# Notes on Signal Modulation

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## 1) Amplitude Modulation

Carrier signal $x_c(t) = C\cos(\omega_c t + \Phi_c)$

Modulation signal $x_m^{AM}(t) = M\cos(\omega_m t + \Phi_m)$

For simplicity, we can assume $\Phi_c = \Phi_m = 0$

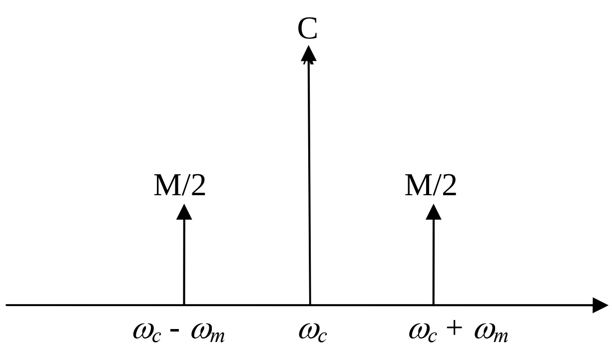
Then, the modulated signal is given by:

$$\begin{aligned} y_{AM}(t) &= [C + x_m^{AM}(t)]\cos(\omega_c t) \\ &= [C + M\cos(\omega_m t)]\cos(\omega_c t) \\ &= C\cos(\omega_c t) + M\cos(\omega_m t)\cos(\omega_c t) \end{aligned}$$

Making use of the trigonometric identity $\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$, we get

$$\boxed{y_{AM}(t) = C\cos(\omega_c t) + \frac{M}{2}\cos\left((\omega_m + \omega_c)t\right) + \frac{M}{2}\cos\left((\omega_m - \omega_c)t\right)}$$

We see the carrier component $\omega_c$ and the two side bands, $\omega_c + \omega_m$ and $\omega_c - \omega_m$. Because the sign of the amplitude of each sign band is the same, we refer to this modulation as an ***even*** function.



## 2) Frequency Modulation

Carrier signal $x_c(t) = C\cos(\omega_c t + \Phi_c)$

Modulation signal $x_m^{FM}(t) = M\cos(\omega_m t + \Phi_m)$

For simplicity, we can assume $\Phi_c = \Phi_m = 0$

Then, the modulated signal is given by:

$$
\begin{aligned}
y_{FM}(t) &= C\cos\left(\int_0^t [\omega_c + x_m^{FM}(\tau)]\mathrm{d}\tau\right) \\
&= C\cos\left(\int_0^t [\omega_c + M\cos(\omega_m \tau)]\mathrm{d}\tau\right) \\
&= C\cos\left(\int_0^t \omega_c \mathrm{d}\tau + M\int_0^t \cos(\omega_m \tau)\mathrm{d}\tau\right) \\
&= C\cos\left(\omega_c t + M\frac{\sin(\omega_m t)}{\omega_m}\right)
\end{aligned}
$$

$M$ is the frequency deviation and $\omega_m$ is the modulation frequency. The ***modulation index*** is then defined as follows:

$$h = \frac{M}{\omega_m} = \frac{2\pi \cdot \Delta f}{2\pi \cdot f_m} = \frac{\Delta f}{f_m}$$

The modulated signal can now be expressed as $y(t) = C\cos\left(\omega_c t + h\sin(\omega_m t)\right)$

In our case, we are interested by small frequency deviations (e.g. several hundred mHz at a modulation frequency of several hundred Hz). This case ($\Delta f << f_m$ or $h << 1$) is referred to as ***narrow band FM***. The expression of the modulated signal can be further expanded using the following trigonometric identity: $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$$y_{FM}(t) = C\cos(\omega_c t)\underbrace{\cos\left(h\sin(\omega_m t)\right)}_{\to 1} - C\sin(\omega_c t)\underbrace{\sin\left(h\sin(\omega_m t)\right)}_{\to h\sin(\omega_m t)}$$

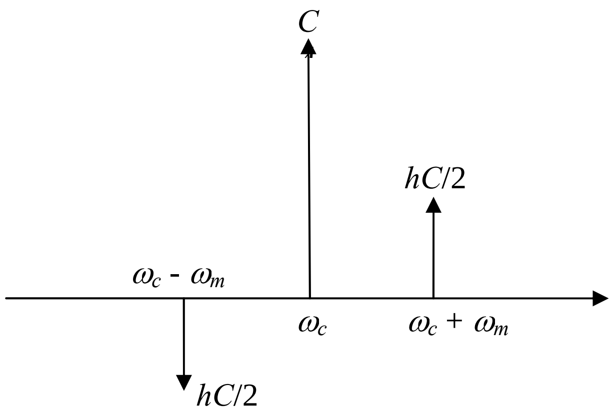
For $h << 1$, $\cos\left(h\sin(\omega_m t)\right) \approx 1$ and $\sin\left(h\sin(\omega_m t)\right) \approx h\sin(\omega_m t)$, so $y(t)$ becomes to the first order approximation:

$$y_{FM}(t) = C\cos(\omega_c t) - C\sin(\omega_c t)h\sin(\omega_m t)$$

Using another trigonometric identity: $\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$, we have:

$$\boxed{y_{FM}(t) = C\cos(\omega_c t) - \frac{C}{2}h\sin\left((\omega_c - \omega_m)t\right) + \frac{C}{2}h\sin\left((\omega_c + \omega_m)t\right)}$$

We clearly see the carrier with amplitude A, the lower side band with amplitude $\frac{C}{2}h$ and a negative sign (i.e. 180° phase offset), and the upper side band with amplitude $\frac{C}{2}h$ as illustrated below. Because the two secondary side bands have opposite phase, this modulation scheme is often referred to as an ***odd*** function.




***Carson's theorem*** states that 98% of the power of the FM signal is in the bandwidth defined as:

$$B_T = 2(\Delta f + f_m) = 2f_m(\frac{\Delta f}{f_m} + 1) = 2f_m(h+1)$$

So in the case of narrow band FM, the modulation bandwidth is given by $B_T \approx 2f_m$

### 3) Phase Modulation

We will demonstrate here that phase modulation is a ***special case of frequency modulation***, in which the frequency modulation is given by the ***time derivative*** of the phase modulation.

Carrier signal $x_c(t) = C\sin(\omega_c t + \Phi_c)$

Modulation signal $x_m^{PM}(t) = M\sin(\omega_m t + \Phi_m)$

For simplicity, we can assume $\Phi_c = \Phi_m = 0$

Then, the modulated signal is expressed as:

$$\begin{aligned} y_{PM}(t) &= C\sin\left(\omega_c t + x_m^{PM}(t)\right) \\ &= C\sin\left(\omega_c t + M\sin(\omega_m t)\right) \end{aligned}$$

The argument of the sine function can be expressed with the following integral form:

$$\begin{aligned} \omega_c t + M\sin(\omega_m t) &= \int_0^t \omega_c \mathrm{d}\tau + M\omega_m \int_0^t \cos(\omega_m \tau)\mathrm{d}\tau \\ &= \int_0^t \left(\omega_c + M\omega_m \cos(\omega_m \tau)\right)\mathrm{d}\tau \end{aligned}$$

The modulated signal becomes:

$$y_{PM}(t) = C\sin\left(\int_0^t \left(\omega_c + M\omega_m \cos(\omega_m \tau)\right)\mathrm{d}\tau\right),$$

As seen in section 2), this expression of $y_{PM}(t)$ is equivalent to a frequency modulated signal with modulation:

$$x_m^{FM}(t) = M\omega_m \cos(\omega_m t) = \frac{d}{dt}\left[M\sin(\omega_m t)\right] = \frac{d}{dt} x_m^{PM}(t)$$

So, a phase modulation is a special case of a frequency modulation which modulation signal is the time derivative of the phase modulation, i.e. $\boxed{x_m^{FM}(t) = \frac{d}{dt} x_m^{PM}(t)}$.