Accelerator Physics Developments for Tevatron Run II

Lecture 4

Longitudinal Instabilities

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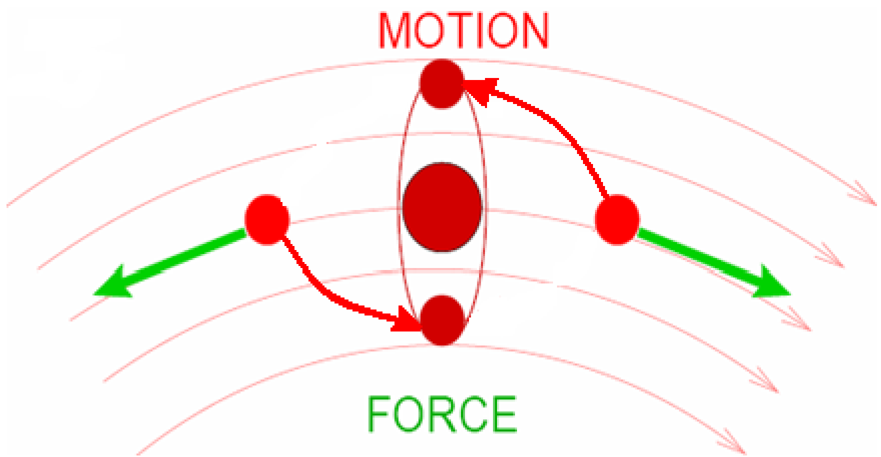
A bit of history:
Negative Mass Instability and J.C.Maxwel

Negative Mass Instability is the first predicted collective instability in ring accelerators.

It is illustrated and explained by Figure



Accidental local bulge of beam density (fluctuation) generates repulsive force which accelerates particles before the fluctuation and decelerates -- after it.

Therefore:

Below transition energy the angular velocity of the particles
increases with acceleration,
and decreases with deceleration
resulting a dissipation of fluctuation.

Above the transition, the situation is opposite (negative effective mass!!!),
resulting growth of the fluctuation -- that is instability.

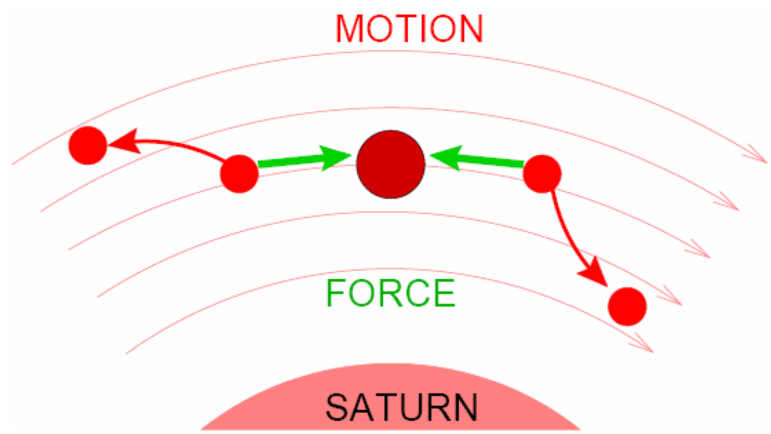
Negative Mass Instability effect is predicted by 2 groups of authors in 1959:

C. Nilsen, A. Sessler, K. Symon and *A. Kolomensky, A. Lebedev.*

However, similar problem was considered 100 years before by J.~C.~Maxwell:

``On the Stability of the Motion of Saturn's Rings'' (1859).}

Applying Fig. to Saturn, we need to change the Beam by the Ring. Effective mass of particles in the ring is negative (Kepler's low!) but the forces between them are attractive.



Therefore, the fluctuation has to disperse (stability).

Coasting beam (in)stability

State of a beam is described by distribution function in longitudinal phase space:

$$\mathcal{F}(t,\theta,p) = \frac{\Delta N}{\Delta\theta\Delta p}$$

According to Liouville theorem, part of the space *ΔθΔp* occupied by *ΔN* particles does not change at motion of this sample:

$$\frac{d}{dt}\mathcal{F}(t,\theta,p) = 0$$

Because azimuth *θ* and momentum *p* depend on time, total time derivative is:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\theta}{dt}\frac{\partial}{\partial\theta} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

where *dθ/dt = Ω(p)* is angular velocity, and *dp/dt = eE(θ)* is electric force.

It results the kinetic equation (Vlasov):

$$\frac{\partial\mathcal{F}}{\partial t} + \Omega(p)\,\frac{\partial\mathcal{F}}{\partial\theta} + eE(t,\theta)\,\frac{\partial\mathcal{F}}{\partial p} = 0$$

Electric field should be obtained by Maxwell equations with the beam current:

$$J(t,\theta) = e\Omega(p_0)\int\mathcal{F}dp$$

Coasting beam is a partial solution of these equations:

$$\mathcal{F} = F(p), \qquad J = J_0 = const, \qquad E = 0$$

To investigate its stability, we consider a small perturbation

$$\mathcal{F}(t, \theta, p) = F(p) + f(t, \theta, p), \qquad f \ll F$$

The small (in the beginning) addition *f* satisfy the equation:

$$\frac{\partial f}{\partial t} + \Omega(p)\,\frac{\partial f}{\partial \theta} + eE(t,\theta)\,\frac{\partial F}{\partial p} = \xcancel{\left[-\,eE(t,\theta)\,\frac{\partial f}{\partial p}\right]} \simeq 0$$

where we neglected the second order term

For continuous beam it is possible to get a solution of this linearized equation in the form

$$f(t,\theta,p) = f_k(\omega,p)\exp(ik\theta - i\omega t)$$

"Electric" values include the same factor

$$\begin{bmatrix} j(t,\theta) \\ E(t,\theta) \end{bmatrix} = \begin{bmatrix} J_k(\omega) \\ E_k(\omega) \end{bmatrix} \times \exp(ik\theta - i\omega t)$$

Then it follows from the linearized Vlasov equation

$$\frac{\partial f}{\partial t} + \Omega(p)\frac{\partial f}{\partial \theta} + eE(t,\theta)\frac{\partial F}{\partial p} = 0 \quad \Rightarrow \quad f_k(\omega,p) \simeq -\frac{eE_k(\omega)F'(p)}{-i\omega + ik\Omega(p)}$$

and corresponding addition to the beam current:

$$J(t,\theta) = e\Omega(p_0)\int \mathcal{F}dp \quad \Rightarrow \quad J_k(\omega) \simeq -ie^2\Omega_0 E_k(\omega)\int \frac{F'(p)\,dp}{\omega - k\Omega(p)}$$

Second *linear* relation of the variables follows from Maxwell equations and can be presented in terms of *longitudinal beam coupling impedance:*

$$E_k(\omega) = -\frac{Z(\omega)}{2\pi R}J_k(\omega)$$

Its substitution to previous equation gives the dispersion equation*:* (*R* is the machine radius).

$$1 = \frac{ie^2\Omega_0 Z(\omega)}{2\pi R}\int \frac{F'(p)dp}{\omega - k\Omega(p)}$$

Longitudinal Instability for Zero Momentum Spread

For a beam without momentum spread:
(N is total number of particles)

$$F(p) = \frac{J_0}{e\Omega_0}\delta(p-p_0)$$

Then solutions of dispersion equation is

$$1 = \frac{ie^2\Omega_0 Z(\omega)}{2\pi R}\int\frac{F'(p)dp}{\omega - k\Omega(p)} \quad\Rightarrow\quad 1 = \frac{ie^2\Omega_0 Z(\omega)}{2\pi R}\frac{J_0}{e\Omega_0}\int\frac{d}{dp}\left(\delta(p-p_0)\right)\frac{dp}{\omega - k\Omega(p)}$$

$$\Rightarrow\quad 1 = -\frac{ie^2\Omega_0 Z(\omega)}{2\pi R}\frac{J_0}{e\Omega_0}\frac{\left(d\Omega/dp\right)_{p=p_0}}{\left(\omega - k\Omega_0\right)^2} \quad\Rightarrow\quad \omega = k\Omega_0\left(1 \pm \sqrt{\frac{ie\eta J_0 Z_k}{2\pi\beta p_0 ck}}\right)$$

Where

$\eta = -d(\ln\Omega)/d(\ln p) = \alpha - 1/\gamma^2$ is the slip-factor , and
correction to the frequency is sufficiently small: $Z_k(\omega) \approx Z_k(k\Omega_0) \equiv Z_k$

- For pure imaginary impedance and $\eta < 0$ the frequency, ω, is real and the motion is stable
- If the impedance has non-zero real part ω has positive imaginary part
  One frequency corresponds to a damped motion
  Another one to an unstable one

Negative Mass Instability

Smooth perfectly conducting beam pipe has purely imaginary space charge impedance:

$$Z_k(\omega) = \frac{icZ_0}{\omega R}\left(k^2 - \frac{\omega^2 c^2}{R^2}\right)\left(\ln\frac{b}{a} + \frac{1}{4}\right) \simeq \frac{ikZ_0}{\beta\gamma^2}\left(\ln\frac{b}{a} + \frac{1}{4}\right)$$

where $Z_0=4\pi/c=377$ Ohm, a and b are the beam and beam pipe radii, and factor ¼ comes due to averaging of longitudinal force across the beam.

For Gaussian beam and round chamber one can write

$$\frac{Z(\omega_k)}{k} \approx i\frac{Z_0}{\beta\gamma^2}\ln\left(\frac{b}{1.5\sigma}\right)$$

Thus, for a zero spread beam we finally can write

$$\omega = k\Omega_0\left(1 \pm \sqrt{-\frac{e\eta J_0 Z_0}{2\pi\beta p_0 ck}\left(\ln\frac{b}{a} + \frac{1}{4}\right)}\right)$$

Below transition ($\eta < 0$):

In the beam frame there are two waves moving along and opposite to the beam velocity

Above transition ($\eta > 0$):

There are two waves: one - damped and another - unstable

Landau Damping

Let's normalize the distribution function by rms momentum spread:

$$\sigma_p{}^2 = \int p^2 F(p)dp \quad \text{so that } x = p/\sigma_p$$

=> $F(p) = \dfrac{1}{\sigma_p}\psi\left(\dfrac{p}{\sigma_p}\right)$ $\quad \int \psi(x)dx = 1 \; . \; \int x^2\psi(x)dx = 1 \; .$

Then the dispersion equation $\quad 1 = \dfrac{ie^2\Omega_0 Z(\omega)}{2\pi R}\displaystyle\int \frac{F'(p)dp}{\omega - k\Omega(p)}$

can be rewritten in the following form: $\quad \left(\varepsilon_n(y) \equiv 1 - iA_n \displaystyle\int_{\delta\to+0} \frac{d\psi/dz}{y+z-i\delta\,\mathrm{sign}(n)}dz\right) = 0$

Where $\quad A_n = \dfrac{eJ_0}{2\pi c p_0 \beta\eta(\sigma_p/p_0)^2}\left(\dfrac{Z_n}{n}\right), \quad y = \dfrac{\delta\omega}{n\Omega_0\eta(\sigma_p/p_0)}, \quad z = \dfrac{p}{\sigma_p}, \quad \delta\omega = \omega - n\Omega_0, \quad \begin{array}{l}\Omega_0 = \Omega(p_0) \\ Z_n = Z(n\Omega_0 + \delta\omega)\end{array}$

- $i\delta$ determines the rule how to cross the pole in the integral
  - It can be obtained by solving the problem with initial boundary conditions using Laplace transform (Landau's rule)
  - It creates the Landau damping
- We solve the dispersion equation for $\delta\omega$ which, in general case, is a complex number

Stability Boundary

- As a rule, the impedance is sufficiently small => The tune shift is small too and one one can neglect a frequency correction in the impedance:

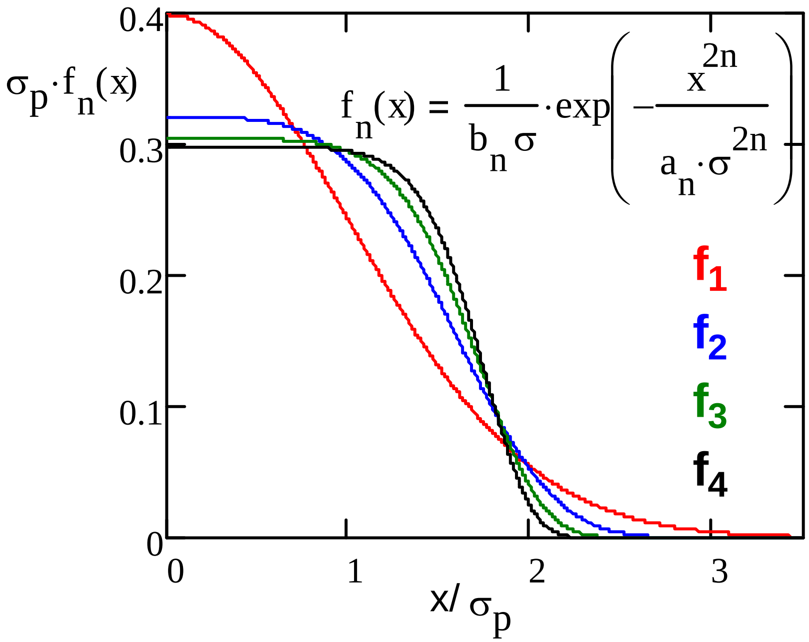
$$Z_k(\omega) = Z_k(k\Omega_0)$$

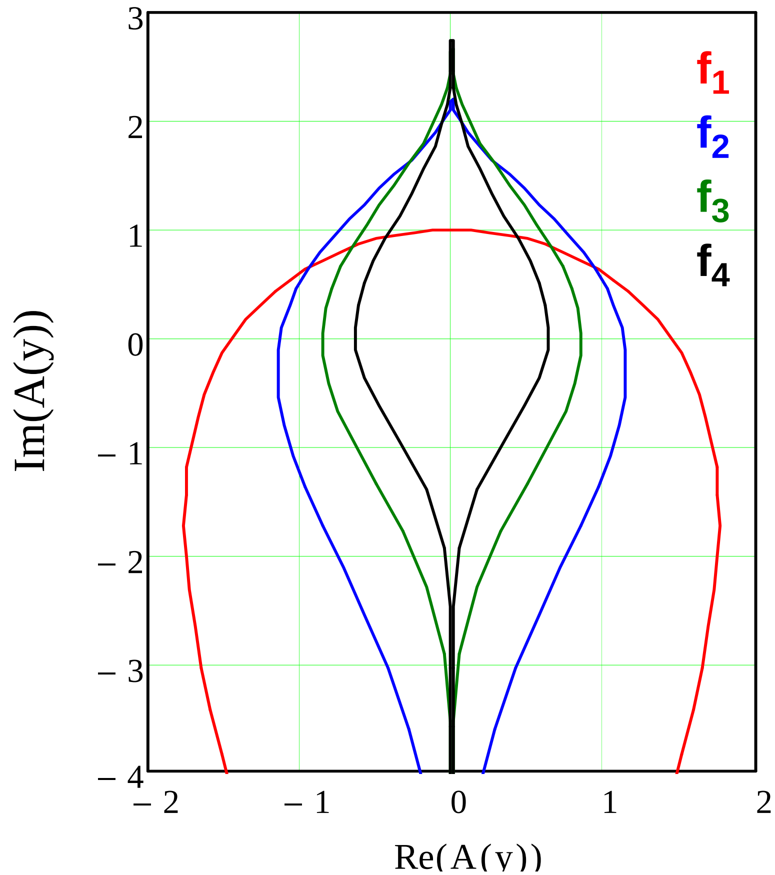
- That allows to find a stability boundary where $\mathrm{Im}(\delta\omega)=0$

$$1 - iA_n \int_{\delta\to+0} \frac{d\psi/dz}{y+z-i\delta\,\mathrm{sign}(n)}dz = 0 \qquad \Rightarrow \qquad A(y) = \left(i \int_{\delta\to+0} \frac{d\psi/dz}{y+z-i\delta\,\mathrm{sign}(n)}dz \right)^{-1}$$

Remind that

$$A_n = \frac{eJ_0}{2\pi c p_0 \beta \eta (\sigma_p/p_0)^2}\left(\frac{Z_n}{n}\right), \quad y = \frac{\delta\omega}{n\Omega_0 \eta(\sigma_p/p_0)}$$



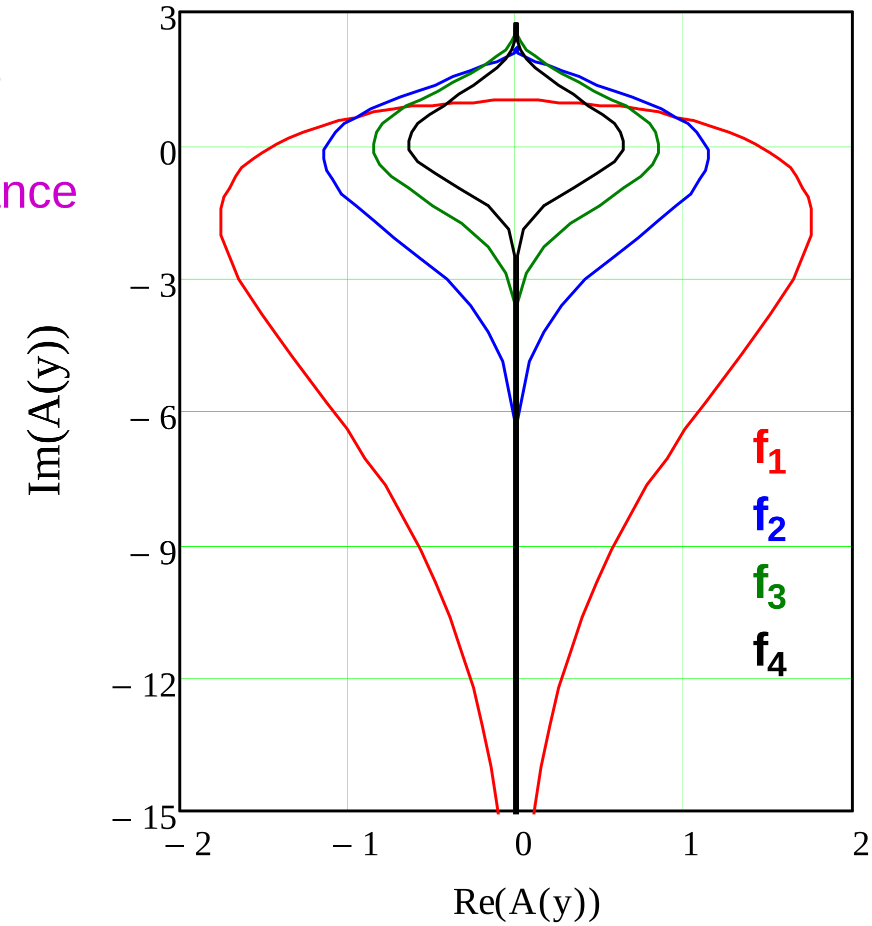


Stability Boundary (continue)

The beam is stable below transition ($\eta < 0$) if only the space charge impedance is present ($Z \propto i$)
For $|A| >> 1$ even a small resistive impedance breaks the beam stability

For simple estimate:

$$|A_n| \equiv \frac{eJ_0}{2\pi c p_0 \beta |\eta| (\sigma_p / p_0)^2} \left| \frac{Z_n}{n} \right| \leq 1$$



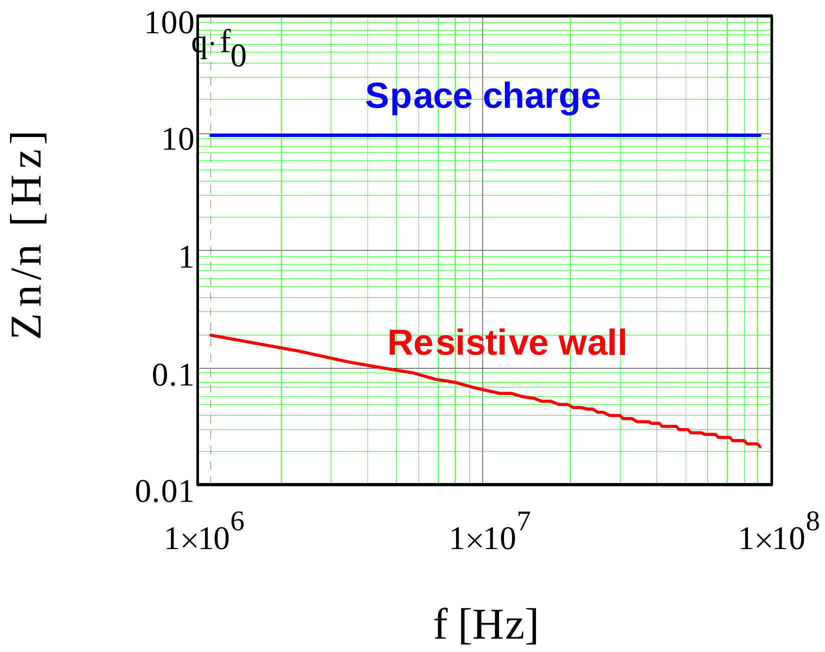
Longitudinal Impedance at Low Energies

- The space charge impedance Z_n/n is constant up to very high frequencies, $f \sim b/\gamma c$, and stability boundary does not depend on frequency
  Both Landau damping and instability growth rates grows proportionally to the frequency
- Space charge impedance
  For round beam & vacuum chamber with radius a

$$\frac{Z(\omega_n)}{n} = i\frac{Z_0}{\beta\gamma^2}\ln\left(\frac{a}{1.5\sigma}\right)$$

- Resistive wall impedance
  For round beam & vacuum chamber

$$\frac{Z(\omega)}{n} = (1 - i\,\mathrm{sign}(\omega))\frac{Z_0\beta c}{2a\sqrt{2\pi\sigma\omega}}$$



Copper chamber, f_0 = 1.13 MHz, a = 4.8 cm, E=8 GeV

***It is important to emphasize*:**

The instability damping appears if the coherent frequency falls to the range of incoherent frequencies of the particles:

$$k\Omega_{min} < \omega < k\Omega_{max}$$

Then phase velocity of the electromagnetic wave coincides with velocity of a group of particles.
Their intense interaction results in attenuation of the wave – and coherent beam perturbation.

This effect was predicted first by L. Landau for electromagnetic waves in plasma. Therefore, the term ***Landau damping*** is used in accelerator physics as well.

Growth Rates

- Well above the instability threshold the Landau damping can be neglected and the growth rates can be described by the model with zero momentum spread.

$$\delta\omega = \pm n\Omega_0 \sqrt{\frac{ie\eta J_0}{2\pi\beta c p_0}\left(\frac{Z_n}{n}\right)}$$

- The rate is proportional to frequency of the harmonic and can be very fast
- If momentum aperture is sufficiently large it does not kill the beam but results an increase of the momentum spread
- If growth rates are larger than the synchrotron frequency the continuous beam theory can be used for the bunched beam
  - Microwave instability
  - It can be damped by a damper at sufficiently low frequencies, < ~1 GHz, It is impossible to do at high frequencies, >~10 GHz

Bunched beam

Linearized Vlasov equation can be used for bunched beam as well, resulting in

$$\frac{df}{dt} = -eE(t,\theta)\,\frac{\partial F}{\partial p}$$

where total time derivative is calculated *along equilibrium phase trajectories*:

$$\frac{d}{dt} = \begin{cases} \partial/\partial t + \Omega(p)\,\partial/\partial\theta & \text{coasing beam} \\ \partial/\partial t + \Omega_s\,\partial/\partial\phi & \text{bunched beam} \end{cases}$$

φ and Ω_s are phase and frequency of synchrotron oscillations.

Therefore, at the functions $\propto exp(-i\omega t)$, the equation of *n*-th bunch is:

$$-i\omega f_n + \Omega_s\,\frac{\partial f_n}{\partial\phi} = -eE_n(\theta)\,\frac{\partial F_n}{\partial p}$$

Expansion $f_n = \sum f_{n,m} exp\,(im\omega)$ results in:

$$f_n = -ie\sum_m \frac{\exp(im\phi)}{\omega - m\Omega_s}\left\langle \exp(-im\phi)E_n(\theta)\,\frac{\partial F_n}{\partial p}\right\rangle$$

where <…> means average on synchrotron phase.

Then the bunch current is:

$$J_n = e\Omega \int f_n dp = -ie^2\Omega \sum_m \int \left\langle \exp(-im\phi) E_n(\theta) \frac{\partial F_n}{\partial p} \right\rangle \frac{\exp(im\phi)\, dp}{\omega - m\Omega_s}$$

Generally, field *E* depends on all the bunches and combines them in a series.

Its solution has two aspects: intra-bunch oscillations (``head-tail'' modes), and bunch-to-bunch ones (``collective modes'').

Short-range (wide-band) impedance affects only on intra-bunch modes, and long-range (narrow-band) one -- both on intra-bunch and collective modes.

General solution of this problem is unknown (in contrast with coasting beam). However, there is an easier and practically important case of space charge dominated impedance and parabolic potential well.

One can expect that it determines the instability threshold (like coasting beam).

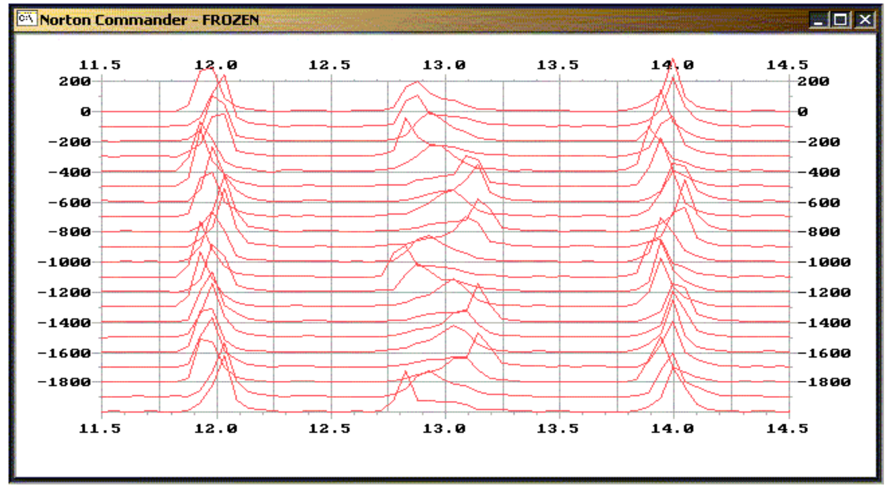
Complications with Bunched Beam Theory

- There are additional complications omitted in the above picture which make the problem even harder
  - Dependence of frequency on amplitude of synchrotron motion
  - Potential well distortions by wake-field of the bunch
- Landau damping for a given mode implies that the frequency of coherent motion is within frequencies of the incoherent motion
  - There is no tune spread in the parabolic potential well

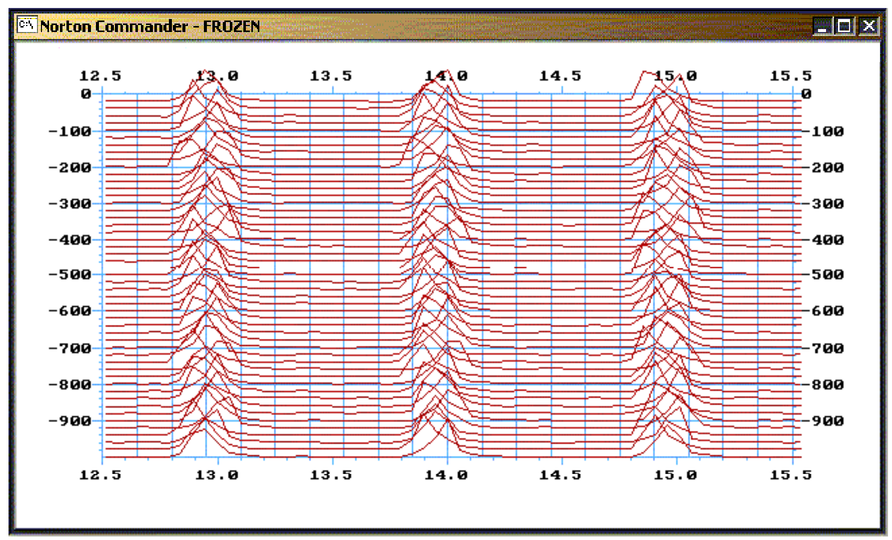
    => Mode 1 (the motion of the bunch as whole) is intrinsically unstable

“Dancing bunches” at Tevatron

Signal of the Tevatron BCM is shown. Three bunches (from 28 uncoalesced bunches at 150 GeV) execute independent coherent oscillations with different amplitudes and phases. There are about 2 periods in 42 ms.



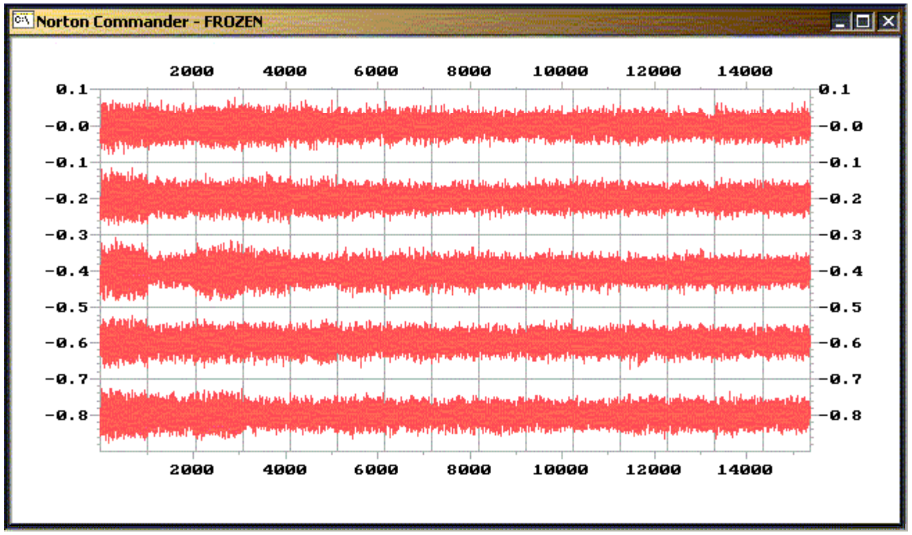
More long picture is presented here
It is wonderful that the amplitude does not change during 2 seconds, though estimated synchrotron frequency spread is about 1 Hz in each bunch.

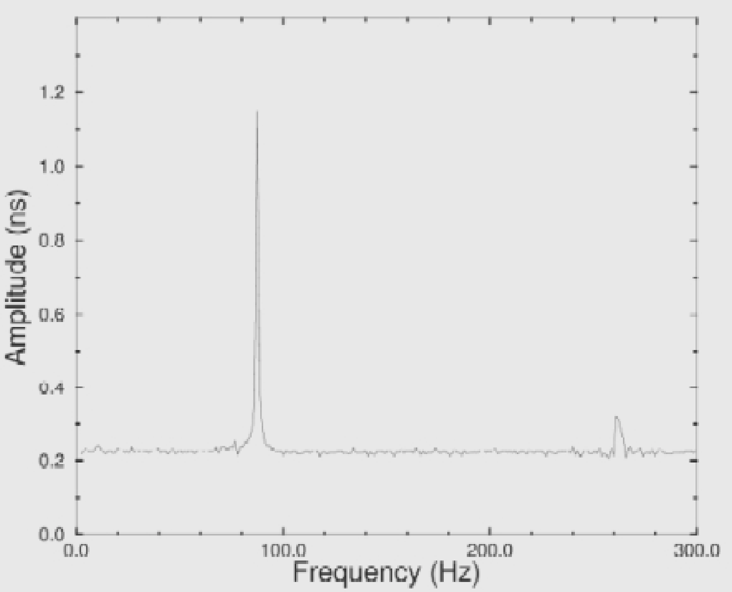


Measured positions of 5 bunch centers *during 15 min* are recreated in the picture.

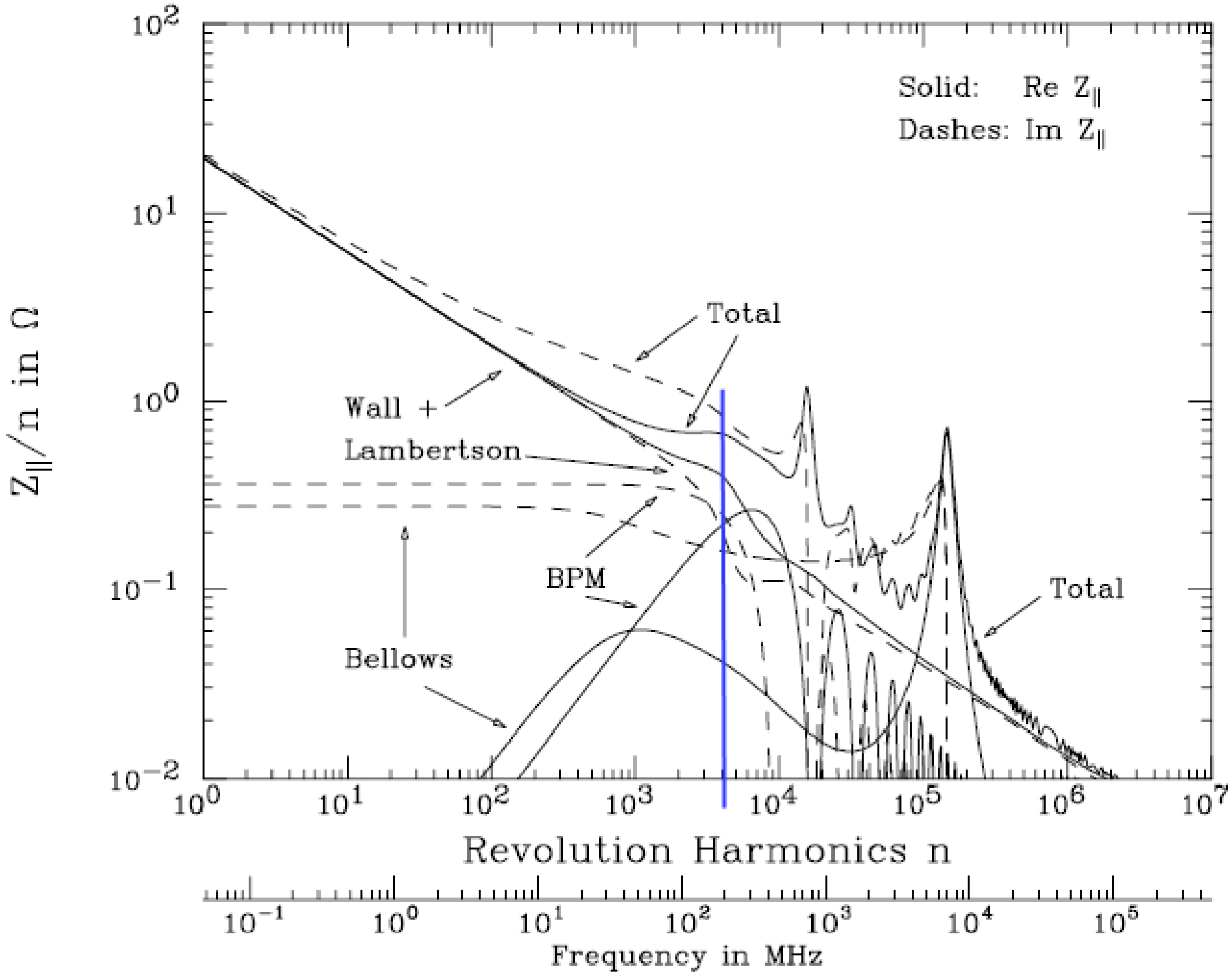
There are strong grounds to believe that the oscillations do not decay at all.

Bunches are weakly coupled resulting energy exchange between bunches at difference frequencies
(τ ~ 2-5 min while T_s ~ 30 ms)





FFT of an oscillating proton bunch centroid.



The real and imaginary parts of $Z_{\parallel}/n$ contributions to the Tevatron vacuum chamber. The capacitive parts are not shown.

At frequencies of interest, ~200 MHz, Im(Z) >> Re(Z)

Inductive impedance (BPMs, discontinuities) can cause the total depression of the decoherence.

The effective voltage induced by long. impedance, ~100 kV, is comparable to the RF voltage ~ 1 MV

$$V = ZI \approx \left(\frac{Z_n}{n}\right)\frac{eN}{\sqrt{2\pi}\sigma_s} n \propto \frac{1}{\Delta\phi^2}, \quad n \approx \left(\frac{C}{5\sigma_s}\right)$$

The coherent tune shift

$$\Omega_c \propto \frac{dV}{ds} \propto \frac{1}{\Delta\phi^3}$$

Comparing it to the synchrotron tune spread
We obtain instability threshold $N \sim \Delta\phi^5$

$$\delta\Omega_s \approx \Omega_s \frac{\Delta\phi^2}{16}$$

For the Tevatron, corresponding condition is: (*N* is number particles per bunch, $2\Delta\varphi$ is the bunch total length in RF radians)

$$\frac{N}{\Delta\phi^5} > 10^{13}$$

Experimental data:
$N_{threshold}=10^{10}$ at $\Delta\varphi=0.2\text{-}0.3$

$$\frac{N}{\Delta\phi^5} = (0.4 - 3) \times 10^{13}$$

Taking into account strong dependence of the criterion on the bunch length, is would be very hard to hope for better agreement.

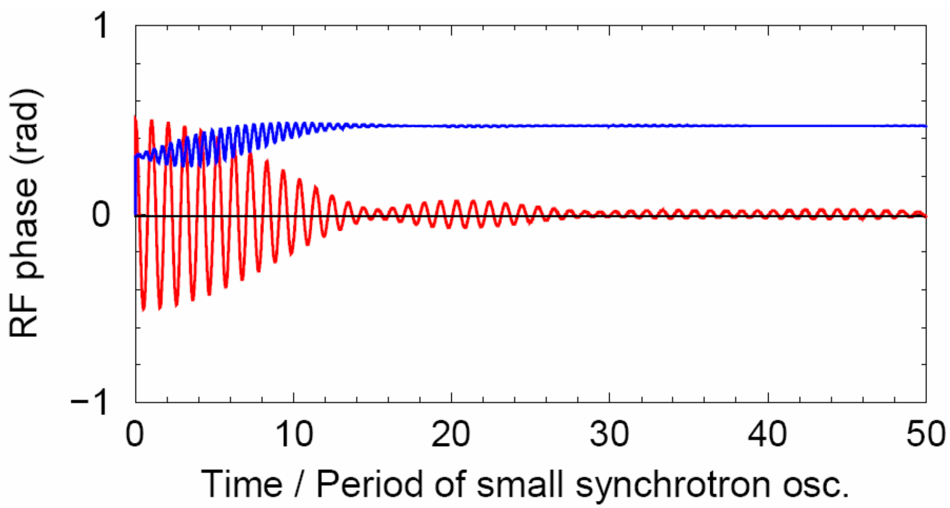
Numerical modeling

Bunch current considered
$$J(\phi) = J_{max}\left(1 - \frac{(\phi-\phi_0)^2}{\Delta\phi^2}\right)$$

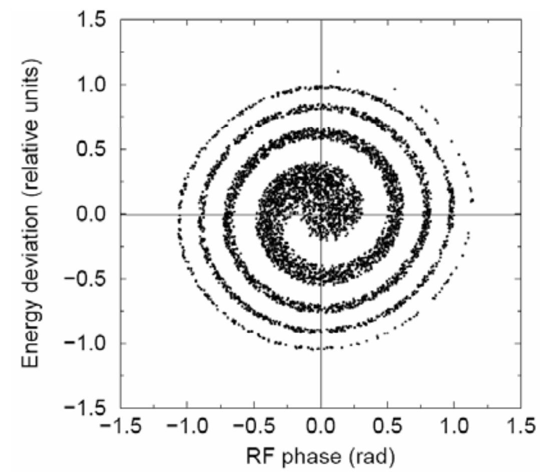
with unknown time-dependent parameters φ_0 and $\Delta\varphi$.

Its electric field is calculated as above, and external field is taken $\infty \sin\varphi$.

Below: the bunch evolution at: $Jmax = 0$, initial sizes $\varphi_0 = \Delta\varphi = 0.5$,



The bunch evolution: red φ_0 blue $\Delta\varphi$.



Lamination after 50 synchrotron periods

So space-charge-like impedance can explain why the bunch oscillations do not damped

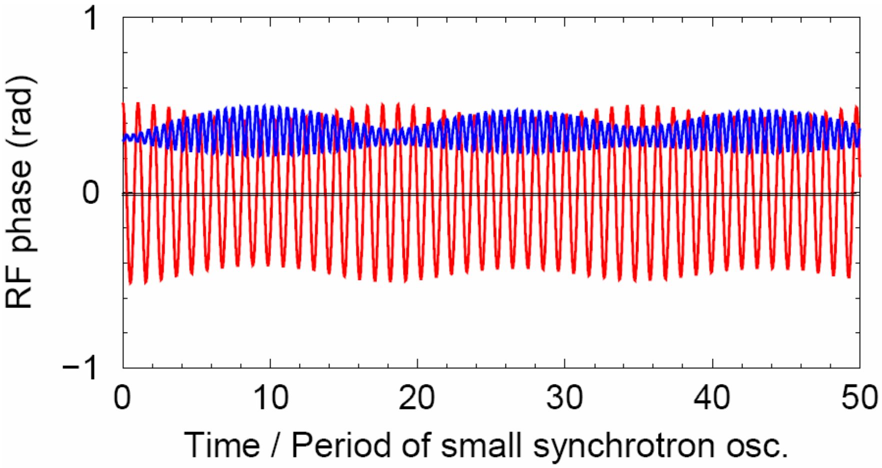
But it is not quite clearly why they do not grow?

It is apparently that the impedance has real part which should provoke an instability above the threshold.

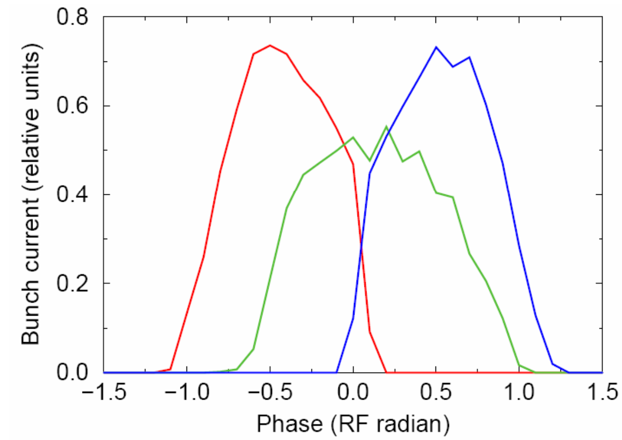
Possibly, it can be explained by very strong dependence of the threshold on the bunch length – $N_{thresh} \propto \Delta\varphi^5$.

Then the bunch lengthening at the instability brings the bunch to the threshold stopping the instability.

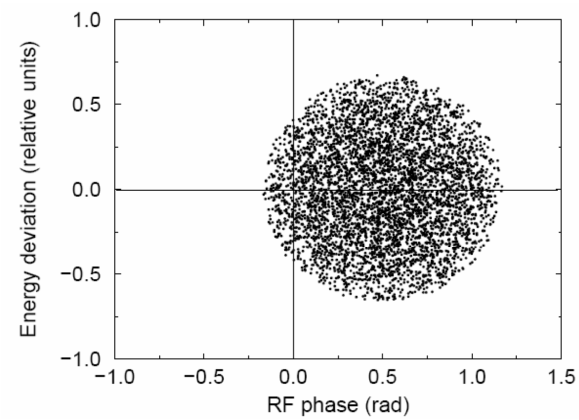
Above the threshold: no damping, no lamination

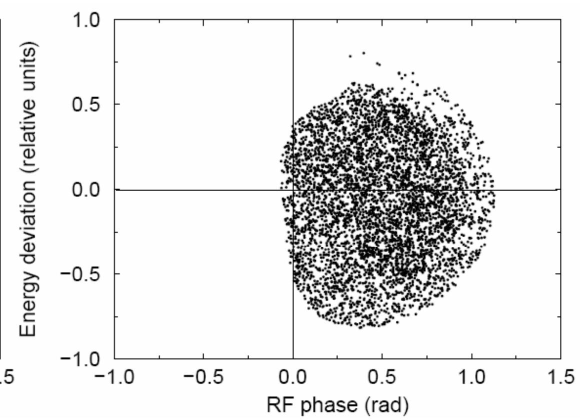


The bunch evolution: red φ_0 blue $\Delta\varphi$.



The bunch profile: 3 limit positions





Phase space: left – in the beginning, right -- after 50 synchrotron periods

Backup viewgraphs

Bunched beam instability

We consider this case more tightly writing for shortness $Z_k = i\,k\mathcal{Z}$.

$$E_n(\theta) = -\frac{\mathcal{Z} J_n'(\theta)}{2\pi R}$$

It is really short-range impedance:
Therefore series brakes up into a set of independent equations.

The problem is simplified if only one term gives a major contribution. It is the term $m=1$ if field $E_n(\theta)$ is about constant in the bunch.

$$J(\theta) = \frac{e^2 \Omega \mathcal{Z} |\eta|}{2\pi R\,\eta} \int \left\langle \frac{\partial F}{\partial p} J'(\theta) \sin\phi \right\rangle \frac{\cos\phi\, dp}{\omega - \Omega_s}$$

After rearrangement, it gives

$$J(\theta) = \frac{e^2 \Omega \mathcal{Z} |\eta| \theta}{\pi^2 R\,\eta} \int\limits_{|\theta|}^{\infty} \frac{F'(A)\, dA}{A^2 \sqrt{A^2 - \theta^2}(\omega - \Omega_s)} \int\limits_{-A}^{A} J'(\theta') \sqrt{A^2 - \theta'^2}\, d\theta'$$

It is exact relation if J is an eigen-function.

$$\int\limits_{-\infty}^{\infty} J^2(\theta)\, d\theta = \frac{e^2 \Omega \mathcal{Z} |\eta|}{\pi^2 R\,\eta} \int\limits_{0}^{\infty} \frac{F'(A)\, dA}{A^2(\omega - \Omega_s)} \left[\int\limits_{-A}^{A} J'(\theta) \sqrt{A^2 - \theta^2}\, d\theta \right]^2$$

But it is rather well approximation with relevant trial function:
$J = \theta$ at $|\theta| < \Delta\theta$:

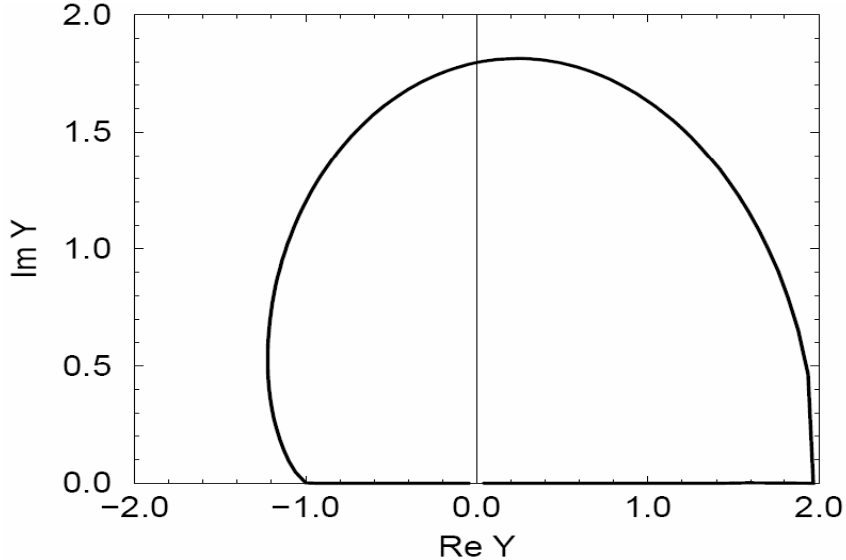
$$1 = \frac{3 e^2 \Omega \mathcal{Z} |\eta|}{8 R \Delta\theta^3 \eta} \int\limits_{0}^{\Delta\theta} \frac{F'(A) A^2\, dA}{\omega - \Omega_s}$$

If synchrotron frequency does not depend on amplitude

$$\omega = \Omega_s - \frac{3e^2 \Omega N \mathcal{Z} |\eta|}{8\pi R \Delta\theta^2 \Delta p\, \eta}$$

More generally, one can solve the equation graphically, like coasting beam problem. Threshold maps are plotted for the distribution function:

$$F = \frac{8 J_{max}}{3\pi e \Omega \Delta p} \left(1 - \frac{A^2}{\Delta\theta^2}\right)^{3/2}$$



According them, instability threshold is (Δp and $\Delta\Omega_s$ are maximal spreads):

$$e J_{max} |\mathcal{Z}| > E\beta^2 |\eta| \left(\frac{\Delta p}{p}\right)^2 \frac{\Delta\Omega_s}{\Omega_s} \times \begin{cases} \pi/2 & \eta > 0 \\ \pi & \eta < 0 \end{cases}$$

Undamped oscillations are possible at these conditions at real $\mathcal{Z}$ (that is at purely imaginary impedance Z_k), and instability is possible if there is a small imaginary addition to $\mathcal{Z}$ (real addition to Z_k).