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**New Formulae for 3DoF Space Charge Field**

Y. Alexahin (FNAL APC)

There is a large number of programs for space charge effect simulations using various methods to calculate the space charge field which can be divided in two main groups: the first uses actual particle distribution obtained on the preceding step of simulations (e.g. PIC, multipole expansion) while the other relies on a smooth approximation (e.g. Gaussian) of particle distribution and analytical formulas for the field.

Here we present new formulas long bunches with charge density described by

. (1)

where *λ* is linear charge density (not necessarily Gaussian).

For a long bunch, *σz* >> max(*σx*, *σy*), we can neglect variation of *λ*(*z*) and use formulas for two-dimensional charge distribution. Then for the transverse electric field the well-known Basetti-Erskine formula [1] can be used.

In this approximation the transverse field (and the associated kick) is proportional to the charge density at the particle location, *λ*(*z*). For symplecticity of the 6D transfer map it must be complemented by a longitudinal kick dependent on the transverse coordinates.

The symplecticity will be guaranteed if the field components are derived from the same potential. In the present report we give the space charge potential of a (transversely) Gaussian bunch in a convenient form for numerical calculations and also provide an alternative to the Erskine-Basetti formula. This approach was successfully used in the beam-beam effect analyses [2].

In the long bunch approximation the time retardation can also be neglected to give direct space charge potential in the form[[1]](#footnote-1)

. (2)

where the Green function can be presented as

 , (3)

*r* =*σy* /*σx*, so far this choice being arbitrary.

Performing in eq.(2) integration by transverse variables we obtain

. (4)

Making use of the formula

 (5)

with subsequent integration by *k*1,2 and finally setting τ = (*t*‑1‑1)/2*r*2 we get

. (6)

We deliberately have chosen an asymmetric form to make eq.(6) easier to treat at least in some cases (see later).

Thus far we have not paid attention to the logarithmic divergence of the integral (6) at *t*=0: it disappears in the formula for the transverse field. But if we need the longitudinal field we have to eliminate this divergence.

The simplest way to regularize the potential (6) is by subtracting unity from the exponential in the integrand to obtain

. (7)

This regularization does not affect the transverse field at all and makes the potential (hence the longitudinal field) identically zero on the beam axis (*x*=*y*=0). If necessary, longitudinal and transverse wakes can be added on top of the field obtained from potential (7).

**Power series**

Integral (7) as well as its derivatives w.r.t. the transverse coordinates can be computed numerically but for small displacements in respective r.m.s. beam sizes

 (8)

a power expansion can be used. Its coefficients are integrals which can be expressed via the Gauss hypergeometric series:

. (9)

with *a* = *r*2 − 1. Then we have for the potential function

. (10)

where *δm,n* is the Kronecker delta: *δm,n* =1 if *m*=*n* and *δm,n* =0 otherwise.

The following recurrence relations can be used to compute coefficients (9)

 (11)

For | *a* | < 1 the first relation should be used as it is presented, in the descending order in *m* in order to avoid loss of precision, while for *a* = *r*2 − 1 > 1 it must be used in the ascending order in *m*. So to obtain the whole set of coefficients just one integral has to be computed (or a hypergeometric function if | *a* | < 1). For *m* >> 1 an asymptotical formula can be obtained by expanding the denominator in eq. (9) in powers of (1−*t*):

. (12)

These coefficients determine not only the potential but also the electric field components, so that all necessary quantities can be obtained simultaneously.

With *M*=50 and *L*=*M-m* in eq. (10) a better than 6-digits precision is obtained for *d*res= 5.6 in the case *a* < 0. For *a* > 1 the error can be by an order of magnitude larger, so it is better to use formulas with interchanged *x* and *y* if *σy* > *σx*.

**Asymptotic expansion for halo particles**

For distant halo particles we can use another approach based on expansion of the Green function in eq. (2). Let us impose the same boundary condition for the potential at the axis as in eq. (7)

. (13)

This could have been done from the very beginning by addition of a term to the Green function which does not depend on the observation point and therefore does not affect the electric field:

, (14)

The contribution to Φ from the added term evaluates analytically:

. (15)

with γ=0.577216… being the Euler gamma.

Introducing vectors  in (*x*, *y*) plane for brevity we can write for the first term in *G*

 (16)

in the case . Performing now integration in eq. (2) we obtain for the total potential function

, (17)

where *r* =*σy* /*σx* is assumed. A few first functions *Pk* computed with the help of *Mathematica* are given in the Appendix.

Derivatives of Φ needed for computation of the transverse kick can be found in a similar fashion:

. (18)

A few of functions *Qk* are also given in the Appendix. The *y*-derivative can be obtained from the above formula by interchange *x* ↔ *y* and *r* ↔ 1/*r*.

These expansions give better than 5-digit precision for large particle displacements in maximal beam size

 (19)

with *σmax* = max(*σx*, *σy*).

**Intermediate case**

With large beam ellipse excentricity it is possible that *d*max for a particle displaced in the direction of smaller beam size is not large enough to make the asymptotic expansion of the previous section work while *d*res is already too large for the power series to convergence. The larger deviation of the aspect ratio *r* from 1 the wider the gap.

To calculate the integral in (7) for particles in this gap let us divide the integration interval in (7) in two parts:

, (20)

with some small *u* (analysis shows that u=0.4 is enough to achieve 6 digit accuracy for *r* > 0.1) and use power series in the first integral:

. (21)

The part of the second integral which contains exponential can be found direct numerical integration. The Simpson rule with 60 grid points appears to be sufficient.

The remaining part of the second integral evaluates analytically:

. (22)

**References**

[1] M. Basetti, G.A. Erskine, CERN-ISR-TH/80-06 (1980)

[2] Y. Alexahin, FERMILAB-TM-2148 (2001).

**Appendix**

P[1] = Pi\*(-1 + r^2)\*(u^2 - v^2)

P[2] = (-3\*Pi\*(-1 + r^2)^2\*(u^4 - 6\*u^2\*v^2 + v^4))/2

P[3] = 5\*Pi\*(-1 + r^2)^3\*(u^6 - 15\*u^4\*v^2 + 15\*u^2\*v^4 - v^6)

P[4] = (-105\*Pi\*(-1 + r^2)^4\*(u^8 - 28\*u^6\*v^2 + 70\*u^4\*v^4 - 28\*u^2\*v^6 +

v^8))/4

P[5] = 189\*Pi\*(-1 + r^2)^5\*(u^10 - 45\*u^8\*v^2 + 210\*u^6\*v^4 -

210\*u^4\*v^6 + 45\*u^2\*v^8 - v^10)

P[6] = (-3465\*Pi\*(-1 + r^2)^6\*(u^12 - 66\*u^10\*v^2 + 495\*u^8\*v^4 -

924\*u^6\*v^6 + 495\*u^4\*v^8 - 66\*u^2\*v^10 + v^12))/

P[7] = 19305\*Pi\*(-1 + r^2)^7\*(u^14 - 91\*u^12\*v^2 + 1001\*u^10\*v^4 -

3003\*u^8\*v^6 + 3003\*u^6\*v^8 - 1001\*u^4\*v^10 + 91\*u^2\*v^12 - v^14)

P[8] = (-2027025\*Pi\*(-1 + r^2)^8\*(u^16 - 120\*u^14\*v^2 + 1820\*u^12\*v^4 -

8008\*u^10\*v^6 + 12870\*u^8\*v^8 - 8008\*u^6\*v^10 + 1820\*u^4\*v^12 -

120\*u^2\*v^14 + v^16))/8

P[9] = 3828825\*Pi\*(-1 + r^2)^9\*(u^18 - 153\*u^16\*v^2 + 3060\*u^14\*v^4 -

18564\*u^12\*v^6 + 43758\*u^10\*v^8 - 43758\*u^8\*v^10 + 18564\*u^6\*v^12 -

3060\*u^4\*v^14 + 153\*u^2\*v^16 - v^18)

P[10] = (-130945815\*Pi\*(-1 + r^2)^10\*(u^20 - 190\*u^18\*v^2 +

4845\*u^16\*v^4 - 38760\*u^14\*v^6 + 125970\*u^12\*v^8 - 184756\*u^10\*v^10 +

125970\*u^8\*v^12 - 38760\*u^6\*v^14 + 4845\*u^4\*v^16 - 190\*u^2\*v^18 +

v^20))/2

Q[1] = 2\*Pi\*(-1 + r^2)\*u\*(-u^2 + 3\*v^2)

Q[2] = 6\*Pi\*(-1 + r^2)^2\*u\*(u^4 - 10\*u^2\*v^2 + 5\*v^4)

Q[3] = 30\*Pi\*(-1 + r^2)^3\*u\*(-u^6 + 21\*u^4\*v^2 - 35\*u^2\*v^4 + 7\*v^6)

Q[4] = 210\*Pi\*(-1 + r^2)^4\*u\*(u^8 - 36\*u^6\*v^2 + 126\*u^4\*v^4 -

84\*u^2\*v^6 + 9\*v^8)

Q[5] = 1890\*Pi\*(-1 + r^2)^5\*u\*(-u^10 + 55\*u^8\*v^2 - 330\*u^6\*v^4 +

462\*u^4\*v^6 - 165\*u^2\*v^8 + 11\*v^10)

Q[6] = 20790\*Pi\*(-1 + r^2)^6\*u\*(u^12 - 78\*u^10\*v^2 + 715\*u^8\*v^4 -

1716\*u^6\*v^6 + 1287\*u^4\*v^8 - 286\*u^2\*v^10 + 13\*v^12)

Q[7] = 270270\*Pi\*(-1 + r^2)^7\*u\*(-u^14 + 105\*u^12\*v^2 - 1365\*u^10\*v^4 +

5005\*u^8\*v^6 - 6435\*u^6\*v^8 + 3003\*u^4\*v^10 - 455\*u^2\*v^12 + 15\*v^14)

Q[8] = 4054050\*Pi\*(-1 + r^2)^8\*u\*(u^16 - 136\*u^14\*v^2 + 2380\*u^12\*v^4 -

12376\*u^10\*v^6 + 24310\*u^8\*v^8 - 19448\*u^6\*v^10 + 6188\*u^4\*v^12 -

680\*u^2\*v^14 + 17\*v^16)

Q[9] = 68918850\*Pi\*(-1 + r^2)^9\*u\*(-u^18 + 171\*u^16\*v^2 - 3876\*u^14\*v^4 +

27132\*u^12\*v^6 - 75582\*u^10\*v^8 + 92378\*u^8\*v^10 - 50388\*u^6\*v^12 +

11628\*u^4\*v^14 - 969\*u^2\*v^16 + 19\*v^18)

Q[10] = 1309458150\*Pi\*(-1 + r^2)^10\*u\*(u^20 - 210\*u^18\*v^2 +

5985\*u^16\*v^4 - 54264\*u^14\*v^6 + 203490\*u^12\*v^8 - 352716\*u^10\*v^10 +

293930\*u^8\*v^12 - 116280\*u^6\*v^14 + 20349\*u^4\*v^16 - 1330\*u^2\*v^18 +

21\*v^20)

1. Gaussian units are used. To convert to SI units the r.h.s. should be divided by 4*πε*0 [↑](#footnote-ref-1)