Analytical insight of a delayed up and down conversion

INTRODUCTION

This study is motivated by the need for a better understanding of the impact of the 8/9π mode present in the cavity probe signal. This document presents the detailed derivations of a standard RF operation, from a mathematical view point, both in time and frequency domain.
The starting point is an RF signal $A(t) = A\cos(\omega_{RF} t + \varphi_0)$ with a random phase φ_0. This signal is down converted to an intermediate frequency ω_{IF} . To keep this derivation as generic as possible, the signal used for the down conversion has a random phase φ_1 with respect to the input signal. Then, the down converted signal is delayed in time (t-t_d) before the final up conversion step. The LO signal used for the up conversion also shows some phase shift φ_2 with respect to the IF signal. All simpler cases (such as φ_0 =0, or φ_1=φ_2) are particular cases of this generic derivation. Figure 1 illustrates the RF operation. For consistency, the subscript 1 corresponds to the down converted signal, the subscript 2 to the time delayed signal, the subscript 3 to the up converted signal.

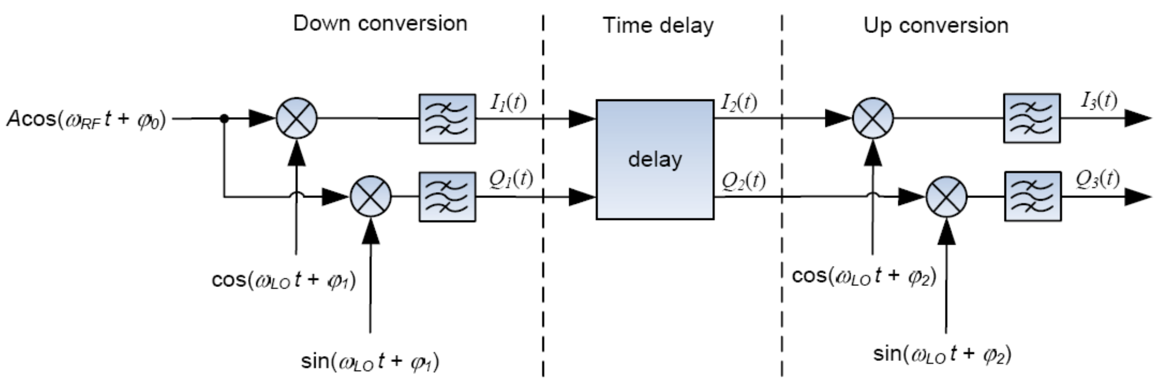

Figure 1: diagram of delayed down conversion followed by an up conversion

In section 1), the expression of (I_1 , Q_1), (I_2 , Q_2) and (I_3 , Q_3) are derived in the time domain. The same derivation is performed in section 2) in the frequency domain.
Section 3) deals with the special case of an input signal with two components at different frequencies, which corresponds to an RF signal with a 8/9π mode content. Each section starts with some mathematical refreshers, useful for subsequent derivations.

Note to the reader:

Although the derivations are conceptually simple, they involve heavy calculations, which are not always trivial to follow. I tried to make this document as complete as possible, not skipping intermediate steps in the derivations, and yet tried to keep them as concise as possible.

1) In the time domain

a. analytical overview

Down converting $A(\omega_{RF} t)$ to ω_{IF} with the upper side band ($\omega_{LO} = \omega_{RF} + \omega_{IF}$) is equivalent to multiplying by $\times e^{j(\omega_{LO} t + \varphi_1)}$, where φ_1 is an arbitrary phase:

$$I(t) = \cos(\omega_{LO} t + \varphi_1) \times A(\omega_{RF} t)$$
$$Q(t) = \sin(\omega_{LO} t + \varphi_1) \times A(\omega_{RF} t)$$

Up converting $A(\omega_{IF} t)$ to ω_{RF} with the upper side band ($\omega_{LO} = \omega_{RF} + \omega_{IF}$) is equivalent to multiplying by $\times e^{j(\omega_{LO} t + \varphi_2)}$ where φ_2 is an arbitrary phase. In terms of *I* and *Q*, we have:

$$I(t) = \cos(\omega_{LO} t + \varphi_2) \times I(\omega_{IF} t)$$
$$Q(t) = \sin(\omega_{LO} t + \varphi_2) \times Q(\omega_{IF} t)$$

A time delay corresponds to substituting t with $t - t_d$ (i.e. $t \mapsto t - t_d$) in all time-dependent factors.

b. down conversion

Starting with a single frequency input signal with a random phase delay φ_0

$$A(t) = A\cos(\omega_{RF} t + \varphi_0)$$

Making use of the following trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right],$$

the down converted signal is now

$$I_1(t) = \cos\left((\overbrace{\omega_{RF} + \omega_{IF}}^{\omega_{LO}})t + \varphi_1 \right) \times A(t)$$
$$= \cos(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_1}_{a}) \times A\cos(\underbrace{\omega_{RF} t + \varphi_0}_{b})$$
$$= \frac{A}{2}\left[\cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_1}_{a} + \underbrace{\omega_{RF} t + \varphi_0}_{b} \right) + \cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_1}_{a} - \underbrace{(\omega_{RF} t + \varphi_0)}_{b} \right) \right]$$
$$= \frac{A}{2}\left[\cos\left((2\omega_{RF} + \omega_{IF})t + \varphi_1 + \varphi_0\right) + \cos\left(\omega_{IF} t + \varphi_1 - \varphi_0\right) \right]$$

And after filtering out the higher frequency term in $2\omega_{RF}+\omega_{IF}$, the down converted signal is:

$$\boxed{I_1(t)=\frac{A}{2}\cos\left(\omega_{IF}t+\varphi_1-\varphi_0\right)}$$

For Q, we make use of the following trigonometric identity

$$\sin(a)\cos(b)=\frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

and we obtain

$$Q_1(t)=\sin\left((\overbrace{\omega_{RF}+\omega_{IF}}^{\omega_{LO}})t+\varphi_1\right)\times A(t)$$

$$=\sin(\underbrace{(\omega_{RF}+\omega_{IF})t+\varphi_1}_{a})\times A\cos(\underbrace{\omega_{RF}t+\varphi_0}_{b})$$

$$=\frac{A}{2}\left[\sin\left(\underbrace{(\omega_{RF}+\omega_{IF})t+\varphi_1}_{a}+\underbrace{\omega_{RF}t+\varphi_0}_{b}\right)+\sin\left(\underbrace{(\omega_{RF}+\omega_{IF})t+\varphi_1}_{a}-\underbrace{(\omega_{RF}t+\varphi_0)}_{b}\right)\right]$$

$$=\frac{A}{2}\left[\sin\left((2\omega_{RF}+\omega_{IF})t+\varphi_1+\varphi_0\right)+\sin\left(\omega_{IF}t+\varphi_1-\varphi_0\right)\right]$$

And after filtering out the higher frequency term in $2\omega_{RF}+\omega_{IF}$, the expression for Q is now :

$$\boxed{Q_1(t)=\frac{A}{2}\sin\left(\omega_{IF}t+\varphi_1-\varphi_0\right)}$$

c. time delay

To account for any time delay t_d taking place at the down converted stage, the I and Q expressions are translated in time by t-t_d :

$$\boxed{\begin{aligned}I_2(t)=I_1(t-t_d)=\frac{A}{2}\left[\cos\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)\right]\\ Q_2(t)=Q_1(t-t_d)=\frac{A}{2}\left[\sin\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)\right]\end{aligned}}$$

d. up conversion

The new expressions after the up conversion are obtained by multiplying I and Q by $\cos(\omega_{LO}t+\varphi_2)$ and $\sin(\omega_{LO}t+\varphi_2)$ respectively, where φ_2 is an arbitrary phase that can be adjusted to compensate for prior time delays.

Here again, we make use of the identity: $\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$,

$$I_3(t) = \cos\left(\overbrace{(\omega_{RF} + \omega_{IF})}^{\omega_{LO}}t + \varphi_2\right) \times I_2(t)$$

$$= \cos(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a}) \times \frac{A}{2}\cos(\underbrace{\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0}_{b})$$

$$= \frac{A}{4}\left[\cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a} + \underbrace{\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0}_{b}\right) + \cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a} - \underbrace{(\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0)}_{b}\right)\right]$$

$$= \frac{A}{4}\left[\cos\left((\omega_{RF} + 2\omega_{IF})t - \omega_{IF}t_d + \varphi_2 + \varphi_1 - \varphi_0\right) + \cos\left(\omega_{RF}t + \omega_{IF}t_d + \varphi_2 - \varphi_1 + \varphi_0\right)\right]$$

And after filtering out the higher frequency term in $\omega_{RF} + 2\omega_{IF}$

$$\boxed{I_3(t) = \frac{A}{4}\left[\cos\left(\omega_{RF}t + \omega_{IF}t_d + \varphi_2 - \varphi_1 + \varphi_0\right)\right]}$$

The phase term φ_2 can be adjusted to compensate for prior time delays by choosing

$$\varphi_2 = \varphi_1 - \varphi_0 - \omega_{IF}t_d$$

Then the final expression for *I* is

$$\boxed{I_3(t) = \frac{A}{4}\cos\left(\omega_{RF}t\right)}$$

Following the same approach, we up-convert the *Q* term by using the following trigonometric identity:

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

we obtain:

$$Q_3(t) = \sin\left(\overbrace{(\omega_{RF} + \omega_{IF})}^{\omega_{LO}}t + \varphi_2\right) \times Q_2(t)$$

$$= \sin(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a}) \times \frac{A}{2}\sin(\underbrace{\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0}_{b})$$

$$= \frac{A}{4}\left[\cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a} - \underbrace{(\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0)}_{b}\right) - \cos\left(\underbrace{(\omega_{RF} + \omega_{IF})t + \varphi_2}_{a} + \underbrace{\omega_{IF}(t - t_d) + \varphi_1 - \varphi_0}_{b}\right)\right]$$

$$= \frac{A}{4}\left[\cos\left(\omega_{RF}t + \omega_{IF}t_d + \varphi_2 - \varphi_1 + \varphi_0\right) - \cos\left((\omega_{RF} + 2\omega_{IF})t - \omega_{IF}t_d + \varphi_2 + \varphi_1 - \varphi_0\right)\right]$$

And after filtering out the higher frequency term in $\omega_{RF}+2\omega_{IF}$

$$\boxed{Q_3(t)=\frac{A}{4}\left[\cos\left(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0\right)\right]}$$

Applying the same choice of $\varphi_2=\varphi_1-\varphi_0-\omega_{IF}t_d$, the expression for Q simplifies to

$$\boxed{Q_3(t)=\frac{A}{4}\cos\left(\omega_{RF}t\right)}$$

The first conclusion from this derivation is that **the up-converted signal does not depend on the choice of intermediate frequency**. Further more, the phase of the LO signal used at the up and down conversion can be adjusted to compensate for delays occurring in previous stages ($\varphi_2=\varphi_1-\varphi_0-\omega_{IF}t_d$), i.e. before the down conversion and during the down converted stage.

Filtering the IF after the down conversion and filtering the RF after the up conversion results in a degradation of the signal amplitude by a factor of 2 at every step. So the reconstructed signal has an amplitude that is 4 times smaller than the original signal.

2) In the frequency domain

The same conclusions can be derived in the frequency domain

a. analytical overview

First, listed below are a few common time domain signals and their frequency domain counterparts:

$$f(t)=\cos(\omega_0 t) \qquad F(\omega)=\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$f(t)=\sin(\omega_0 t) \qquad F(\omega)=\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

$$f(t)=e^{j\omega_0 t} \qquad F(\omega)=2\pi\delta(\omega-\omega_0)$$

A time shift in the time domain $t\mapsto t-t_0$ translates as $F(\omega)\mapsto F(\omega)e^{-j\omega t_0}$ in the frequency domain:

$$f(t-t_0) \qquad F(\omega)\mapsto F(\omega)e^{-j\omega t_0}$$

$$f(t+t_0) \qquad F(\omega)\mapsto F(\omega)e^{+j\omega t_0}$$

So the Fourier transform of $A\cos(\omega_{RF}t+\varphi_0)=A\cos\left(\omega_{RF}(t+\frac{\varphi_0}{\omega_{RF}})\right)$ is

$$A(\omega)=\pi[\delta(\omega-\omega_{RF})+\delta(\omega+\omega_{RF})]e^{j\omega\frac{\varphi_0}{\omega_{RF}}}$$

Similarly, the frequency equivalent of $\cos(\omega_{LO} t + \varphi_1) = \cos\left(\omega_{LO}(t + \frac{\varphi_1}{\omega_{LO}})\right)$ is

$$\pi[\delta(\omega - \omega_{LO}) + \delta(\omega + \omega_{LO})]e^{j\omega\frac{\varphi_1}{\omega_{LO}}}$$

A down conversion in the time domain is obtained by multiplying signals. In the frequency domain, this translates into a convolution of the Fourier transformed signals.

$$f(t)g(t) \qquad \frac{1}{2\pi}F(\omega) \otimes G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\Omega)G(\omega - \Omega)\mathrm{d}\Omega$$

Finally, here are two useful properties of the delta function:

$$\delta(\Omega - a) = \delta(a - \Omega)$$

$$\int_{-\infty}^{+\infty} F(\Omega)\delta(\Omega - a)\mathrm{d}\Omega = F(a)$$

We'll make use of these two properties in the following derivation. For a down conversion in the time domain, the input component *A*(*t*) is multiplied by $\cos(\omega_{LO} t + \varphi_1)$. In the frequency domain:

$$I_{DC}(\omega) = \frac{1}{2\pi}A(\omega) \otimes \pi[\delta(\omega - \omega_{LO}) + \delta(\omega + \omega_{LO})]e^{j\omega\frac{\varphi_1}{\omega_{LO}}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(\Omega)\pi[\delta(\omega - \omega_{LO} - \Omega) + \delta(\omega + \omega_{LO} - \Omega)]e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\mathrm{d}\Omega$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} A(\Omega)e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\delta\big(\Omega - (\omega - \omega_{LO})\big)\mathrm{d}\Omega + \frac{1}{2}\int_{-\infty}^{+\infty} A(\Omega)e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\delta\big(\Omega - (\omega + \omega_{LO})\big)\mathrm{d}\Omega$$

$$= \frac{1}{2}A(\omega - \omega_{LO})e^{j(\omega-(\omega-\omega_{LO}))\frac{\varphi_1}{\omega_{LO}}} + \frac{1}{2}A(\omega + \omega_{LO})e^{j(\omega-(\omega+\omega_{LO}))\frac{\varphi_1}{\omega_{LO}}}$$

$$= \frac{1}{2}A(\omega - \omega_{LO})e^{j\varphi_1} + \frac{1}{2}A(\omega + \omega_{LO})e^{-j\varphi_1}$$

As expected, a down conversion corresponds to a frequency shift by ω_0 and its image $-\omega_0$
The down converted imaginary part is obtained in the time domain through a multiplication by $\sin(\omega_{LO} t + \varphi_1)$. In the frequency domain, this corresponds to

$$
\begin{aligned}
Q_{DC}(\omega) &= \frac{1}{2\pi} A(\omega) \otimes \frac{\pi}{j}[\delta(\omega-\omega_{LO}) - \delta(\omega+\omega_{LO})]e^{j\omega\frac{\varphi_1}{\omega_{LO}}} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(\Omega)\frac{\pi}{j}[\delta(\omega-\omega_{LO}-\Omega) - \delta(\omega+\omega_{LO}-\Omega)]e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\mathrm{d}\Omega \\
&= \frac{1}{2j}\int_{-\infty}^{+\infty} A(\Omega)e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\delta\big(\Omega-(\omega-\omega_{LO})\big)\mathrm{d}\Omega - \frac{1}{2j}\int_{-\infty}^{+\infty} A(\Omega)e^{j(\omega-\Omega)\frac{\varphi_1}{\omega_{LO}}}\delta\big(\Omega-(\omega+\omega_{LO})\big)\mathrm{d}\Omega \\
&= \frac{1}{2j} A(\omega-\omega_{LO})e^{j(\omega-(\omega-\omega_{LO}))\frac{\varphi_1}{\omega_{LO}}} - \frac{1}{2j} A(\omega+\omega_{LO})e^{j(\omega-(\omega+\omega_{LO}))\frac{\varphi_1}{\omega_{LO}}} \\
&= \frac{1}{2j} A(\omega-\omega_{LO})e^{j\varphi_1} - \frac{1}{2j} A(\omega+\omega_{LO})e^{-j\varphi_1}
\end{aligned}
$$

The same formula can be derived for the up conversion but this time convoluting $I(\omega)$ and $Q(\omega)$ by $\cos(\omega_{LO}t+\varphi_2)$ and $\sin(\omega_{LO}t+\varphi_2)$ respectively

$$
\begin{aligned}
I_{UC}(\omega) &= \frac{1}{2} I(\omega-\omega_{LO})e^{j\varphi_2} + \frac{1}{2} I(\omega+\omega_{LO})e^{-j\varphi_2} \\
Q_{UC}(\omega) &= \frac{1}{2j} Q(\omega-\omega_{LO})e^{j\varphi_2} - \frac{1}{2j} Q(\omega+\omega_{LO})e^{-j\varphi_2}
\end{aligned}
$$

After these mathematical refreshers, we can now derive the equations for the whole signal chain.

b. down conversion

We start from the same input function $A(t) = A\cos(\omega_{RF}t+\varphi_0)$. Its Fourier transform is:

$$
A(\omega) = A\pi[\delta(\omega-\omega_{RF}) + \delta(\omega+\omega_{RF})]e^{j\omega\frac{\varphi_0}{\omega_{RF}}}
$$

As we've seen before, a down conversion with an arbitrary phase φ_1 will yield:

$$
\begin{aligned}
I_1(\omega) &= \frac{1}{2} A(\omega-\omega_{LO})e^{j\varphi_1} + \frac{1}{2} A(\omega+\omega_{LO})e^{-j\varphi_1} \\
&= \frac{A\pi}{2}\underbrace{\big[\delta(\omega-\omega_{RF}-\omega_{LO}) + \delta(\omega+\omega_{RF}-\omega_{LO})\big]e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}}_{A(\omega-\omega_{LO})}e^{j\varphi_1} \\
&\quad + \frac{A\pi}{2}\underbrace{\big[\delta(\omega-\omega_{RF}+\omega_{LO}) + \delta(\omega+\omega_{RF}+\omega_{LO})\big]e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}}_{A(\omega+\omega_{LO})}e^{-j\varphi_1} \\
&= \frac{A\pi}{2}\big[\delta(\omega-2\omega_{RF}-\omega_{IF}) + \delta(\omega-\omega_{IF})\big]e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} \\
&\quad + \frac{A\pi}{2}\big[\delta(\omega+\omega_{IF}) + \delta(\omega+2\omega_{RF}+\omega_{IF})\big]e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}
\end{aligned}
$$

Just like in the time domain, the down conversion produces harmonics at higher frequencies ($2\omega_{RF}+\omega_{IF}$ and its image $-2\omega_{RF} - \omega_{IF}$) which can be filtered out:

$$\boxed{I_1(\omega) = \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} + \delta(\omega+\omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]}$$

Similarly, for $Q(\omega)$:

$$
\begin{aligned}
Q_1(\omega) &= \frac{1}{2j}A(\omega-\omega_{LO})e^{j\varphi_1} - \frac{1}{2j}A(\omega+\omega_{LO})e^{-j\varphi_1} \\
&= \frac{A\pi}{2j}\underbrace{\left[\delta(\omega-\omega_{RF}-\omega_{LO})+\delta(\omega+\omega_{RF}-\omega_{LO})\right]e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}}_{A(\omega-\omega_{LO})}e^{j\varphi_1} \\
&\quad -\frac{A\pi}{2j}\underbrace{\left[\delta(\omega-\omega_{RF}+\omega_{LO})+\delta(\omega+\omega_{RF}+\omega_{LO})\right]e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}}_{A(\omega+\omega_{LO})}e^{-j\varphi_1} \\
&= \frac{A\pi}{2j}\left[\delta(\omega-2\omega_{RF}-\omega_{IF})+\delta(\omega-\omega_{IF})\right]e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} \\
&\quad -\frac{A\pi}{2j}\left[\delta(\omega+\omega_{IF})+\delta(\omega+2\omega_{RF}+\omega_{IF})\right]e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}
\end{aligned}
$$

And after filtering out the higher frequency terms:

$$
\boxed{Q_1(\omega) = \frac{A\pi}{2j}\left[\delta(\omega-\omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} - \delta(\omega+\omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]}
$$

As a quick check, we inverse transform these two expressions to see their time domain counterparts. Let's rearrange the terms to evidence some standard Fourier expressions:

$$
\begin{aligned}
I_1(\omega) &= \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} + \delta(\omega+\omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right] \\
&= \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{-j\frac{\omega_{LO}}{\omega_{RF}}\varphi_0}e^{j\varphi_1} + \delta(\omega+\omega_{IF})e^{j\frac{\omega_{LO}}{\omega_{RF}}\varphi_0}e^{-j\varphi_1}\right]e^{j\omega\frac{\varphi_0}{\omega_{RF}}} \\
&= \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{-j\frac{\omega_{RF}+\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j\varphi_1} + \delta(\omega+\omega_{IF})e^{j\frac{\omega_{RF}+\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j\varphi_1}\right]e^{j\omega\frac{\varphi_0}{\omega_{RF}}} \\
&= \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j\varphi_0}e^{j\varphi_1} + \delta(\omega+\omega_{IF})e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j\varphi_0}e^{-j\varphi_1}\right]e^{j\omega\frac{\varphi_0}{\omega_{RF}}} \\
&= \frac{A\pi}{2}\delta(\omega-\omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}\underbrace{e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j(\varphi_1-\varphi_0)}}_{\text{constant}} + \frac{A\pi}{2}\delta(\omega+\omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}\underbrace{e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j(\varphi_1-\varphi_0)}}_{\text{constant}}
\end{aligned}
$$

We've seen that $F(\omega) = 2\pi\delta(\omega - \omega_0)$ corresponds to $f(t) = e^{j\omega_0 t}$ and that multiplying by a complex exponential is equivalent to a time shift, hence:

$$\frac{\pi}{2}\delta(\omega - \omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}} \quad \Rightarrow \quad \frac{1}{4}e^{j\omega_{IF}(t+\frac{\varphi_0}{\omega_{RF}})} \quad \text{in the time domain}$$

and

$$\frac{\pi}{2}\delta(\omega + \omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}} \quad \Rightarrow \quad \frac{1}{4}e^{-j\omega_{IF}(t+\frac{\varphi_0}{\omega_{RF}})} \quad \text{in the time domain}$$

So, $I_1(\omega)$ $\Rightarrow$ $I_1(t) = \frac{A}{4}e^{j\omega_{IF}(t+\frac{\varphi_0}{\omega_{RF}})}\underbrace{e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j(\varphi_1-\varphi_0)}}_{\text{constant}} + \frac{A}{4}e^{-j\omega_{IF}(t+\frac{\varphi_0}{\omega_{RF}})}\underbrace{e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j(\varphi_1-\varphi_0)}}_{\text{constant}}$

Rearranging terms and making use of the trigonometric identity $\cos(a) = \frac{1}{2}\left[e^{ja} + e^{-ja}\right]$, we have

$$\begin{aligned}
I_1(t) &= \frac{A}{4}e^{j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} + \frac{A}{4}e^{-j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} \\
&= \frac{A}{4}\left[e^{j(\omega_{IF}t+\varphi_1-\varphi_0)} + e^{-j(\omega_{IF}t+\varphi_1-\varphi_0)}\right] \\
&= \frac{A}{2}\cos\left(\omega_{IF}t + \varphi_1 - \varphi_0\right)
\end{aligned}$$

This is identical to the expression found in the time domain
For the imaginary part, following the same steps as for $I_1(\omega)$:

$$\begin{aligned}
Q_1(\omega) &= \frac{A\pi}{2j}\left[\delta(\omega - \omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1} - \delta(\omega + \omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right] \\
&= \frac{A\pi}{2j}\delta(\omega - \omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}\underbrace{e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j(\varphi_1-\varphi_0)}}_{\text{constant}} - \frac{A\pi}{2j}\delta(\omega + \omega_{IF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}\underbrace{e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j(\varphi_1-\varphi_0)}}_{\text{constant}}
\end{aligned}$$

The inverse Fourier and the identity $\sin(a) = \frac{1}{2j}\left[e^{ja} - e^{-ja}\right]$ yield:

$$\begin{aligned}
Q_1(t) &= \frac{A}{4j}e^{j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} - \frac{A}{4j}e^{-j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} \\
&= \frac{A}{4j}\left[e^{j(\omega_{IF}t_0+\varphi_1-\varphi_0)} - e^{-j(\omega_{IF}t+\varphi_1-\varphi_0)}\right] \\
&= \frac{A}{2}\sin\left(\omega_{IF}t_0 + \varphi_1 - \varphi_0\right)
\end{aligned}$$

This is identical to the expression found in the time domain

c. phase shift

Any time delay t_d that takes place at the intermediate frequency ω_{IF} corresponds to a phase shift in the frequency domain ($\times e^{-j\omega t_d}$).

$$\boxed{\begin{aligned} I_2(\omega) &= I_1(\omega)\times e^{-j\omega t_d} \\ &= \frac{A\pi}{2}\left[\delta(\omega-\omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1}+\delta(\omega+\omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]\times e^{-j\omega t_d}\end{aligned}}$$

and

$$\boxed{\begin{aligned} Q_2(\omega) &= Q_1(\omega)\times e^{-j\omega t_d} \\ &= \frac{A\pi}{2j}\left[\delta(\omega-\omega_{IF})e^{j(\omega-\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1}-\delta(\omega+\omega_{IF})e^{j(\omega+\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]\times e^{-j\omega t_d}\end{aligned}}$$

For completeness, we inverse Fourier transform these expressions:

$$I_2(\omega)=\frac{A\pi}{2}\delta(\omega-\omega_{IF})e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}}-t_d\right)}\underbrace{e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j(\varphi_1-\varphi_0)}}_{\text{constant}}+\frac{A\pi}{2}\delta(\omega+\omega_{IF})e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}}-t_d\right)}\underbrace{e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j(\varphi_1-\varphi_0)}}_{\text{constant}}$$

becomes in the time domain:

$$\begin{aligned} I_2(t) &= \frac{A}{4}e^{j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\omega_{IF}t_d-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)}+\frac{A}{4}e^{-j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\omega_{IF}t_d-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} \\ &= \frac{A}{4}\left[e^{j(\omega_{IF}t-\omega_{IF}t_d+\varphi_1-\varphi_0)}+e^{-j(\omega_{IF}t-\omega_{IF}t_d+\varphi_1-\varphi_0)}\right] \\ &= \frac{A}{2}\cos\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)\end{aligned}$$

And for Q

$$Q_2(\omega)=\frac{A\pi}{2j}\delta(\omega-\omega_{IF})e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}}-t_d\right)}\underbrace{e^{-j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{j(\varphi_1-\varphi_0)}}_{\text{constant}}-\frac{A\pi}{2j}\delta(\omega+\omega_{IF})e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}}-t_d\right)}\underbrace{e^{j\frac{\omega_{IF}}{\omega_{RF}}\varphi_0}e^{-j(\varphi_1-\varphi_0)}}_{\text{constant}}$$

becomes in the time domain:

$$\begin{aligned} Q_2(t) &= \frac{A}{4j}e^{j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\omega_{IF}t_d-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)}-\frac{A}{4j}e^{-j\left(\omega_{IF}t+\omega_{IF}\frac{\varphi_0}{\omega_{RF}}-\omega_{IF}t_d-\frac{\omega_{IF}}{\omega_{RF}}\varphi_0+\varphi_1-\varphi_0\right)} \\ &= \frac{A}{4j}\left[e^{j(\omega_{IF}t-\omega_{IF}t_d+\varphi_1-\varphi_0)}-e^{-j(\omega_{IF}t-\omega_{IF}t_d+\varphi_1-\varphi_0)}\right] \\ &= \frac{A}{2}\sin\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)\end{aligned}$$

Note that the expressions indeed do correspond to the ones obtained in the time domain.

d. up conversion

The last step consists of performing an up conversion back to the RF frequency. The phase φ_2 associated with the up converting signal is purposely kept different from φ_1 to keep this calculation as generic as possible. In reality, this phase parameter can be adjusted to compensate for delays occurring before the down conversion, and during the intermediate frequency stage, as we will show.

As seen before, the up conversion is obtained using these formulas:

$$I_{UC}(\omega) = \frac{1}{2} I(\omega - \omega_{LO}) e^{j\varphi_2} + \frac{1}{2} I(\omega + \omega_{LO}) e^{-j\varphi_2}$$
$$Q_{UC}(\omega) = \frac{1}{2j} Q(\omega - \omega_{LO}) e^{j\varphi_2} - \frac{1}{2j} Q(\omega + \omega_{LO}) e^{-j\varphi_2}$$

For *I*

$$I_3(\omega) = \frac{1}{2} I_2(\omega - \omega_{LO}) e^{j\varphi_2} + \frac{1}{2} I_2(\omega + \omega_{LO}) e^{-j\varphi_2}$$

Substituting the expression of I_2 in the formula above, we get:

$$I_3(\omega) = \frac{A\pi}{4} \underbrace{\left[\delta(\omega - \omega_{IF} - \omega_{LO}) e^{j(\omega - 2\omega_{LO})\frac{\varphi_0}{\omega_{RF}}} e^{j\varphi_1} + \delta(\omega + \omega_{IF} - \omega_{LO}) e^{j\omega\frac{\varphi_0}{\omega_{RF}}} e^{-j\varphi_1} \right] e^{-j(\omega - \omega_{LO})t_d}}_{I_2(\omega - \omega_{LO})} \times e^{j\varphi_2}$$
$$+ \frac{A\pi}{4} \underbrace{\left[\delta(\omega - \omega_{IF} + \omega_{LO}) e^{j\omega\frac{\varphi_0}{\omega_{RF}}} e^{j\varphi_1} + \delta(\omega + \omega_{IF} + \omega_{LO}) e^{j(\omega + 2\omega_{LO})\frac{\varphi_0}{\omega_{RF}}} e^{-j\varphi_1} \right] e^{-j(\omega + \omega_{LO})t_d}}_{I_2(\omega + \omega_{LO})} \times e^{-j\varphi_2}$$

After filtering out the higher frequency terms, the expression simplifies to

$$I_3(\omega) = \frac{A\pi}{4} \left[\delta(\omega - \omega_{RF}) e^{j\omega\frac{\varphi_0}{\omega_{RF}}} e^{-j\varphi_1} \right] e^{-j(\omega - \omega_{LO})t_d} \times e^{j\varphi_2} + \frac{A\pi}{4} \left[\delta(\omega + \omega_{RF}) e^{j\omega\frac{\varphi_0}{\omega_{RF}}} e^{j\varphi_1} \right] e^{-j(\omega + \omega_{LO})t_d} \times e^{-j\varphi_2}$$

This can be rearranged as

$$\boxed{I_3(\omega) = \frac{A\pi}{4} \left[\delta(\omega - \omega_{RF}) e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}} - t_d\right)} e^{j(\varphi_2 - \varphi_1 + \omega_{LO}t_d)} + \delta(\omega + \omega_{RF}) e^{j\omega(\frac{\varphi_0}{\omega_{RF}} - t_d)} e^{-j(\varphi_2 - \varphi_1 + \omega_{LO}t_d)} \right]}$$

Following the same step for *Q*, we have

$$Q_3(\omega) = \frac{1}{2j} Q_2(\omega - \omega_{LO}) e^{j\varphi_2} - \frac{1}{2j} Q_2(\omega + \omega_{LO}) e^{-j\varphi_2}$$

Substituting the expression of Q in the formula above:

$$Q_3(\omega)=\frac{1}{2j}\frac{A\pi}{2j}\underbrace{\left[\delta(\omega-\omega_{IF}-\omega_{LO})e^{j(\omega-2\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1}-\delta(\omega+\omega_{IF}-\omega_{LO})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]}_{Q_2(\omega-\omega_{LO})}e^{-j(\omega-\omega_{LO})t_d}\times e^{j\varphi_2}$$

$$-\frac{1}{2j}\frac{A\pi}{2j}\underbrace{\left[\delta(\omega-\omega_{IF}+\omega_{LO})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1}-\delta(\omega+\omega_{IF}+\omega_{LO})e^{j(\omega+2\omega_{LO})\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]}_{Q_2(\omega+\omega_{LO})}e^{-j(\omega+\omega_{LO})t_d}\times e^{-j\varphi_2}$$

After filtering out the higher frequency terms, the expression simplifies to

$$Q_3(\omega)=-\frac{A\pi}{4}\left[-\delta(\omega-\omega_{RF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}e^{-j\varphi_1}\right]e^{-j(\omega-\omega_{LO})t_d}\times e^{j\varphi_2}+\frac{A\pi}{4}\left[\delta(\omega+\omega_{RF})e^{j\omega\frac{\varphi_0}{\omega_{RF}}}e^{j\varphi_1}\right]e^{-j(\omega+\omega_{LO})t_d}\times e^{-j\varphi_2}$$

This can be rearranged into

$$\boxed{Q_3(\omega)=\frac{A\pi}{4}\left[\delta(\omega-\omega_{RF})e^{j\omega(\frac{\varphi_0}{\omega_{RF}}-t_d)}e^{j(\varphi_2-\varphi_1+\omega_{LO}t_d)}+\delta(\omega+\omega_{RF})e^{j\omega(\frac{\varphi_0}{\omega_{RF}}-t_d)}e^{-j(\varphi_2-\varphi_1+\omega_{LO}t_d)}\right]}$$

To validate this result, we can inverse Fourier transform to compare with the expression obtained in the time domain:

$$\begin{aligned}I_3(t)&=\frac{A}{8}\left[e^{j\left(\omega_{RF}t+\omega_{RF}\frac{\varphi_0}{\omega_{RF}}-\omega_{RF}t_d+\varphi_2-\varphi_1+\omega_{LO}t_d\right)}+e^{j\left(-\omega_{RF}t-\omega_{RF}\frac{\varphi_0}{\omega_{RF}}+\omega_{RF}t_d-\varphi_2+\varphi_1-\omega_{LO}t_d\right)}\right]\\&=\frac{A}{8}\left[e^{j(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)}+e^{-j(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)}\right]\\&=\frac{A}{4}\cos(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)\end{aligned}$$

And for Q

$$\begin{aligned}Q_3(t)&=\frac{A}{8}\left[e^{j\left(\omega_{RF}t+\omega_{RF}\frac{\varphi_0}{\omega_{RF}}-\omega_{RF}t_d+\varphi_2-\varphi_1+\omega_{LO}t_d\right)}+e^{j\left(-\omega_{RF}t-\omega_{RF}\frac{\varphi_0}{\omega_{RF}}+\omega_{RF}t_d-\varphi_2+\varphi_1-\omega_{LO}t_d\right)}\right]\\&=\frac{A}{8}\left[e^{j(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)}+e^{-j(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)}\right]\\&=\frac{A}{4}\cos(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0)\end{aligned}$$

Here too, if the up conversion process allows for tuning the phase, we can choose φ_2 to cancel out previous delays: $\varphi_2 = \varphi_1 - \varphi_0 - \omega_{IF} t_d$

$$I_3(t) = \frac{A}{4}\cos(\omega_{RF} t)$$

$$Q_3(t) = \frac{A}{4}\cos(\omega_{RF} t)$$

In the frequency domain

$$I_3(\omega) = \frac{A\pi}{4}\left[\delta(\omega-\omega_{RF})e^{j\omega\left(\frac{\varphi_0}{\omega_{RF}}-t_d\right)}e^{j(-\varphi_0+\omega_{RF}t_d)} + \delta(\omega+\omega_{RF})e^{j\omega(\frac{\varphi_0}{\omega_{RF}}-t_d)}e^{-j(-\varphi_0+\omega_{RF}t_d)}\right]$$

$$Q_3(\omega) = \frac{A\pi}{4}\left[\delta(\omega-\omega_{RF})e^{j\omega(\frac{\varphi_0}{\omega_{RF}}-t_d)}e^{j(-\varphi_0+\omega_{RF}t_d)} + \delta(\omega+\omega_{RF})e^{j\omega(\frac{\varphi_0}{\omega_{RF}}-t_d)}e^{-j(-\varphi_0+\omega_{RF}t_d)}\right]$$

And at $\omega = \omega_{RF}$

$$I_3(\omega_{RF}) = \frac{A\pi}{4}e^{j[\varphi_0-\omega_{RF}t_d-\varphi_0+\omega_{RF}t_d]} = \frac{A\pi}{4}$$

$$Q_3(\omega_{RF}) = \frac{A\pi}{4}e^{j[\varphi_0-\omega_{RF}t_d-\varphi_0+\omega_{RF}t_d]} = \frac{A\pi}{4}$$

These two derivations 1) and 2) are essentially the same. There is no surprise in that they yield the same results and the same conclusions.

3) With a dual input frequency signal

We now consider the case of an input signal with dual frequency content:

$$A\cos(\omega_{RF}^A t + \varphi_0^A) + B\cos(\omega_{RF}^B t + \varphi_0^B)$$

We can write

$$\omega_{RF}^A = \omega_{RF} \qquad \varphi_0^A = \varphi_0$$

$$\omega_{RF}^B = \omega_{RF}^A + \Delta\omega = \omega_{RF} + \Delta\omega \qquad \varphi_0^B = \varphi_0^A + \Delta\varphi = \varphi_0 + \Delta\varphi$$

The expression for the input signal becomes:

$$\boxed{A\cos(\omega_{RF} t + \varphi_0) + B\cos\big((\omega_{RF} + \Delta\omega)t + (\varphi_0 + \Delta\varphi)\big)}$$

For simplicity, this will case be only be derived in the time domain.

a. down conversion

At the down conversion stage, the input signal is multiplied by $\cos(\omega_{LO}t+\varphi_1)$ and $\sin(\omega_{LO}t+\varphi_1)$ for I and Q respectively:

$$
\begin{aligned}
I_1(t) &= \cos(\omega_{LO}t+\varphi_1)\times\left[A\cos(\omega_{RF}t+\varphi_0)+B\cos\left((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi)\right)\right]\\
Q_1(t) &= \sin(\omega_{LO}t+\varphi_1)\times\left[A\cos(\omega_{RF}t+\varphi_0)+B\cos\left((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi)\right)\right]
\end{aligned}
$$

Solving for $I_1(t)$, and using the identity $\cos(a)\cos(b)=\frac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$, we get

$$
\begin{aligned}
I_1(t) &= A\cos(\omega_{LO}t+\varphi_1)\cos(\omega_{RF}t+\varphi_0)+B\cos(\omega_{LO}t+\varphi_1)\cos\left((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi)\right)\\
&= \frac{A}{2}\left[\cos\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}+\underbrace{\omega_{RF}t+\varphi_0}_{b}\right)+\cos\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}-\underbrace{(\omega_{RF}t+\varphi_0)}_{b}\right)\right]\\
&\quad+\frac{B}{2}\left[\cos\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}+\underbrace{(\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi)}_{b}\right)+\cos\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}-\underbrace{((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi))}_{b}\right)\right]\\
&= \frac{A}{2}\left[\cos\left(\underbrace{(\omega_{LO}+\omega_{RF}}_{2\omega_{RF}+\omega_{IF}})t+\varphi_1+\varphi_0\right)+\cos\left(\underbrace{(\omega_{LO}-\omega_{RF}}_{\omega_{IF}})t+\varphi_1-\varphi_0\right)\right]\\
&\quad+\frac{B}{2}\left[\cos\left(\underbrace{(\omega_{LO}+\omega_{RF}+\Delta\omega}_{2\omega_{RF}+\omega_{IF}+\Delta\omega})t+\varphi_1+\varphi_0+\Delta\varphi\right)+\cos\left(\underbrace{(\omega_{LO}-\omega_{RF}-\Delta\omega}_{\omega_{IF}-\Delta\omega})t+\varphi_1-\varphi_0-\Delta\varphi\right)\right]
\end{aligned}
$$

Assuming $\Delta\omega << \omega_{IF}$, we can legitimately assume that the term in $2\omega_{RF}+\omega_{IF}+\Delta\omega$ gets filtered out with the term in $2\omega_{RF}+\omega_{IF}$. Hence, after filtering the IF, the remaining expression for I is

$$
\boxed{I_1(t)=\frac{A}{2}\cos\left(\omega_{IF}t+\varphi_1-\varphi_0\right)+\frac{B}{2}\cos\left((\omega_{IF}-\Delta\omega)t+\varphi_1-\varphi_0-\Delta\varphi\right)}
$$

Similarly, for Q and using the identity $\sin(a)\cos(b)=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$, we have

$$Q_1(t) = A\sin(\omega_{LO}t+\varphi_1)\cos(\omega_{RF}t+\varphi_0) + B\sin(\omega_{LO}t+\varphi_1)\cos((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi))$$

$$= \frac{A}{2}\left[\sin\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}+\underbrace{\omega_{RF}t+\varphi_0}_{b}\right)+\sin\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}-\underbrace{(\omega_{RF}t+\varphi_0)}_{b}\right)\right]$$

$$+\frac{B}{2}\left[\sin\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}+\underbrace{(\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi)}_{b}\right)+\sin\left(\underbrace{\omega_{LO}t+\varphi_1}_{a}-\underbrace{((\omega_{RF}+\Delta\omega)t+(\varphi_0+\Delta\varphi))}_{b}\right)\right]$$

$$= \frac{A}{2}\left[\sin\left(\underbrace{(\omega_{LO}+\omega_{RF})}_{2\omega_{RF}+\omega_{IF}}t+\varphi_1+\varphi_0\right)+\sin\left(\underbrace{(\omega_{LO}-\omega_{RF})}_{\omega_{IF}}t+\varphi_1-\varphi_0\right)\right]$$

$$+\frac{B}{2}\left[\sin\left(\underbrace{(\omega_{LO}+\omega_{RF}+\Delta\omega)}_{2\omega_{RF}+\omega_{IF}+\Delta\omega}t+\varphi_1+\varphi_0+\Delta\varphi\right)+\sin\left(\underbrace{(\omega_{LO}-\omega_{RF}-\Delta\omega)}_{\omega_{IF}-\Delta\omega}t+\varphi_1-\varphi_0-\Delta\varphi\right)\right]$$

After filtering the higher frequency terms:

$$\boxed{Q_1(t)=\frac{A}{2}\sin(\omega_{IF}t+\varphi_1-\varphi_0)+\frac{B}{2}\sin((\omega_{IF}-\Delta\omega)t+\varphi_1-\varphi_0-\Delta\varphi)}$$

As a quick check, we notice that if B = 0, we get the expressions derived in part 1). Also, if $\Delta\omega$ = 0 and $\Delta\varphi$ = 0, then we have twice the same signal.

b. time delay

The time delayed expressions are simply obtained by substituting t with t - t_d :

$$\boxed{I_2(t)=\frac{A}{2}\cos(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0)+\frac{B}{2}\cos((\omega_{IF}-\Delta\omega)(t-t_d)+\varphi_1-\varphi_0-\Delta\varphi)}$$

$$\boxed{Q_2(t)=\frac{A}{2}\sin(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0)+\frac{B}{2}\sin((\omega_{IF}-\Delta\omega)(t-t_d)+\varphi_1-\varphi_0-\Delta\varphi)}$$

c. up conversion

For the final step, we multiply $I_2(t)$ and $Q_2(t)$ by $\cos(\omega_{LO}t+\varphi_2)$ and $\sin(\omega_{LO}t+\varphi_2)$ respectively and make use of $\cos(a)\cos(b)=\frac{1}{2}[\cos(a+b)+\cos(a-b)]$ twice.

$$
\begin{aligned}
I_3(t) &= \cos(\omega_{LO}t+\varphi_2)I_2(t) \\
&= \frac{A}{2}\cos(\omega_{LO}t+\varphi_2)\cos\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)+\frac{B}{2}\cos(\omega_{LO}t+\varphi_2)\cos\left((\omega_{IF}-\Delta\omega)(t-t_d)+\varphi_1-\varphi_0-\Delta\varphi\right) \\
&= \frac{A}{4}\left[\cos\left(\underbrace{(\omega_{LO}+\omega_{IF})}_{2\omega_{RF}+\omega_{IF}}t-\omega_{IF}t_d+\varphi_2+\varphi_1-\varphi_0\right)+\cos\left(\underbrace{(\omega_{LO}-\omega_{IF})}_{\omega_{RF}}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0\right)\right] \\
&+\frac{B}{4}\left[\begin{array}{l}\cos\left(\underbrace{(\omega_{LO}+\omega_{IF}-\Delta\omega)}_{2\omega_{RF}+\omega_{IF}-\Delta\omega}t-(\omega_{IF}-\Delta\omega)t_d+\varphi_2+\varphi_1-\varphi_0-\Delta\varphi\right) \\ +\cos\left(\underbrace{(\omega_{LO}-\omega_{IF}+\Delta\omega)}_{\omega_{RF}+\Delta\omega}t+(\omega_{IF}-\Delta\omega)t_d+\varphi_2-\varphi_1+\varphi_0+\Delta\varphi\right)\end{array}\right]
\end{aligned}
$$

Filtering out the terms in $\omega_{LO}+\omega_{IF}-\Delta\omega$ and in $\omega_{LO}+\omega_{IF}$ (assuming $\Delta\omega << \omega_{IF}$), the expression above simplifies to:

$$
\boxed{I_3(t)=\frac{A}{4}\cos\left(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0\right)+\frac{B}{4}\cos\left((\omega_{RF}+\Delta\omega)t+(\omega_{IF}-\Delta\omega)t_d+\varphi_2-\varphi_1+\varphi_0+\Delta\varphi\right)}
$$

For $Q_3(t)$, we multiply $Q_2(t)$ by $\sin(\omega_{LO}t+\varphi_2)$ and use the following identity twice:

$$
\sin(a)\sin(b)=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]
$$

$$
\begin{aligned}
Q_3(t) &= \sin(\omega_{LO}t+\varphi_2)Q_2(t) \\
&= \frac{A}{2}\sin(\omega_{LO}t+\varphi_2)\sin\left(\omega_{IF}(t-t_d)+\varphi_1-\varphi_0\right)+\frac{B}{2}\sin(\omega_{LO}t+\varphi_2)\sin\left((\omega_{IF}-\Delta\omega)(t-t_d)+\varphi_1-\varphi_0-\Delta\varphi\right) \\
&= \frac{A}{4}\left[\cos\left(\underbrace{(\omega_{LO}-\omega_{IF})}_{\omega_{RF}}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0\right)-\cos\left(\underbrace{(\omega_{LO}+\omega_{IF})}_{2\omega_{RF}+\omega_{IF}}t-\omega_{IF}t_d+\varphi_2+\varphi_1-\varphi_0\right)\right] \\
&+\frac{B}{4}\left[\begin{array}{l}\cos\left(\underbrace{(\omega_{LO}-\omega_{IF}+\Delta\omega)}_{\omega_{RF}+\Delta\omega}t+(\omega_{IF}-\Delta\omega)t_d+\varphi_2-\varphi_1+\varphi_0+\Delta\varphi\right) \\ -\cos\left(\underbrace{(\omega_{LO}+\omega_{IF}-\Delta\omega)}_{2\omega_{RF}-\Delta\omega}t-(\omega_{IF}-\Delta\omega)t_d+\varphi_2+\varphi_1-\varphi_0-\Delta\varphi\right)\end{array}\right]
\end{aligned}
$$

Filtering out the terms in $\omega_{LO}+\omega_{IF}-\Delta\omega$ and in $\omega_{LO}+\omega_{IF}$ (assuming $\Delta\omega << \omega_{IF}$), the expression above simplifies to:

$$
\boxed{Q_3(t)=\frac{A}{4}\cos\left(\omega_{RF}t+\omega_{IF}t_d+\varphi_2-\varphi_1+\varphi_0\right)+\frac{B}{4}\cos\left((\omega_{RF}+\Delta\omega)t+(\omega_{IF}-\Delta\omega)t_d+\varphi_2-\varphi_1+\varphi_0+\Delta\varphi\right)}
$$

As explained earlier, we can choose φ_2 to cancel out previous delays: $\varphi_2 = \varphi_1 - \varphi_0 - \omega_{IF} t_d$

$$\boxed{I_3(t) = \frac{A}{4}\cos(\omega_{RF} t) + \frac{B}{4}\cos((\omega_{RF} + \Delta\omega)t - \Delta\omega t_d + \Delta\varphi)}$$

Note that the value for φ_2 chosen here is identical to the one used in part 1). Similarly, this value will simplify $Q_3(t)$:

$$\boxed{Q_3(t) = \frac{A}{4}\sin(\omega_{RF} t) + \frac{B}{4}\sin((\omega_{RF} + \Delta\omega)t - \Delta\omega t_d + \Delta\varphi)}$$

We can easily check that if $B = 0$, we get the expressions derived in part 1) and $\Delta\omega = 0$ and $\Delta\varphi = 0$, then we have twice the same signal.

CONCLUSION

The document has shown the complete derivation of a single frequency input signal, going through the process of a down conversion, experiencing some delay at the intermediate frequency stage, and being up converted back to the RF frequency. The analysis was carried both in the time and in the frequency domain.
The same derivation was performed for a dual frequency input signal.

This analysis clearly shows that the output signal is independent of the choice of intermediate frequency after up conversion in the dual frequency case, as in the case of a single frequency input signal.

This also shows that a phase shift that occurs at the nominal operating frequency ω_{RF} due to system delay can be compensated for at the time of the up conversion by a single phase adjustment.

In the dual frequency case, while the nominal frequency signal can be phase adjusted, the second component ($\omega_{RF} + \Delta\omega$) will show a phase shift $-\Delta\omega\, t_d$ inversely proportional to the frequency offset ($\Delta\omega$) of the input signal.

So, for a calibrated system, in the case of a cavity signal with some 8/9π content, the phase shift of the 8/9π mode is only a function of its frequency offset ($\Delta\omega$) and of the delay (t_d) but is independent of the choice of IF.