(s) = 2.8$ mm.

Courant-Snyder Invariant

- In general,

$$\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi}
\end{aligned}$$

$$\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

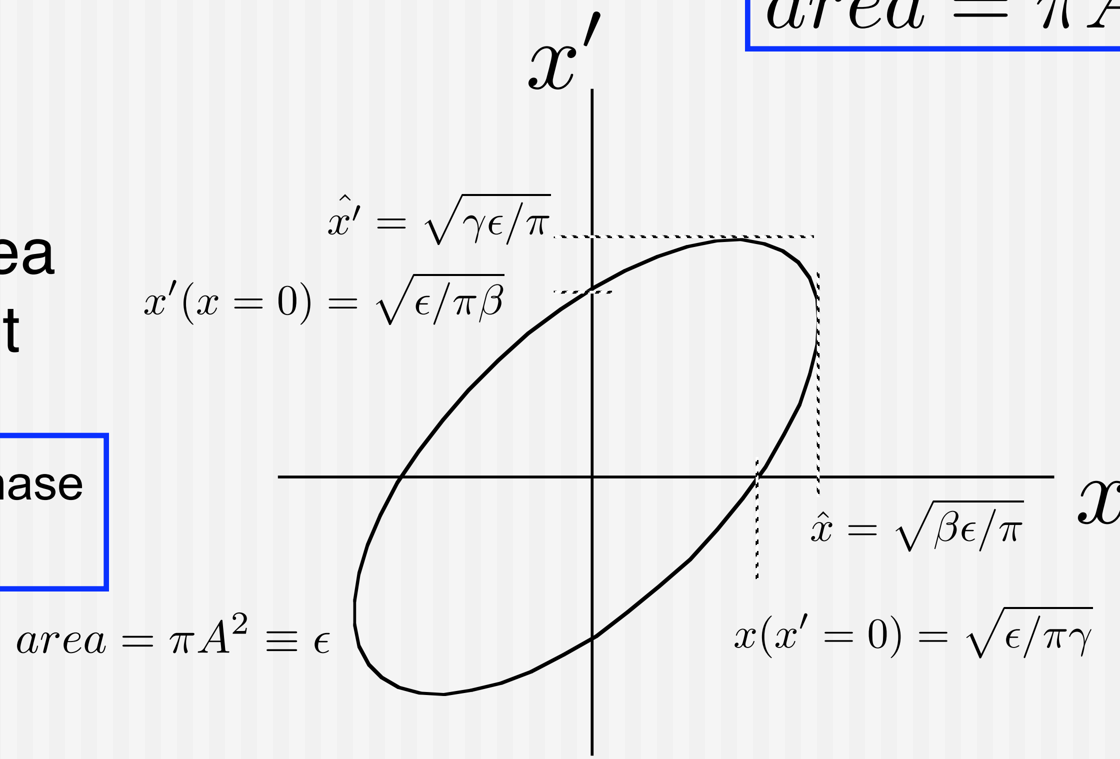
While C-S parameters evolve along the beam line, the combination above remains constant.

Properties of the Phase Space Ellipse

- The eqn. for the C-S invariant is that of an ellipse. If compute the area of the ellipse, find that

$$\boxed{area = \pi A^2}$$

- i.e., while the ellipse changes shape along the beam line, its area remains constant

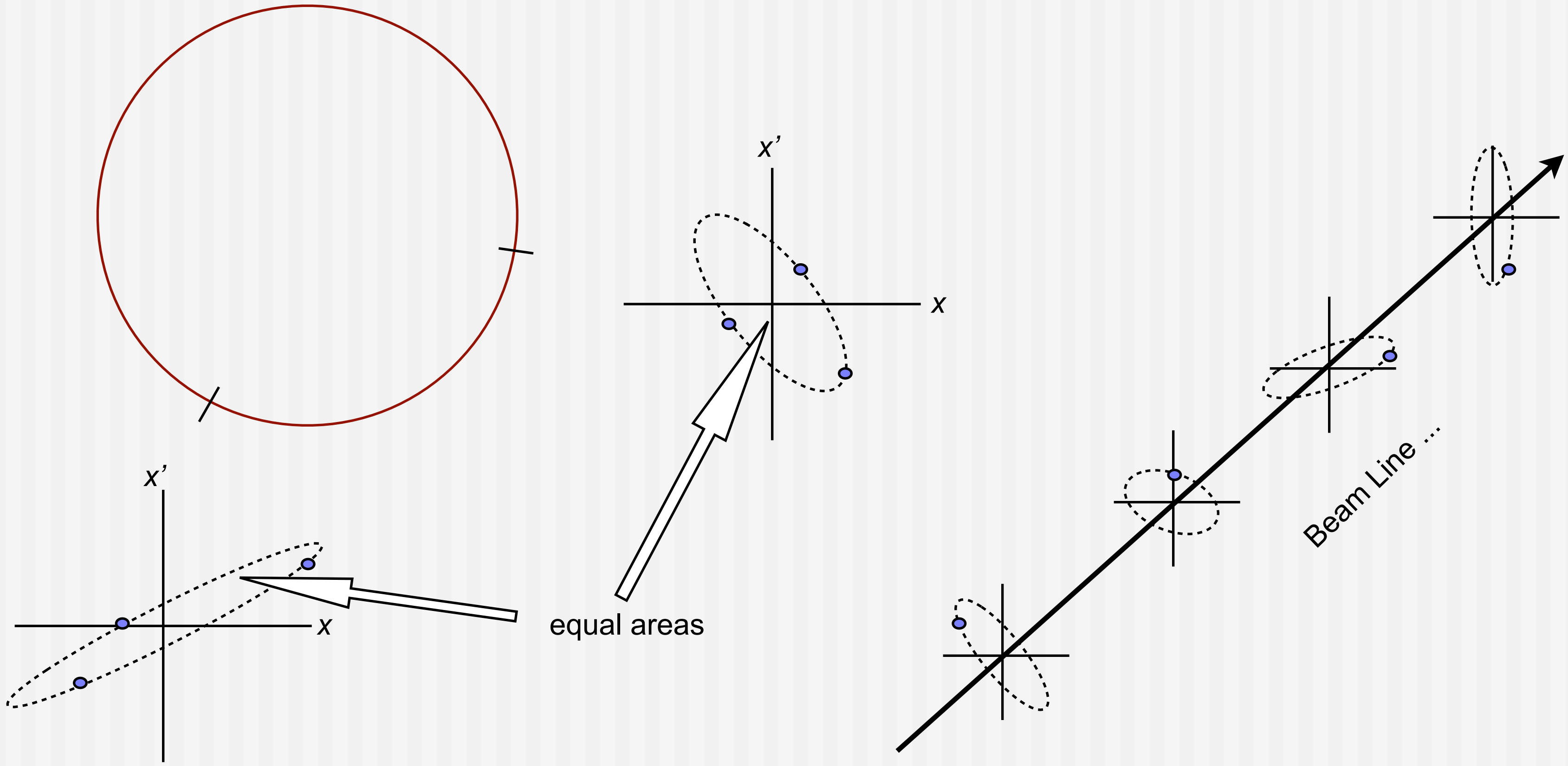


Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$

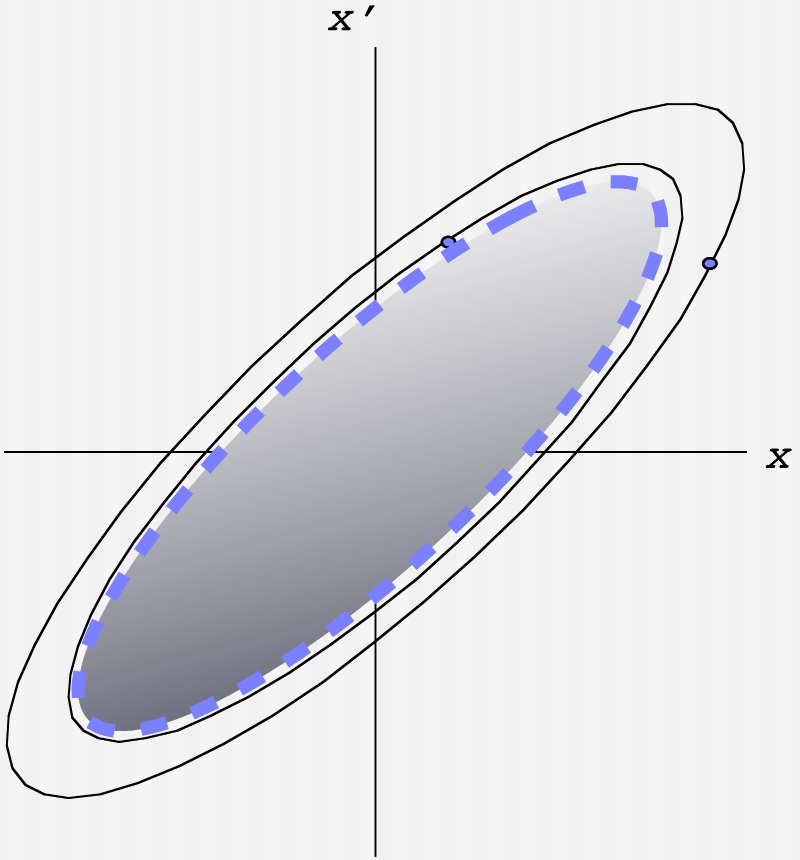
Motion in Phase Space

- Follow phase space trajectory...



Beam Emittance

- Phase space area which contains a certain fraction of the beam particles
- Popular Choices:
  - 95%
  - 39%
  - 15%
  - ...more on this subject coming up...



Adiabatic Damping from Acceleration

- Transverse oscillations imply transverse momentum. As accelerate, momentum is “delivered” in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.
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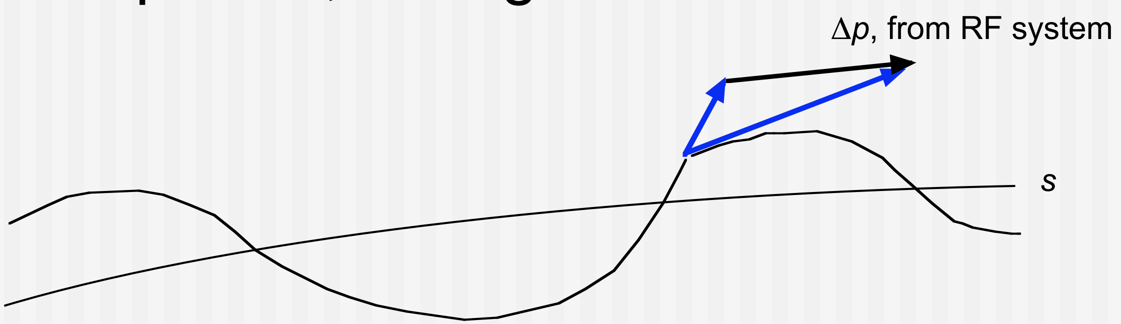


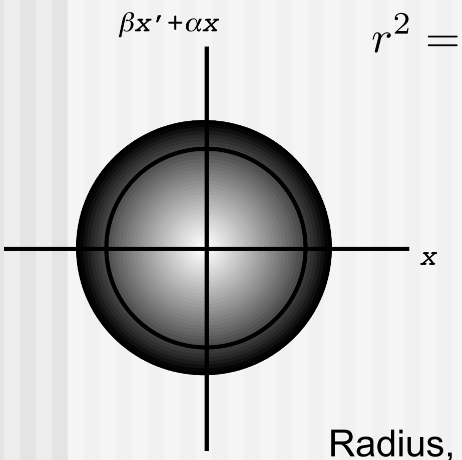
- The coordinates x-x' are not canonical conjugates, but x-p_x are; thus, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Normalized Beam Emittance

- Hence, as particles are accelerated, the area in x-x' phase space is not preserved, while area in x-p_x is preserved. Thus, we define a "normalized" beam emittance, as
- $$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)$$
-
- In principle, the normalized beam emittance should be preserved during acceleration, and hence along the chain of accelerators (at FNAL: Linac, Booster, Main Injector, etc.). Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.

Gaussian Beam in a Periodic System

- Imagine a synchrotron in which the transverse distribution of circulating particles has reached an equilibrium with a Gaussian profile in transverse coordinate *x* with zero mean and standard deviation σ.
  - The distribution can be described as follows:



$$r^2 = x^2 + (\beta x' + \alpha x)^2$$

$$\rho(r,\theta) r dr d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\, dr d\theta$$

$$\int_0^{2\pi} \int_0^{a} \rho\, r dr d\theta = f$$

Radius, *a*, containing fraction, *f*, of particles, corresponding to phase space area with emittance, ϵ:

$$a^2 = -2\sigma^2 \ln(1-f) = \epsilon\beta/\pi$$

Gaussian Emittance

- So, the normalized emittance that contains a fraction *f* of a Gaussian beam is:

$$\epsilon_N = \frac{-2\pi \ln(1-f)\sigma^2(s)}{\beta(s)}(\beta\gamma)$$

Lorentz!

- Common values of *f* :

f	$\epsilon_N/(\beta\gamma)$
95%	$6\pi\sigma^2/\beta$
86.5%	$4\pi\sigma^2/\beta$
39%	$\pi\sigma^2/\beta$
15%	σ^2/β

Typically used at Fermilab

Emittance Measurements

- Typical practice, in a synchrotron, is to measure rms beam size (assumed Gaussian), at a location where β is presumed to be known, and thus emittance can be deduced.
- While Gaussian description is often good approximation of the distribution, not necessarily true. Also possible to define the emittance in terms of *2nd* moments of the (arbitrary) distribution.

Emittance in Terms of Moments

- For each particle, $x = A\sqrt{\beta}\sin\psi \qquad x' = \frac{A}{\sqrt{\beta}}(\cos\psi - \alpha\sin\psi)$
- Average over the (stationary) distribution...

$$x^2 = A^2\beta\sin^2\psi \qquad x'^2 = \frac{A^2}{\beta}(\cos^2\psi + \alpha^2\sin^2\psi - \alpha\sin 2\psi)$$

$$\langle x^2\rangle = \frac{1}{2}\langle A^2\rangle\beta \qquad \langle x'^2\rangle = \frac{\langle A^2\rangle}{2\beta}(1+\alpha^2) = \frac{1}{2}\langle A^2\rangle\gamma$$

and ...

$$\begin{aligned} xx' &= A^2(\frac{1}{2}\sin 2\psi - \alpha\sin^2\psi) \\ \langle xx'\rangle &= -\frac{1}{2}\langle A^2\rangle\alpha \end{aligned}$$

$\beta\gamma - \alpha^2 = 1$

From which the average of all particle emittances will be $\pi\langle A^2\rangle = 2\pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$

and the “normalized rms emittance” can be defined as:

$$\boxed{\epsilon_N = \pi(\beta\gamma)\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}}$$

TRANSPORT of Beam Moments

- For simplicity, define $\tilde{\epsilon} \equiv \frac{1}{2}\langle A^2 \rangle$; then,

$$\tilde{\epsilon} J = \begin{pmatrix} \tilde{\epsilon}\alpha & \tilde{\epsilon}\beta \\ -\tilde{\epsilon}\gamma & -\tilde{\epsilon}\alpha \end{pmatrix} = \begin{pmatrix} -\langle xx' \rangle & \langle x^2 \rangle \\ -\langle x'^2 \rangle & \langle xx' \rangle \end{pmatrix}$$

- Correlation Matrix:

$$\Sigma \equiv \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix} = -\tilde{\epsilon} J S, \quad where \;\; S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- then,

$$\begin{aligned} \Sigma_2 = -\tilde{\epsilon} J_2 S &= -\tilde{\epsilon} M J_1 M^{-1} S \\ &= M(-\tilde{\epsilon} J_1 S) S^{-1} M^{-1} S \\ &= M \Sigma_1 S^{-1} M^{-1} S \\ &= M \Sigma_1 M^T \end{aligned}$$

Here, M is from point 1 to point 2 along the beam line

Summary

- So, can look at propagation of amplitude function through beam line given matrices of individual elements. Beam size at a particular location determined by
- $$x_{rms}(s) = \sqrt{\beta(s)\epsilon_N/\pi(\beta\gamma)}$$
- Or, given an initial particle distribution, can look at propagation of second moments (position, angle) given the same element matrices.
- Have neglected:
  - dispersion of trajectories due to momentum (next time)
  - hor-ver coupling (typically zero, by design)

Propagation of Amplitude Function Mismatch

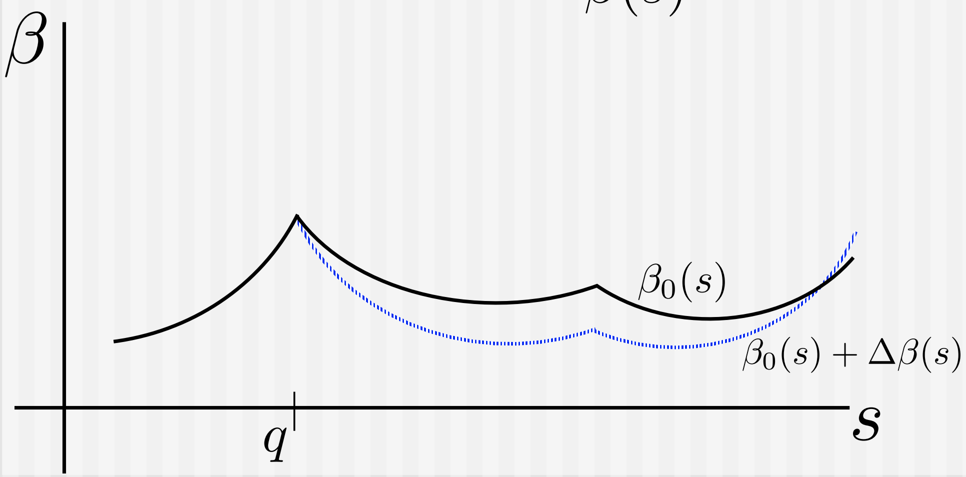
- Effects of a local gradient error
  - Suppose gradient of a thin lens is changed by amount $\Delta B'$ ...

Define: $q \equiv \dfrac{\Delta B' \ell}{B\rho}$

At the source, $\Delta\alpha = \beta q$

Downstream of the source, $$\frac{\Delta\beta(s)}{\beta(s)} = -q\beta_0 \sin 2\Delta\psi_0 + \frac{1}{2}(q\beta_0)^2(1 - \cos 2\Delta\psi_0)$$

$$\approx -q\beta_0 \sin 2\Delta\psi_0$$ (for small q)



Mismatch Invariant

- Consider two solutions to $\beta'' + 4K\beta = const.$ through a focusing system
  - for example, one may be the periodic solution, the other a perturbed solution
- Then,

propagate original solution

$$\begin{aligned} J_{02} &= M J_{01} M^{-1} \\ J_{02} + \Delta J_2 &= M(J_{01} + \Delta J_1) M^{-1} \\ \Delta J_2 &= M \Delta J_1 M^{-1} \\ \det \Delta J_2 &= \det M \det \Delta J_1 \det M^{-1} \\ \det \Delta J_2 &= \det \Delta J_1 \end{aligned}$$

propagate perturbed solution

Thus, $\det \Delta J$ for two solutions is a constant along a beamline

Expressions for Determinant of ΔJ

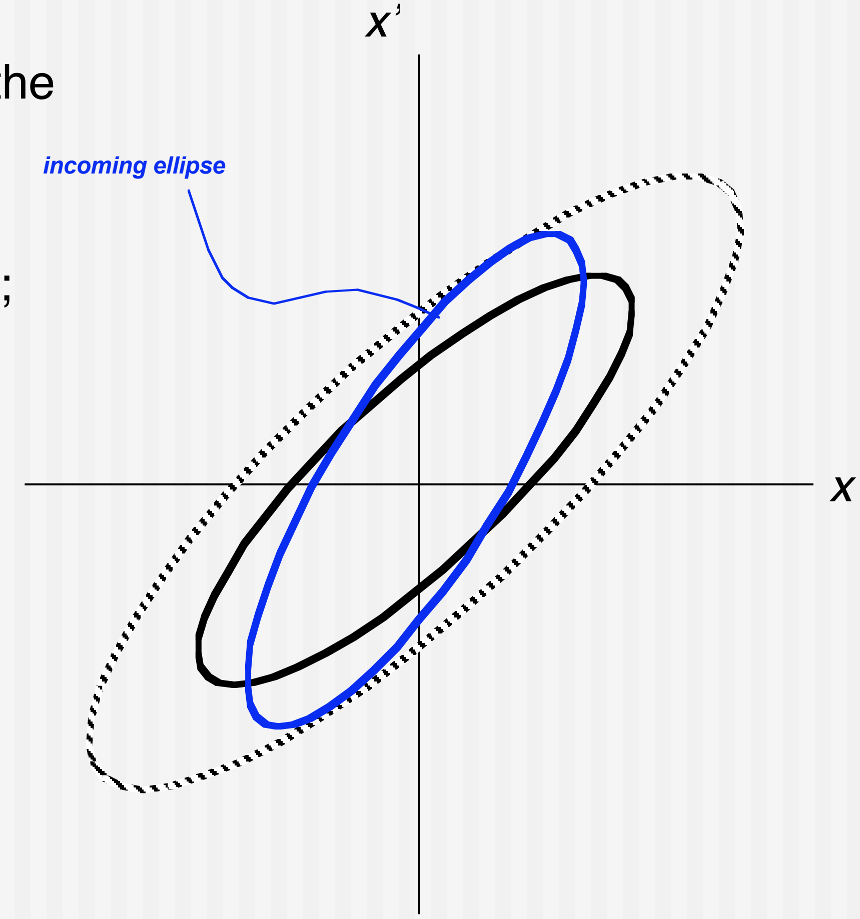
$$
\begin{aligned}
\det \Delta J &= \det(J_1 - J_0) \\
&= \begin{vmatrix} \Delta\alpha & \Delta\beta \\ -\Delta\gamma & -\Delta\alpha \end{vmatrix} \\
&= -\Delta\alpha^2 + \Delta\beta\Delta\gamma \\
&= 2 - (\beta_0\gamma_1 + \beta_1\gamma_0 - 2\alpha_0\alpha_1) \\
&= -\frac{\left(\frac{\Delta\beta}{\beta_0}\right)^2 + \left(\Delta\alpha - \alpha_0\frac{\Delta\beta}{\beta_0}\right)^2}{1 + \frac{\Delta\beta}{\beta_0}} < 0
\end{aligned}
$$

Some Examples...

- Injection Mismatch and Emittance Dilution
- Adjustment of Quadrupole in Beam Line
- Tune Shift in a Synchrotron
- Half-integer Stopband in a Synchrotron

Injection Mismatch and Emittance Dilution

- Suppose beam arrives through a transfer line into a synchrotron, but the beta function of the line is not matched to the periodic beta function of the ring...
- Particles will follow phase space trajectories dictated by the ring lattice; actual nonlinearities of the real accelerator will cause their motion to decohere
- Net result: emittance dilution

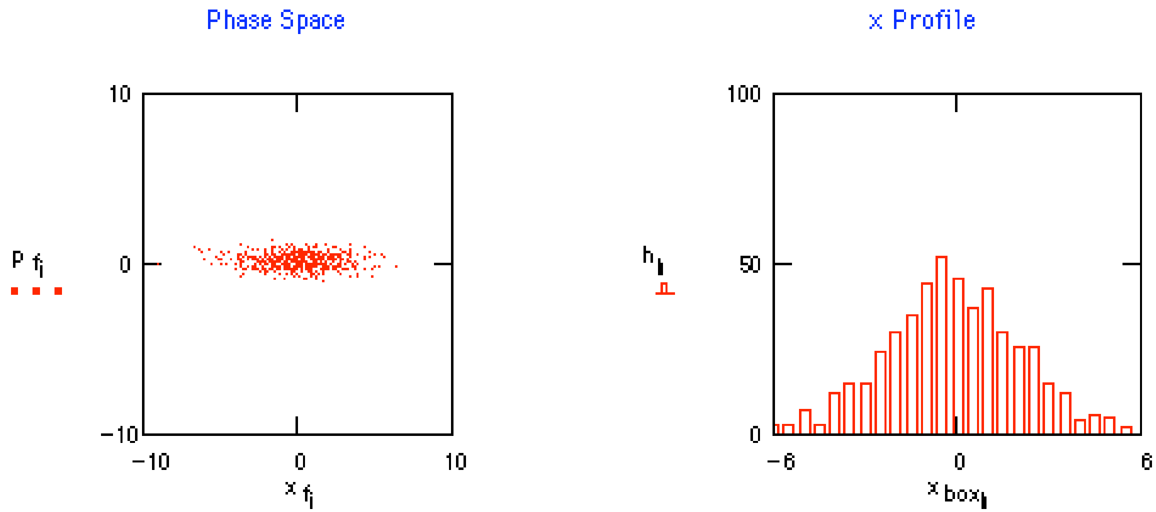


if $\epsilon \sim \langle x^2 \rangle$, then

$$\boxed{\epsilon/\epsilon_0 = 1 - \frac{1}{2}\det \Delta J}$$

Let's Role the Video Tape...

Matched: circular
Mismatched: elliptical



$\mathrm{mean}(x_f) = 0.056$ $\mathrm{stdev}(x_f) = 2.283$

Emittance Increase: $\mathrm{stdev}(x_f)^2 = 5.21$

Predicted "typical" values:

(Steering Mismatch) $1 + \frac{1}{2}\cdot\Delta x^2 = 1$

FRAME = 0 (Amplitude function Mismatch) $1 + \frac{1}{2}\cdot\frac{\delta\beta^2}{1+\delta\beta} = 2.6$

Decoherence occurs in a "real" accelerator due to inherent non-linear fields; particle motion gets out of phase, results in an apparent increase of phase space area

Quad Error/Adjustment

- Suppose a beam line is perfectly matched to a circular accelerator downstream.
- Suppose now adjust a thin lens quad in the beam line, with nominal focal length *F*; just after the lens, $\Delta\alpha = \beta_0 q = (\beta_0/F)(\Delta B'/B')$
- $\Delta\beta = 0$
- The mismatch invariant, ΔJ, is constant downstream, so...
  - the beta function distortion through the rest of the line will have amplitude: $\Delta\beta/\beta \approx \beta_0 q$
  - at the source, and at injection point to the accelerator, $\Delta J = -\Delta\alpha^2$
  - and the resulting emittance growth will be

$$\boxed{\epsilon/\epsilon_0 = 1 + \frac{1}{2}(\beta_0/F)^2(\Delta I/I)^2}$$

Suppose $\beta_0 = 45$ m, $F = 15$ m, and a 5% change is made; then $\Delta\beta/\beta \approx 15\%$, but $\epsilon/\epsilon_0 \approx 1.01$

Tune Shift in a Synchrotron

- Insert "thin quad" at one point in the synchrotron:

$$M = M_q M_0 = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$\Longrightarrow \quad \mathrm{tr} M = a + d - bq = \mathrm{tr} M_0 - q\, \beta_0 \sin 2\pi\nu_0$$

$$\cos 2\pi(\nu_0 + \Delta\nu) = \cos 2\pi\nu_0 - \beta_0 q \sin 2\pi\nu_0$$

For small changes or small errors:

$$\boxed{\Delta\nu \approx \frac{1}{4\pi}\beta_0 q}$$

Note: will also generate a distortion of amplitude function...

Half-Integer Stopband

- Actual tune change due to gradient error:

$$\boxed{\begin{aligned}\cos 2\pi\nu &= \cos 2\pi\nu_0 \\ &\quad -\beta_0 q \sin 2\pi\nu_0\end{aligned}}$$

For given gradient error, or distribution of errors, as approach tune with half-integer value the lattice will become unstable -- the “stopband width” is the spread of unstable tune values.