# *Matching Beams for the Nonlinear Integrable Optics*

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## *Linear Matching & Its Generalization*

Generating matched distributions for the nonlinear integrable optics is fundamentally different from a matched beam in linear lattices. This topic has been discussed in several locations[1]. Because it is such an important aspect of simulating the integrable optics, it deserves its own technical note. Much of this is based on the work in the 2015 IPAC proceeding.

[1] Stephen D. Webb, David L. Bruhwiler, Dan T. Abell, Kirill Danilov, John R. Cary, Sergei Nagaitsev, Alexander Valishev, Viatcheslav Danilov, and Andrei Shishlo. Simulating High-Intensity Proton Beams in Nonlinear Lattices with PyORBIT. In *Proceedings of IPAC 2012*, number WEPPR012, 2012a; Stephen D. Webb, David L. Bruhwiler, Sergei Nagaitsev, Viatcheslav Danilov, Alexander Valishev, Dan T. Abell, Andrei Shishlo, Kirill Danilov, and John R. Cary. Suppressing Transverse Beam Halo with Nonlinear Magnetic Fields. In *Proceedings of IPAC 2013*, number THYB101, 2013; Stephen D. Webb, David L. Bruhwiler, Sergei Nagaitsev, Alexander Valishev, Viatcheslav Danilov, and Rami A. Kishek. Experiences Simulating Nonlinear Integrable Optics. In *Proceedings of IPAC 2015*, number MOPMA029, 2015; and Stephen D. Webb, David L. Bruhwiler, Dan T. Abell, Andrei Shishlo, Viatcheslav Danilov, Sergei Nagaitsev, Alexander Valishev, Kirill Danilov, and John R. Cary. Effects of nonlinear decoherence on halo formation. arXiv:1205:7083, 2012b

Beam matching in single-particle linear lattices is frequently understood as producing beams with the proper RMS shapes. This is correct for linear lattices because the Hamiltonian for such a system is given by

$$H = \frac{1}{2}\left(\hat{p}^2 + \hat{q}^2\right) \tag{1}$$

in the normalized coördinates. Because $H$ is an invariant of the single-turn map, and is also a quadratic form in the $p$'s and $q$'s, its distribution can be specified entirely in terms of RMS quantities. On a related note, the Courant-Snyder invariant $J = 2H$. Again, because $J$ is a quadratic form in the coördinates and momenta, the RMS quantities of an ensemble are conserved.

We can thus understand the vertical and horizontal emittance, $\epsilon_x$ and $\epsilon_y$, as being related to the average of single-particle Courant-Snyder invariants $J_x$ and $J_y$, viz. $\epsilon_i = \langle J_i \rangle$. Thus, a gaussian distribution would be given by

$$f(J_x, J_y) = \frac{N}{\epsilon_x \epsilon_y} e^{-J_x/\epsilon_x} e^{-J_y/\epsilon_y}. \tag{2}$$

These are in the normalized coördinates, given by

$$\hat{q}_i = \frac{q_i}{\sqrt{\beta_i(s)}} \tag{3a}$$

$$\hat{p}_i = \sqrt{\beta_i(s)}p_i + \frac{\alpha_i(s)}{\sqrt{\beta_i(s)}}q_i \tag{3b}$$

for the Twiss $\alpha$'s and $\beta$'s. The action-angle variables are then defined as

$$\hat{z}_i = \sqrt{2J_i}\cos\Psi_i \tag{4a}$$

$$\hat{p}_i = \sqrt{2J_i}\sin\Psi_i \tag{4b}$$

and thus the Hamiltonian becomes $H = \sum_i \nu_i J_i$ for the single particle tunes $\nu_i$.

The reason we can define independent vertical and horizontal emittances is that $J_x$ and $J_y$ are independent variables, and that (in the uncoupled case) the two transverse degrees of freedom factor into two one-dimensional Hamiltonians. We can then define the vertical and horizontal emittances in terms of the individual action variables, which are conserved quantities in the linear limit.

This is what beam matching is: making the particle phase space distribution a pure function of some set of invariants of the motion. Because these invariants are defined in terms of the lattice Twiss parameters, they are intimately related to the process. The "beam Twiss parameters" defined in terms of RMS quantities make sense for linear lattices because the invariants are quadratic forms of the canonical coördinates and momenta, and so the quantity $\sigma_{\hat{p}}^2 + \sigma_{\hat{q}}^2$ will be conserved and directly related to the Twiss parameters.

FOR THE NONLINEAR INTEGRABLE OPTICS[2], NONE OF THIS IS TRUE ANYMORE. The Hamiltonian strongly couples the transverse degrees of freedom, and so attempting to inject a beam matched to the linear lattice in the usual way will quickly evolve[3] into a distribution very different from a typical linear matched. distribution. This assumes the distribution places all the particles inside the hyperbolic fixed points of the elliptic potential. Particles starting outside those points will sweep to infinity/be lost to the beam pipe.

[2] V. Danilov and S. Nagaitsev. Nonlinear accelerator lattices with one and two analytic invariants. *Phys. Rev. ST Acc. Beams*, 13(084002), 2010

[3] Strictly speaking, filament, due to the large tune spread in these lattices. The beam envelope does not "breath" like with linear mismatch, again due to the large tune spread.

This is a problem for understanding tracking results – without a properly matched distribution, it is impossible to separate the mismatch dynamics from any actual loss of stability due to external perturbations. However, if we understand beam matching as creating a distribution that is a pure function of the invariants of the motion, we can quickly generalize our notion of matching and produce matched beams.

Danilov and Nagaitsev provided a transverse Hamiltonian with two invariants of the motion. In the work cited above, we used the Hamiltonian

$$H = \frac{1}{2}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{x}^2 + \hat{y}^2\right) + tV(\hat{x}, \hat{y}) \tag{5}$$

as our choice invariant. We could also have used $I_1$ and $I_2$ to define the distribution, but the Hamiltonian has a particular advantage. For the generalization of the K.-V. distribution[4], with $f(H) = \delta(H - \epsilon)$, the beam edge is defined by

$$\frac{1}{2}(\hat{x}^2 + \hat{y}^2) + tV(\hat{x}, \hat{y}) = \epsilon. \tag{6}$$

[4] I. M. Kapchinskij and V. V. Vladimirskij. Limitations of proton beam current in a strong focusing linear accelerator associated with the beam space charge. In *Proc. of Int'l. Conf. on High Energy Acc.*, pages 274–288. CERN, 1959

In this context, $\epsilon$ is the generalized emittance of the beam. It has the same properties of the emittance for traditional strong focusing lattices: it is related to the beam envelope size, has the same units of action (m-rad), and is an invariant turn-by-turn for an ideal lattice. To generate a matched distribution, we compute macroparticle initial conditions that are a pure function of the Hamiltonian.

## *An Algorithm for Generating a Matching Distribution*

THE STARTING POINTS FOR THE MATCHED DISTRIBUTION IS THE K.-V. DISTRIBUTION, which is a delta function in the action:

$$f(\hat{p}, \hat{q}) = \delta(H(\hat{p}, \hat{q}) - \epsilon). \tag{7}$$

The simplest way to do this is two-fold. First, develop a bounding box on the $\hat{x}$-$\hat{y}$ plane and randomly and uniformly select particle coördinates in that area. If

$$\frac{1}{2}(\hat{x}^2 + \hat{y}^2) + tV(\hat{x}, \hat{y}) \leq \epsilon \tag{8}$$

then the particle is "allowed" and kept. Otherwise the point is discarded. The momentum can be decomposed into polar coördinates, so that

$$\hat{p}_x = \hat{P} \cos\theta \tag{9a}$$

$$\hat{p}_y = \hat{P} \sin\theta \tag{9b}$$

Taking the difference so that

$$\hat{P} = \sqrt{2\left(\epsilon - \left[\frac{1}{2}(\hat{x}^2 + \hat{y}^2) + tV(\hat{x}, \hat{y})\right]\right)} \tag{10}$$

and $\theta$ is selected uniformly, we can compute the $\hat{p}_x$-$\hat{p}_y$ distribution accordingly from the (now determined) $\hat{P}$ and the (randomly generated) $\theta$. This will generate a K.-V.-like distribution, which is unphysical but a good start.

To compute a more general distribution, note that any distribution can be written

$$f(H) = \int d\epsilon' \, f(\epsilon')\delta(H - \epsilon'). \tag{11}$$

By approximating the integral as a Riemann sum, we can compute any arbitrary distribution as the sum of K.-V. distributions. Thus, one picks a $\Delta\epsilon$ that is small compared to the emittance parameter for the beam (i.e. for an exponential $f \propto e^{-H/\epsilon_0}$ then $\Delta\epsilon \ll \epsilon_0$) and rewrites the Riemann sum as

$$f(H) \approx \sum_n \Delta\epsilon N_{tot.} f_{K.-V.}((n + 1/2)\Delta\epsilon) \tag{12}$$

for a "second order" in $\Delta\epsilon$ approximation of the distribution, where $N_{tot.}$ is the total number of (macro-) particles and $f_{K.V.}$ is a delta function distribution normalized to unity. One may want to apply a correcting factor for getting a proper fixed number of macroparticles to account for the error in the method.

Thus, under this method we can generate any distribution by generating $N_{tot.}f((n+1/2)\Delta\epsilon)$ macroparticles for each K.-V. distribution with emittance $\epsilon = (n+1/2)\Delta\epsilon$ using the above algorithm. This will allow us to generate more physical distributions, such as gaussians or waterbags.

## *Additional Tools*

Because $H$ and $I$ are invariants of the ideal motion, their behavior can be used as a diagnostic of the single-particle dynamics which is more revealing than the RMS beam sizes in phase space may be. During the Phase I SBIR that RadiaSoft worked on for this project, such tools were developed. A Python class which computes the invariants and the single-particle kick is available through GitHub (`https://gist.github.com/afda4f4eb3d09d46a4fb`). Other tools which build on these three functions are available upon request (swebb_at_radiasoft.net).

## *References*

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