A Model Ring With Exactly Solvable Nonlinear Motion

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1. Introduction

Recently, a concept of nonlinear accelerator lattices with two analytic invariants has been proposed [1]. Based on further studies [2], the Integrable Optics Test Accelerator (IOTA) was designed and and is being constructed at the Fermi National Accelerator Laboratory. Sush a nonlinear lattice may be helpful in suppression of the collective instabilities by introducing a relatively large tune spread in a beam, while reducing phase-space area occupied by chaotic trajectories.

[1] V. Danilov and S. Nagaitsev, *"Nonlinear Accelerator Lattices with One and Two Analytic Invariants"*, **Phys. Rev. ST Accel. Beams 13**, 084002 (2010).
http://prst-ab.aps.org/abstract/PRSTAB/v13/i8/e084002

[2] *"Proposal for an Accelerator R&D User Facility at Fermilab's Advanced Superconducting Test Accelerator (ASTA)"*.
http://apc.fnal.gov/programs2/ASTA_TEMP/index.shtml

Example: 1-D Nonautonomus Henon Map

Even a simple 1-D model of sextupole does not provide integrability:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ \epsilon\, 3\, q & 1 \end{pmatrix}$$

Video: Stroboscopic Poincare cross-section shows the shrink of phase space volume with regular trajectories due to increasing strength of nonlinearity.

2. Concept of 2D paraxial nonlinear integrable optics

Desired spread of frequencies can be achieved by adding to the Hamiltonian an additional nonlinear potential:

$$\mathcal{K}[p_x, p_z, x, z; s] = \underbrace{\sum_{q=x,z} \left[\frac{p_q^2}{2} + g_q(s) \frac{q^2}{2} \right]}_{\mathcal{K}_0[p_x, p_z, x, z; s]} + \boxed{V(x, z, s)}.$$

In general, the new equations of motion do not necessarily provide two (and even a one) analytic invariants as it was in the case of linear lattice. Below, we will consider one of the possibile ways of how to modify a paraxial Hamiltonian preserving the integrability at the same time **[Danilov, Nagaitsev]**.

2.1 First integral of motion

Consider an accelerator lattice which provides axially symmetric linear focusing, i.e. $\forall\, s : g_x(s) = g_z(s)$

Step 1: Change of independent variable

The betatron phase advance $\psi(s)$ can be choosen as a new independent variable. The new Hamiltonian $\tilde{\mathcal{K}}$ would yield the same physical motion as $\mathcal{K}$ if their gradients are proportional:

$$\frac{\partial \tilde{\mathcal{K}}}{\partial p_q} = \lambda^{-1}(s) \frac{\partial \mathcal{K}}{\partial p_q}, \qquad \frac{\partial \tilde{\mathcal{K}}}{\partial q} = \lambda^{-1}(s) \frac{\partial \mathcal{K}}{\partial q}, \qquad \text{which gives}$$

$$\tilde{\mathcal{K}}[p_x, p_z, x, z; \psi] = \beta\,[s(\psi)]\; \mathcal{K}[p_x, p_z, x, z; s(\psi)].$$

Step 2: Transformation to normalized coordinates

Subsequent canonical transformation, $(p, q) \to (\mathcal{P}_q, \eta_q)$, moves the time dependence into the nonlinear term.

$$\eta_q = q/\sqrt{\beta_q},$$

$$\mathcal{P}_q = p_q\sqrt{\beta_q} - q\,\frac{\beta_q'}{2\,\beta_q^{3/2}},$$

where $' \stackrel{\text{def}}{=} d/d\psi$.

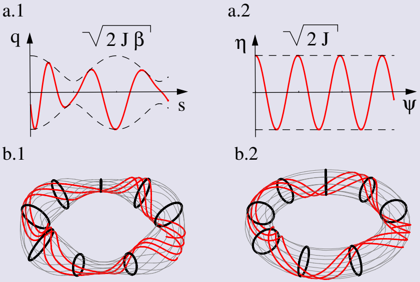


Figure: Particle trajectory (a.1,2) and the evolution of the phase-space ellipse along the accelerator circumference (b.1,2) in old and new canonical variables respectively.

Step 3: Special “time”-dependence

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + \beta\left[s(\psi)\right]\, V(\mathbf{q}(\eta, \psi), \psi)$$

One can see that at least one integral of motion, the Hamiltonian by itself, can be ensured by a special “time”-dependence of the non-linear potential which compensate a modulation by the β-function:

$$\boxed{\beta\left[s(\psi)\right]\, V(\mathbf{q}(\eta, \psi), \psi) = U(\eta_x, \eta_z).}$$

2.2 Second integral of motion

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + U(\eta_x, \eta_y)$$

A presence of a second integral can be guaranteed by the choice of new generalized coordinates where variables can be separated.

Harmonic condition

Additional constraint on a potential $U(\eta_x, \eta_z)$ to satisfies the Laplas equation essentially reduce the number of possible choises among the whole possible functions: only three different families of such a integrable lattices were found for the invariant in the form

$$W = A(x, z)\mathcal{P}_x^2 + B(x, z)\mathcal{P}_x\mathcal{P}_z + C(x, z)\mathcal{P}_z^2 + D(x, z).$$

3. Variables separation in polar coordinates

In the normalized polar coordinates (r, θ):

$$\eta_x = r\cos\theta, \qquad \mathcal{P}_x = p_r\cos\theta - \frac{p_\theta}{r}\sin\theta,$$
$$\eta_z = r\sin\theta, \qquad \mathcal{P}_z = p_r\sin\theta + \frac{p_\theta}{r}\cos\theta,$$

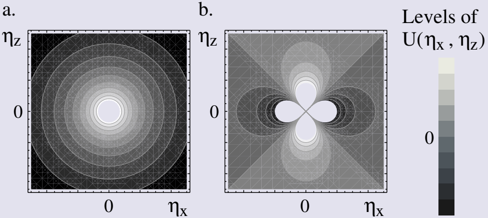
the variables separation is possible for the potentials in the form:

$$U(r,\theta) = f(r) + \frac{h(\theta)}{r^2}$$

where $f(r)$ and $h(\theta)$ are arbitrary functions.

Harmonic potentials in polar normalized coordinates

- $B \ln r$ — potential of the straight wire carries constant current **(special "time"-dependence can not be ensured)**
- $\dfrac{A \sin(2\theta + \varphi)}{r^2}$ — point-like magnetic quadrupole **(s-independent potential remains so after the transformation to the normalized coordinates and time)**



4. Transverse motion model

Finally we have a Hamiltonian

$$\mathcal{H}[p_r, p_\theta, r, \theta; \psi] = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{r^2}{2} + \frac{A\sin(2\theta + \varphi)}{r^2},$$

with two invariants of motion:

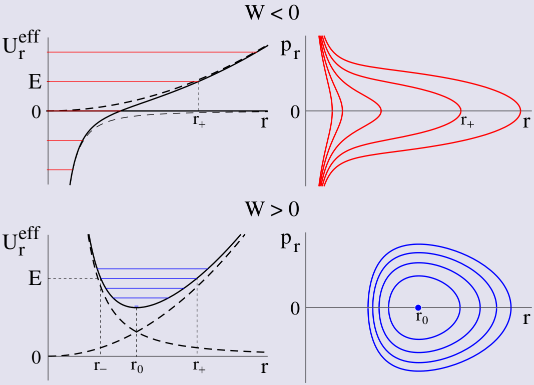
- energy

$$\boxed{E = \frac{p_r^2 + r^2}{2} + \frac{W}{r^2}}$$

- effective angular momentum

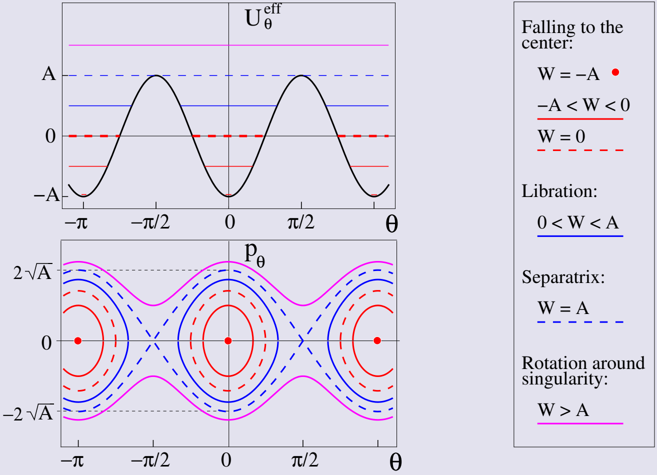
$$\boxed{W = \frac{p_\theta^2}{2} + A\sin(2\theta + \varphi)}$$

Radial motion



$$J_r(E) = \frac{1}{2\pi} \oint p_r \, dr = \frac{E - \sqrt{2W}}{2} \qquad \omega_r = \frac{\partial \mathcal{H}}{\partial J_r} = 2$$

Angular motion



Trajectories classification

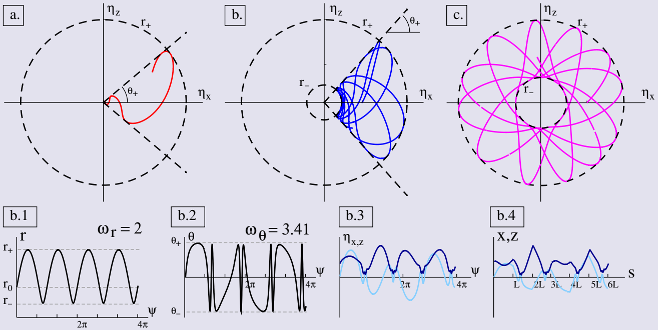
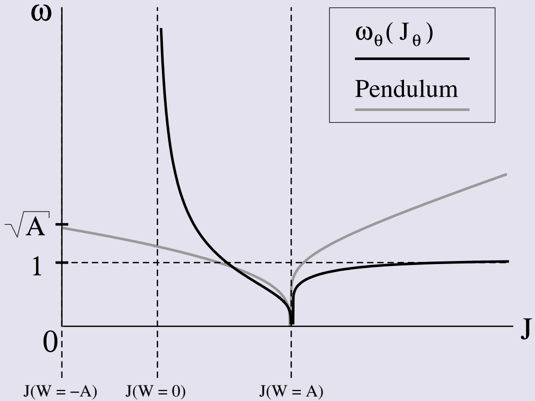


Figure: Particle trajectory in the normalized coordinates for
(a.) falling to the center ($-A < W < 0$)
(b.) libration ($0 < W < A$)
(c.) rotation around the origin ($W > A$).

Frequency dependence of the amplitude for the angular motion



$$\omega_\theta = \frac{1}{\sqrt{2W}} \left(\frac{\partial J_\theta}{\partial W} \right)^{-1} = \frac{1}{\sqrt{2W}} \times \begin{cases} 2\ \omega_{\text{pend}} \\ \omega_{\text{pend}} \end{cases},$$

5. Model ring

Guide to the design

- The use of light particles, like e^-, is as impractical since they will be lost on the inner apperture; thus, below we will consider the design of an accelerator ring for protons, since radiation effects are negligible for them.
- The concept of nonlinear integrable optics under consideration requires an axially symmetric focusing in the accelerator ring. A superperiod of such a lattice can be realized with a drift sapace of length L, where the nonlinear lens is located, and an optics insert (so-called T-insert), which is equivalent to the thin axially symmetric lens with the focal length equal to $1/k$.

For all further simulations we will use a parameters designed for IOTA ring:

Linear Lattice Parameters		
# of superperiods		4
# of nonlinear lenses		2
Circumference, Π	(m)	38.7
Bending diploe field, B	(T)	0.7
Drift space length, L	(cm)	200
T-insert strength parameter, k	(cm^{-1})	$\in (0; 0.02)$

Beam at the Injection		
Beam full energy, $\mathcal{E}_{\mathrm{eq}}$	(MeV)	938.75
Full momentum, P_{eq}	(MeV/c)	60
Normalized emittance, $\epsilon^{\perp}_{\mathrm{norm}}$	(cm rad)	2×10^{-5}

IOTA Layout

$\beta_{\min} \approx 100$ cm, $\beta_{\max} \approx 200$ cm, $\nu \in (0; 2)$

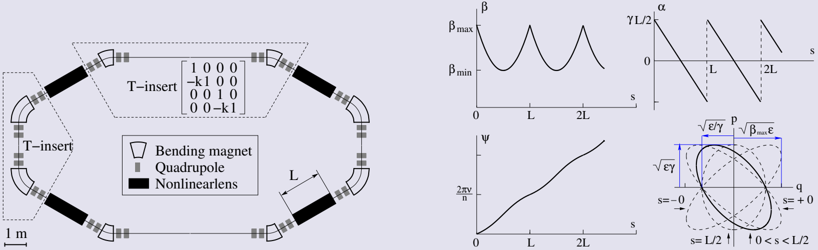
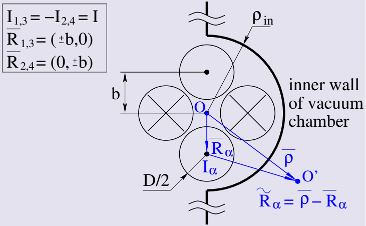


Figure: IOTA ring layout (left) and optical functions behavior in $\frac{F}{2}O\frac{F}{2}$ lattice.

Nonlinear lens parameters

The design of proposed nonlinear lenses bring in two major inevitable perturbations. They are associated with the special longitudinal dependence of the field, and, the physical realization of poles of the lens.



$$
\begin{aligned}
A_s &= -\frac{\mu_0}{2\pi} \sum_{\alpha=1,2,3,4} I_\alpha \ln |\widetilde{R}_\alpha| \\
&= \frac{\mu_0 I}{\pi} \left(b^2 \frac{\cos 2\theta}{\rho^2} + \frac{b^6}{3} \frac{\cos 6\theta}{\rho^6} + O\left[\left(\frac{b}{\rho} \right)^{10} \right] \right)
\end{aligned}
$$

	Supercond.	Water cooling
Full momentum		
P_{eq}, (MeV/c)	60	30
Diametr of the wire		
D, (mm)	6	7
Current density		
ρ_I, (A/mm^2)	100	10
Total current		
I, (A)	2827	385
Inner radius of vacuum pipe		
ρ_{in}, (cm)	0.85	1
Outer radius of vacuum pipe		
ρ_{out}, (cm)	4	4
Strength of nonlinear lens		
A, (cm^2)	10.8×10^{-4}	4×10^{-4}

6. Numerical integration

The use of a simple numerical integrators leads to the smear of the trajectory and requires a very large number of slices.

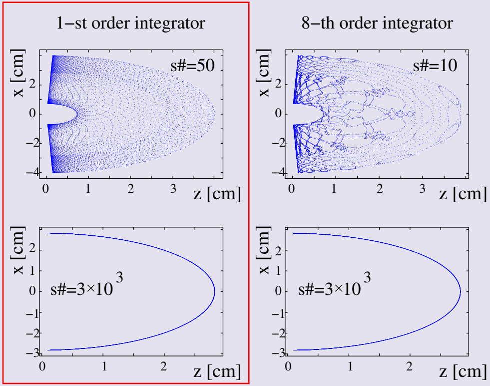


Figure: Turn-by-turn map for betatron frequency $\nu = 0.5$, when particle should have the same radial displacement at fixed observation point.

Explicit symplectic integrators and Yoshida's algorithm

Consider the Hamiltonian which can be split into N solvable parts $H(\mathbf{q},\mathbf{p}) = H_1 + \ldots + H_N$ and has a Lie map $M(t) = \exp(t : -H :)$.

• Therefore, one can construct a second-order integrator using a symmetrized Lie map product (where $K_k = \exp(t : -H_k :)$):

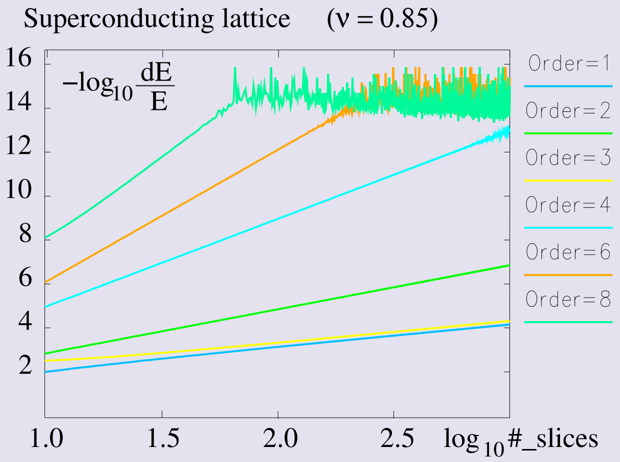
$$M_2 = K_1(t/2) \ldots K_N(t) \ldots K_1(t/2) = M(t) + o(t^3).$$

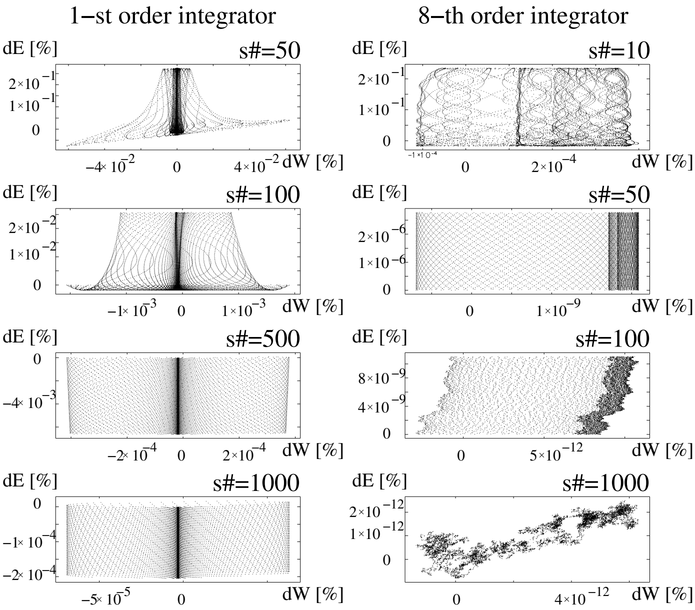
• Yoshida's algorithm allows the recursive construction of $(2n+2)$-th-order integrator as:

$$\boxed{M_{2n+2}(t) = M_{2n}(\chi_1 t) M_{2n}(\chi_0 t) M_{2n}(\chi_1 t)},$$

$$\text{where } \chi_0 = -\frac{2^{1/(2n+1)}}{2 - 2^{1/(2n+1)}} \quad \text{and} \quad \chi_1 = \frac{1}{2 - 2^{1/(2n+1)}}.$$

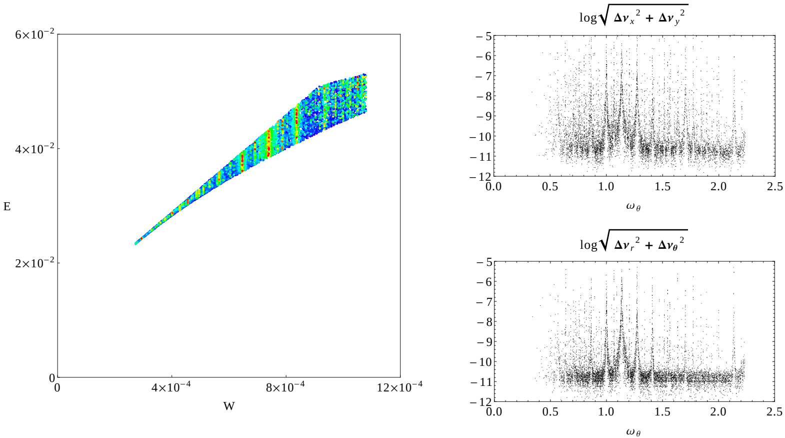
Rate of convergence



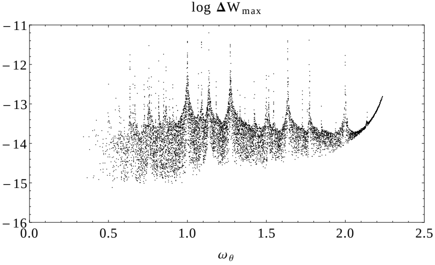

1−st order integrator
8−th order integrator
dE [%]
s#=50
s#=10
dW [%]
s#=100
s#=50
s#=500
s#=100
s#=1000
s#=1000

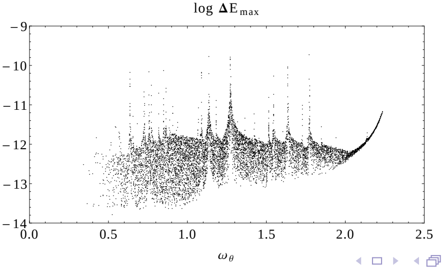
7. Simulation of a monochromatic motion

Frequency Map Analysis (ideal kick):



Diffusion of invariants (ideal kick):





Beam motion (exact kick):

Poincare sections (exact kick):

8. Results and Conclusions

1. The first paraxial nonlinear exactly integrable system has been studied: analytical expressions for dynamical variables change over the time as well as amplitude dependence of frequency are obtained.
2. The possibility to create such a nonlinear lens were demonstrated on the example of IOTA ring for 60 MeV protons.
3. Numerical methods for the simulation of perturbed nonlinear system were discussed (requires further study).
4. This system is of particular interest since it has an unusual feature for accelerator physics: it has no equilibrium orbit. In addition, this system is interesting in that it has the degeneracy.

Thank you for your attention.