Envelope Optics

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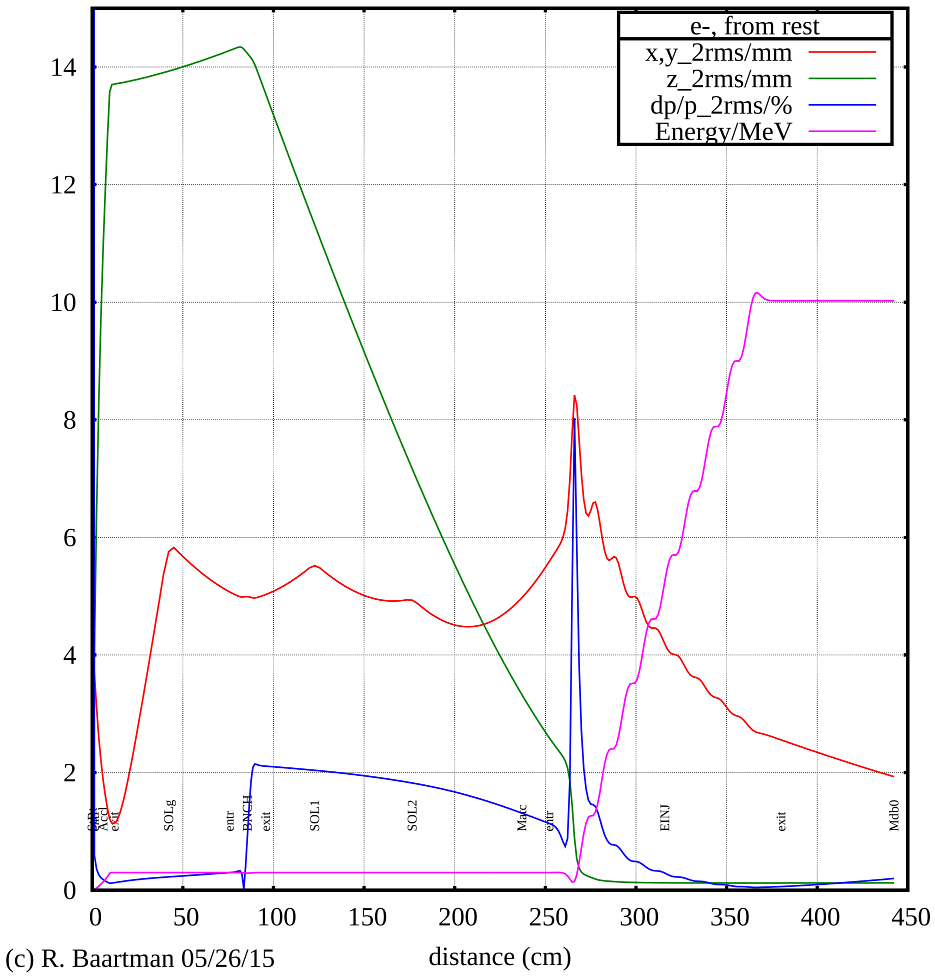
Intro: Motivating Example



Electron gun, solenoids, buncher, linac; 15 pC bunch charge.

How to:

Find peak energy gain phase?

Solenoid match?

Vary with bunch charge?

Buncher phase?

For different cavity excitation?

Etc.

Outline

1. Theory: Hamiltonian, Space Charge, Transfer Matrices
2. Example 1: FNAL Booster
3. Example 2: Injecting into TRIUMF cyclotron
4. Example 3: Linear Accelerators

EE

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

96

CERN/SI/Int. DL/70-12
18.11.1970

RMS ENVELOPE EQUATIONS WITH SPACE CHARGE

Frank J. Sacherer

ABSTRACT

The envelope equations for a continuous beam with circular symmetry but otherwise arbitrary charge distribution have been derived by Lapostolle and Gluckstern. Their results are extended in this report to continuous beams with elliptical symmetry and to bunched beams with ellipsoidal form.

Statistical Approach to Beam Dynamics

If there is a distribution of particles, one would like to calculate the final distribution from the initial. The behaviour of the beam centroid

$$\langle \mathbf{X} \rangle = \sum_{i=1}^{N} \mathbf{X}/N \tag{1}$$

(where N is the number of particles, and $\mathbf{X}$ is the column vector $(x, P_x, y, P_y, z, P_z)^T$ as in eqn. 5) is determined by the same transfer matrix $\mathbf{M}$ as for an individual particle. This is the equation of 'first moments'. At the next level, one would like to calculate the evolution of the beam widths, or, 'second moments' given by

$$\boldsymbol{\sigma} \equiv \frac{1}{N} \sum_{i=1}^{N} \mathbf{X}\mathbf{X}^T \tag{2}$$

For example, $\sigma_{11} = \langle x^2 \rangle$, $\sigma_{12} = \langle xP_x \rangle$, $\sigma_{13} = \langle xy \rangle$, ... For a distribution of particles so dense that we do not see graininess on any scale of our diagnostics, the sums go over into integrals. For example,

$$\sigma_{12} = \int\int\int\int\int\int xP_x\, f(x, P_x, y, P_y, z, P_z)\; dx\, dP_x\, dy\, dP_y\, dz\, dP_z,$$

where f is the distribution in phase space, normalized so that its integral over all 6 phase space dimensions is 1.

Here, s is the independent variable, $z = \beta c \Delta t$, $P_z = (\beta c)^{-1} \Delta E$.

By direct substitution into the definition of $\boldsymbol{\sigma}$, we find

$$\boldsymbol{\sigma}_{\rm f} = \mathbf{M}\boldsymbol{\sigma}_{\rm i}\mathbf{M}^T \tag{3}$$

As well, recalling the infinitesimal transfer matrix $\mathbf{F}$ where $\mathbf{X}' = \mathbf{F}\mathbf{X}$ and the transfer matrix of an infinitesimal length ds is $\mathbf{M} = \mathbf{I} + \mathbf{F}ds$, we find directly

$$\boldsymbol{\sigma}' = \mathbf{F}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{F}^T. \tag{4}$$

This is the **envelope equation**. For the full 6D case, it represents 21 equations. (Because σ is symmetric.)

What is F? Infinitesimal Transfer Matrix

The general Hamiltonian can be Taylor-expanded by orders in the 6 dependent variables[1],

$$H(x_1, x_2, x_3, x_4, x_5, x_6; s) = \sum_i \left.\frac{\partial H}{\partial x_i}\right|_0 x_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 H}{\partial x_i \partial x_j}\right|_0 x_i x_j + \ldots$$

The subscript 0 means that the derivatives are evaluated on the reference trajectory $\forall i,\ x_i = 0$. (Keep in mind though that these partial derivatives in general are functions of the independent variable t or s.)

Terms of first order are eliminated by transforming to a coordinate system measured with respect to the reference trajectory. The remaining terms are second order and higher, and for linear motion, we simply truncate at the second order.

[1] In this shorthand, $x_1 = x, x_2 = P_x, x_3 = y, \ldots$

Then the Hamiltonian looks like $H = Ax^2 + BxP_x + Cxy + \ldots + UP_z^2$: there are 21 independent terms. $A = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}$, and so on; all derivatives are evaluated on the reference trajectory, and may be a function of the independent variable. We know the equations of motion from the Hamiltonian to be: $x' = \partial H/\partial P_x$, $P_x' = -\partial H/\partial x$, etc., where primes denote derivatives w.r.t. the independent variable. Therefore the equations of motion:

$$\begin{pmatrix} x' \\ P_x' \\ y' \\ P_y' \\ z' \\ P_z' \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial P_x \partial x} & \frac{\partial^2 H}{\partial P_x^2} & \frac{\partial^2 H}{\partial P_x \partial y} & \frac{\partial^2 H}{\partial P_x \partial P_y} & \frac{\partial^2 H}{\partial P_x \partial z} & \frac{\partial^2 H}{\partial P_x \partial P_z} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial P_x} & -\frac{\partial^2 H}{\partial x \partial y} & -\frac{\partial^2 H}{\partial x \partial P_y} & -\frac{\partial^2 H}{\partial x \partial z} & -\frac{\partial^2 H}{\partial x \partial P_z} \\ \frac{\partial^2 H}{\partial P_y \partial x} & \frac{\partial^2 H}{\partial P_y \partial P_x} & \frac{\partial^2 H}{\partial P_y \partial y} & \frac{\partial^2 H}{\partial P_y^2} & \frac{\partial^2 H}{\partial P_y \partial z} & \frac{\partial^2 H}{\partial P_y \partial P_z} \\ -\frac{\partial^2 H}{\partial y \partial x} & -\frac{\partial^2 H}{\partial y \partial P_x} & -\frac{\partial^2 H}{\partial y^2} & -\frac{\partial^2 H}{\partial y \partial P_y} & -\frac{\partial^2 H}{\partial y \partial z} & -\frac{\partial^2 H}{\partial y \partial P_z} \\ \frac{\partial^2 H}{\partial P_z \partial x} & \frac{\partial^2 H}{\partial P_z \partial P_x} & \frac{\partial^2 H}{\partial P_z \partial y} & \frac{\partial^2 H}{\partial P_z \partial P_y} & \frac{\partial^2 H}{\partial P_z \partial z} & \frac{\partial^2 H}{\partial P_z^2} \\ -\frac{\partial^2 H}{\partial z \partial x} & -\frac{\partial^2 H}{\partial z \partial P_x} & -\frac{\partial^2 H}{\partial z \partial y} & -\frac{\partial^2 H}{\partial z \partial P_y} & -\frac{\partial^2 H}{\partial z^2} & -\frac{\partial^2 H}{\partial z \partial P_z} \end{pmatrix} \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{pmatrix} \tag{5}$$

or,

$$\mathbf{X}' = \mathbf{F}\mathbf{X},$$

where $\mathbf{F}$ is called the 'infinitesimal transfer matrix'. Of the 36 elements of $\mathbf{F}$ there are only 21 independent ones.

Example: Quadrupole

A particular case is where the beamline consists only of elements that keep all 3 degrees of freedom independent of each other, and there is only a focusing force $K(s)$ that varies with s. In other words, the Hamiltonian is 6,

$$H = \frac{P_x^2}{2} + K(s)\frac{x^2}{2} + \frac{P_y^2}{2} - K(s)\frac{y^2}{2} + \frac{P_z^2}{2\gamma^2} \tag{6}$$

so

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Space Charge part of $\mathbf{F}$

The beam is in bunches rather than continuous, so we need the electric field of an ellipsoidal distribution of charge. For this case as well, it turns out, surprisingly (Sacherer, 1971), that the RMS linear part of the space charge self-field depends mainly on the RMS size of the distribution and only very weakly on its exact form. To within a few percent, the RMS linear part of space charge is the same as that for a uniformly populated ellipsoid. The space charge infinitesimal transfer matrix is now

$$\mathbf{F}_{\rm sc} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ K_{x\rm sc} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{y\rm sc} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{z\rm sc} & 0 \end{pmatrix} \tag{8}$$

where

$$K_{x\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{a^3}\,g\left(\frac{b^2}{a^2},\frac{c^2}{a^2}\right) \tag{9}$$

$$K_{y\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{b^3}\,g\left(\frac{c^2}{b^2},\frac{a^2}{b^2}\right) \tag{10}$$

$$K_{z\mathrm{sc}} = \frac{Q}{4\pi\epsilon_0(mc^2/e)\beta^2\gamma^3}\,\frac{1}{c^3}\,g\left(\frac{a^2}{c^2},\frac{b^2}{c^2}\right) \tag{11}$$

where Q is the bunch charge, the ellipsoid semi-axes in the x, y, z directions are a, b, c, and the function g is

$$g(u,v) = \frac{3}{2}\int_0^\infty (1+s)^{-3/2}(u+s)^{-1/2}(v+s)^{-1/2}ds \tag{12}$$

This is from the family of Carlson elliptic integrals.

Arbitrary bunch distributions, orientations

For arbitrary distributions of the type $f(x, y, z) = f\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$, replace a, b, c with the RMS values according to the values they have for the uniform case, namely, $a^2 = 5\sigma_{11}$, $b^2 = 5\sigma_{33}$. Because of relativity, c^2 is a special case: $c^2 = 5\gamma^2\sigma_{55}$.

Notice the recursiveness.

For arbitrary orientations, have to apply a rotation matrix to F, thus making also $F_{23}, F_{25}, F_{41}, F_{45}, F_{61}, F_{63}$ also non-zero.

For further reading, again refer to Sacherer (1971), but also de Jong (1983).

Elaborated for the case with space charge, DC, uncoupled, it becomes the (better-known) Kapchinsky-Vladimirsky eqns.

What about: `TRANSPORT, TRACE3D`?

If all elements are integrable then the transfer matrices $\mathbf{M}$ are known, and they are simply multiplied together to find the matrix of the whole beamline or synchrotron, and the final beam is found from the initial as in 3. This is the traditional approach, e.g. `TRANSPORT`.

To incorporate space charge, elements were subdivided and appropriate thin defocus lenses inserted.

In `TRACE3D`, there are space charge **impulses** applied in the approximation of long bunches.

7.0 SPACE-CHARGE IMPULSES

Approximate expressions for the electric field components that are due to a uniformly charged ellipsoid, as given by Lapostolle[13] are as follows:

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c\gamma^2}\frac{(1-f)}{r_x(r_x+r_y)r_z}x \quad ,$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c\gamma^2}\frac{(1-f)}{r_y(r_x+r_y)r_z}y \quad ,$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{3I\lambda}{c}\frac{f}{r_x r_y r_z}z \quad ,$$

where r_x, r_y, and r_z are the semiaxes of the ellipsoid, I is the average electrical current (assuming that a bunch occurs in every period of the RF), λ is the free-space wavelength of the RF, c is the velocity of light, and ε_0 is the permittivity of free space. The form factor f is a function of $p \equiv \gamma r_z \big/ \sqrt{r_x r_y}$. Values for f are given in Table III for specific values of p and $1/p$.

TABLE III: Space-Charge Form Factor

[13] P. M. Lapostolle, CERN report AR/Int. SG/65-15, Geneva, Switzerland (July 1965).

TRANSOPTR

These techniques are approximate and non-adaptive: Why not use the equations of motion directly? There are only 21 of them. In TRANSOPTR, 4 is solved with a Runge Kutta integrator. This allows not only space charge, but any general case with no closed-form solution to eom's, e.g. varying axial fields, linacs, short-soft-edge quads,...

Original version written by Mark deJong, Ed Heighway.

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A FIRST ORDER SPACE CHARGE OPTION FOR TRANSOPTR

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Summary

Theory

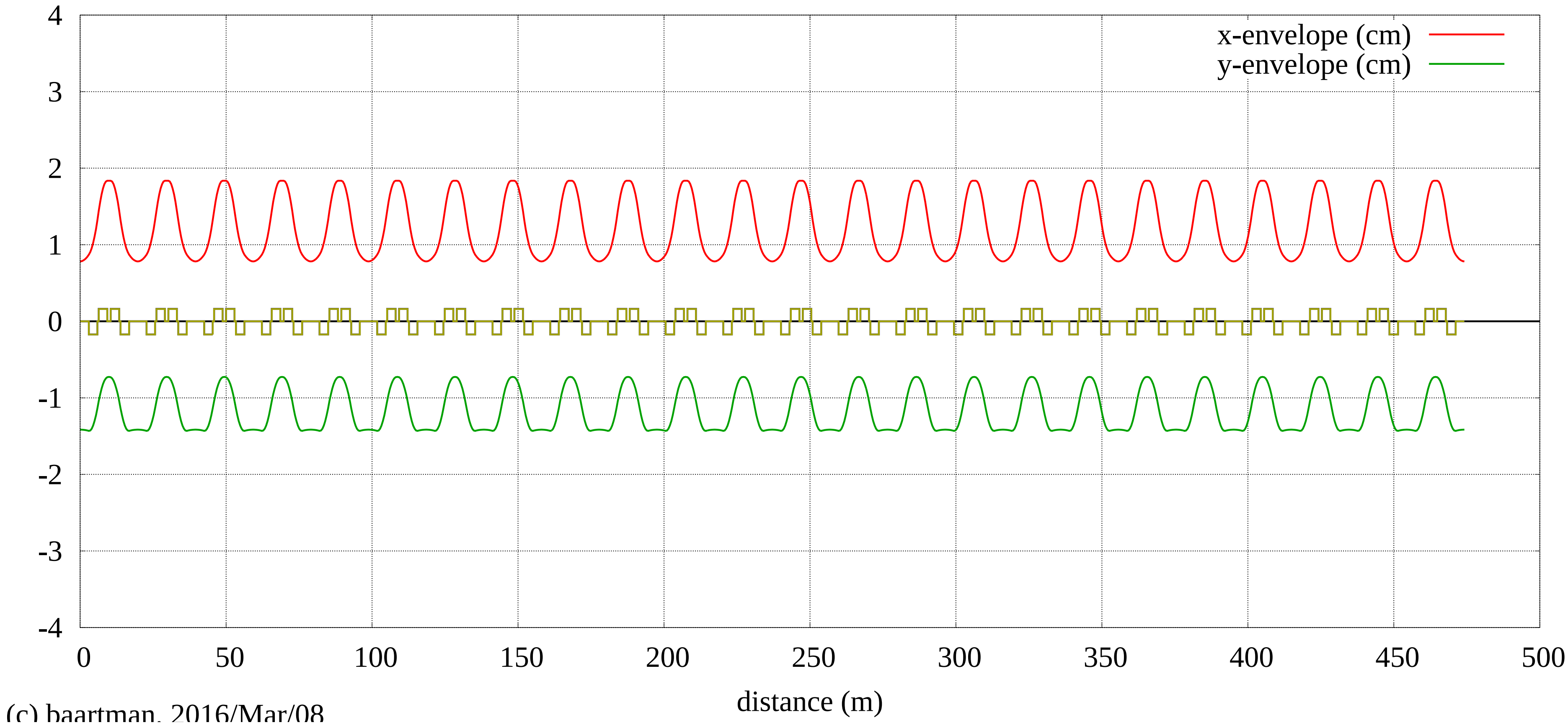
Example 1: FNAL Booster

Combined function magnets, field index n, radius ρ,

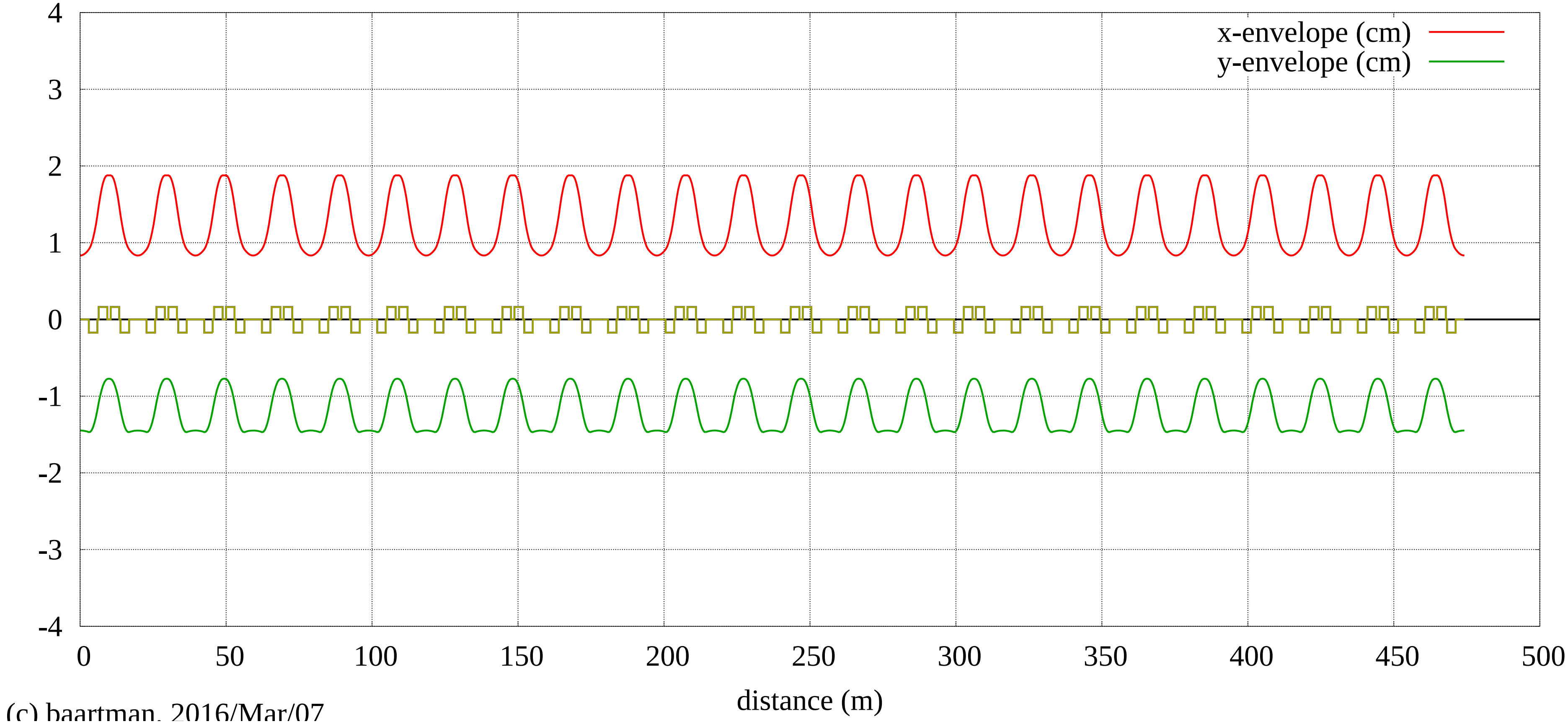
$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1-n}{\rho^2} & 0 & 0 & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{n}{\rho^2} & 0 & 0 & 0 \\ -\frac{1}{\rho} & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

With a synchrotron, have to find periodic envelope. Do this by fitting input beam to equal output after 1 turn.

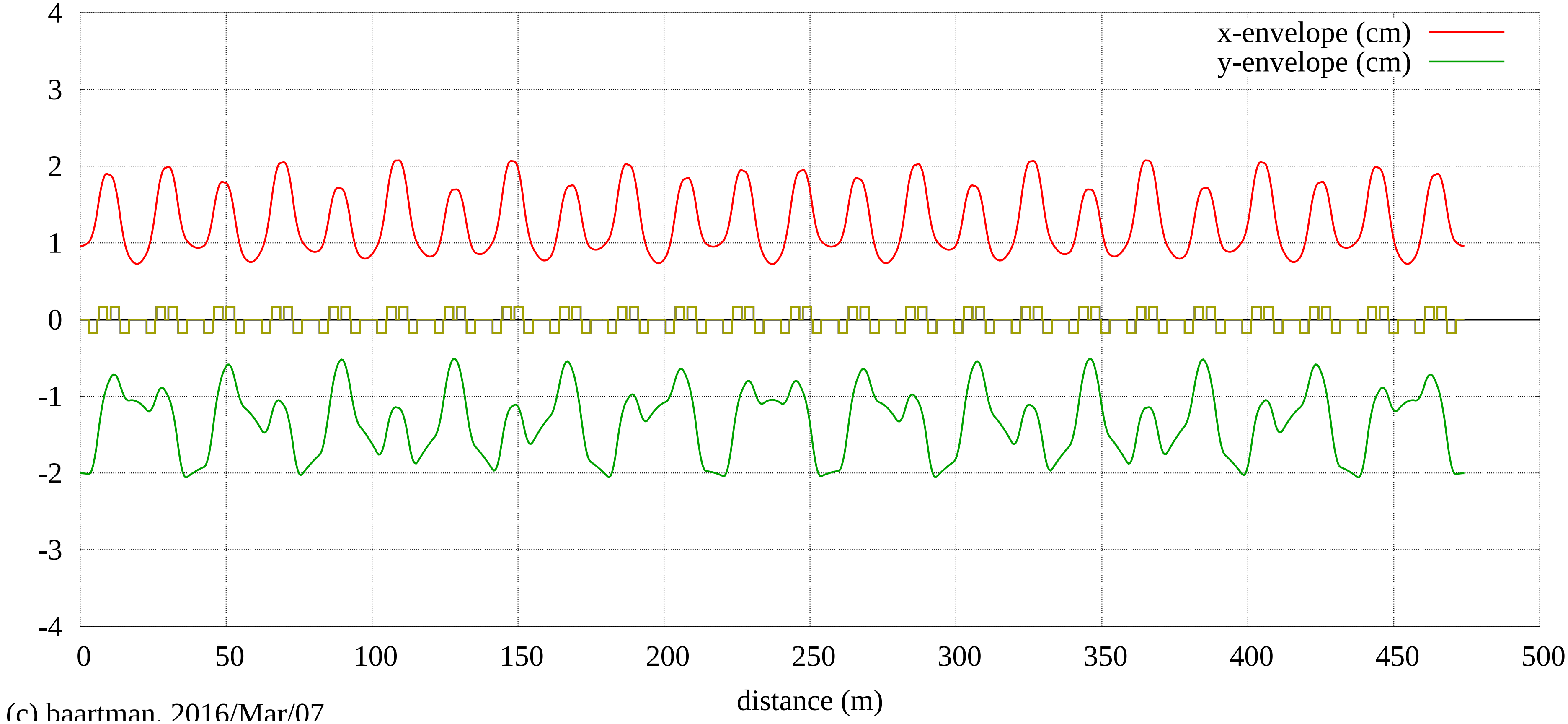
Matched solution, no space charge, tunes=$(6.7, 6.8)$

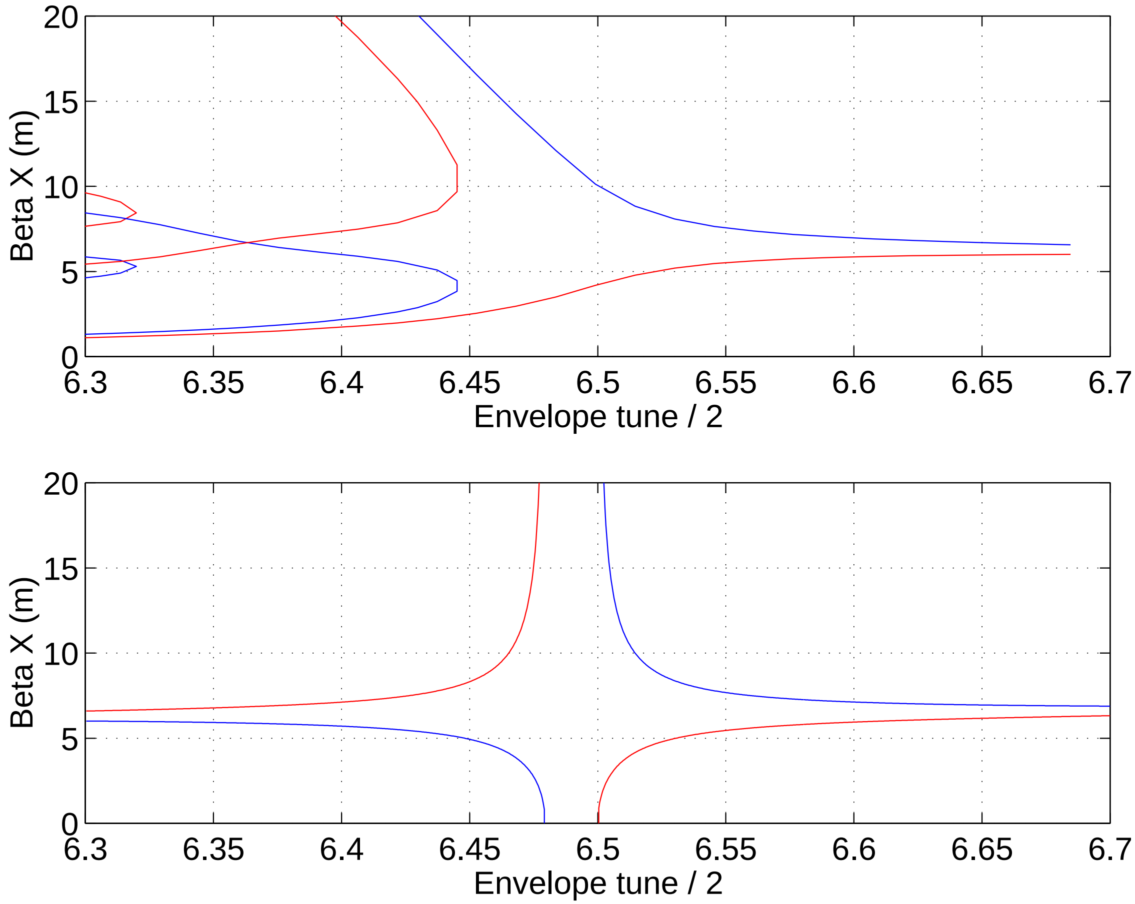


Add space charge, 2.6Amp. Matched solution.



Another Matched solution?



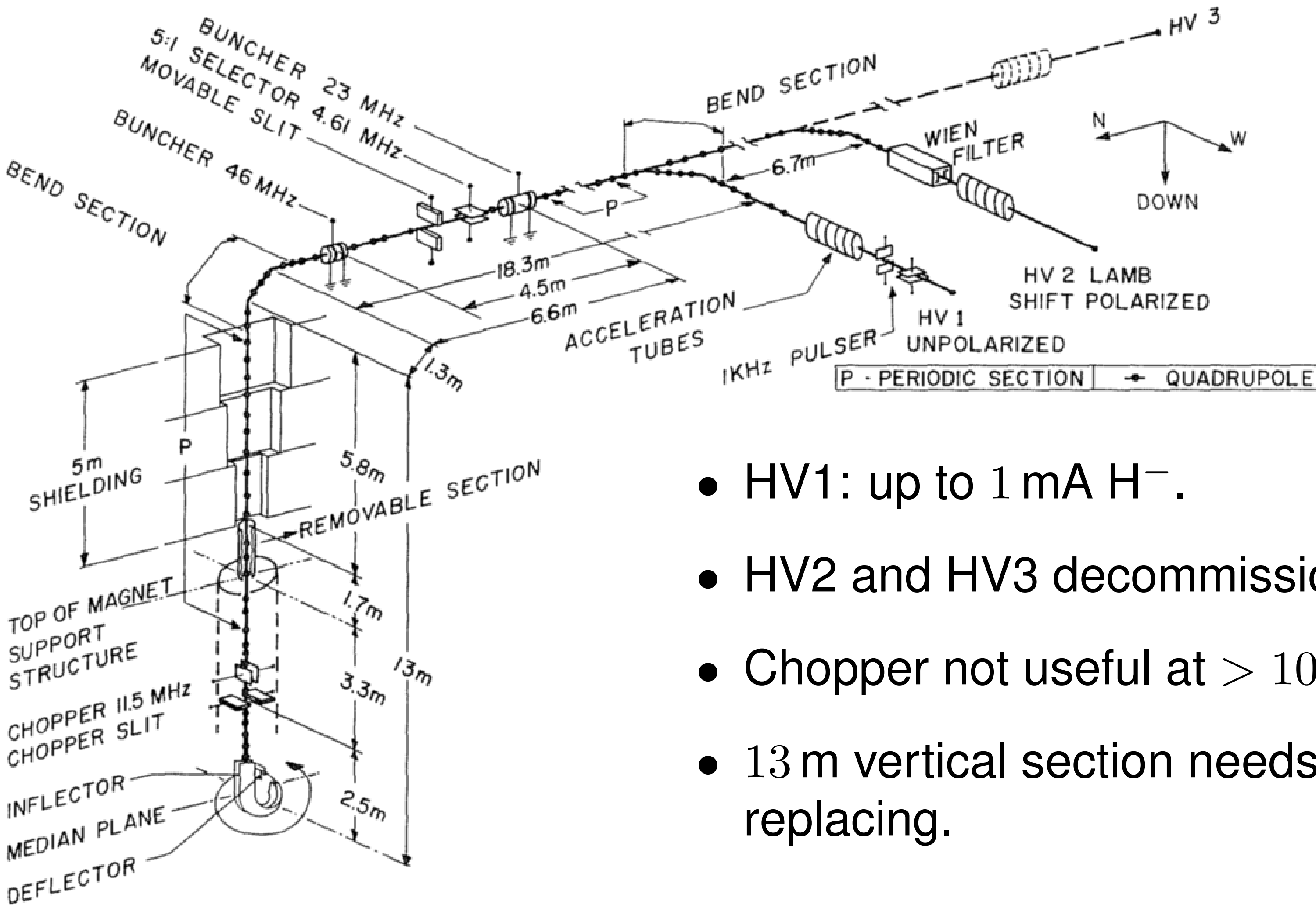


Added a thin lens to excite the 1/2-integer resonance. Bottom: Without space charge, moving bare tune. Top: Moving tune using space charge depression; bare tune fixed.

Reference: M. D'yachkov, R. Baartman and F.W. Jones–*Multiturn Simulation of Coherent Betatron Resonance With Space Charge* Proc. PAC2001.

Animated gif

Example 2: Injection Line to Cyclotron



- HV1: up to 1 mA H^-.
- HV2 and HV3 decommissioned,
- Chopper not useful at $> 10\,\mu$A.
- 13 m vertical section needs replacing.

Some Particulars

1. All electrostatic; 100s of delicate, uncooled electrodes.

2. Typically, $300 - 400\,\mu$A, or ~ 100 Watts of beam; can easily melt a quadrupole (which, after all, is just 4 small pieces of aluminum extrusion).

3. H^-, so vacuum must be better than 10^{-7} ($f_{\rm lost} = P/(2 \times 10^{-5}\,{\rm Torr})$). Even so, it easily sheds electrons and these are electrically indistinguishable from beam particles, confusing the diagnostics.

4. Large energy spread from bunchers, so beamline must be achromatic. (DC beam is bunched to a peak of ~ 5 mA.)

5. Space charge dominated at typical high intensity operation: 5 mA peak means space charge forces are larger than average quadrupole focusing force.

6. But the space charge force is intrisically nonlinear, generating beam "halo".

7. Centre of cyclotron field is 3 kG: almost the whole vertical line can be thought of as in the fringing field of a (poorly designed) solenoid: strong coupling between transverse planes.

8. Beam is injected into the spiral inflector: possibly the most optically complicated element ever devised. (Also insulated, uncooled, can melt.)

9. Must (try to) match to the first turn of the cyclotron where essentially all vertical focusing comes from RF: tail of the bunch is much more strongly focused than the head. Space charge defocusing causes progressive loss of the head.

10. Old line took ~ 5 years before $100\,\mu$A cyclotron running was routine.

Why a New Vertical Section?

- Insulators dirty, shorting. Had to ground some electrodes.
- Vacuum bad, somewhat leaky (o-rings). $> 1\%$ loss but prefer $\sim 0.1\%$.
- Very poor alignment making quads very difficult to tune.
- Insufficient diagnostics (no BPMs).
- In spite of much effort, optics never understood, polarities doubtful.

So in 2011 replaced the whole 13 metre section. **Complete re-design of optics**. (But how?)

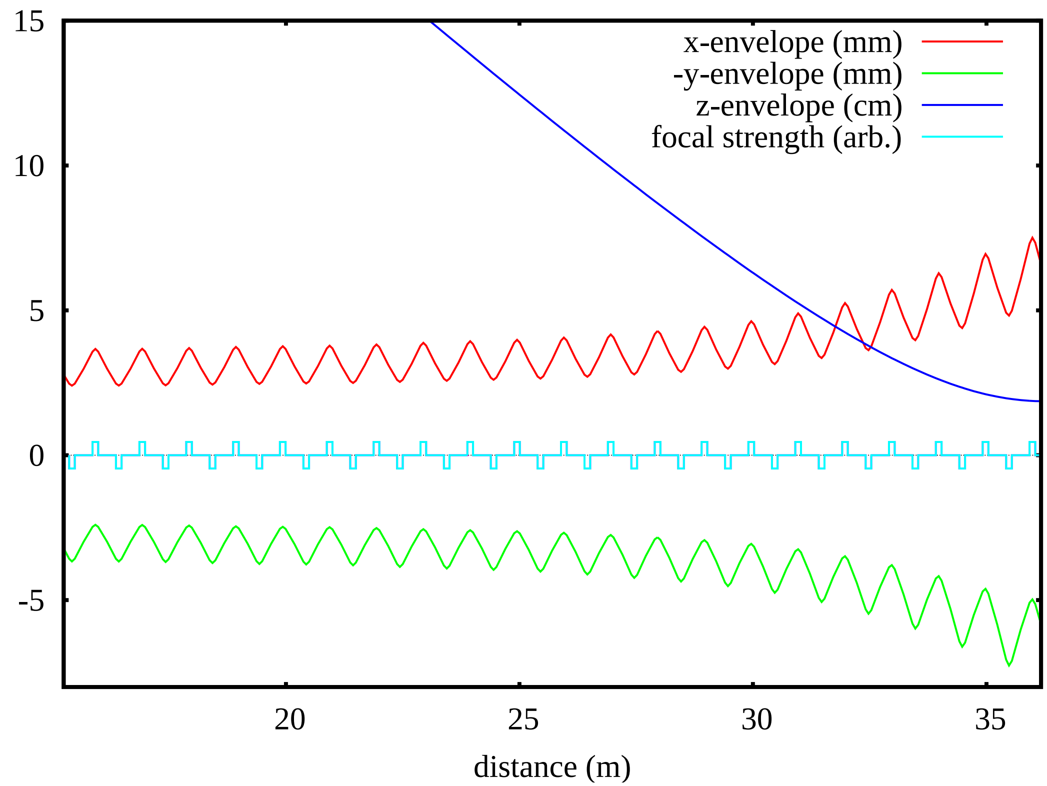
Beam Dynamics Complexity

1. Intense 3D space charge (up to 5 mA peak at 300 keV)
2. Bunching into a 36° phase acceptance (roughly 30mm long bunch)
3. Strong x-y coupling due to cyclotron's axial field
4. Strong and complicated x-y-z coupling in the inflector.
5. Vertical acceptance depends upon particle's phase because all focusing comes from RF on first few turns.
6. Cannot match, even in principle, so what now? How to optimize?.

In spite of this, we successfully designed, built, commissioned a totally new section **without using multi-particle simulations**. Used only the statistical approach sometimes called "envelope equation".

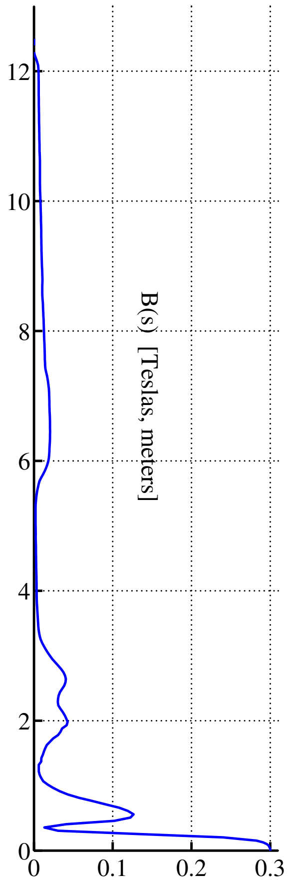
1,2. Bunching into a 36° phase acceptance

Ignore the details of 2-harmonic bunching, take only the linear part. I.e. launch the beam at buncher with a negative correlation between phase and energy. $r_{56} = -1$, $\sqrt{5\sigma_{55}} = \beta\lambda/2$, and $\sqrt{5\sigma_{66}} \propto V_{\rm buncher}$ optimized to give minimum bunch length at injection gap.



Test calculation of bunching beam in a periodic section. Final bunch is 12 mm dia. by 34 mm long. ($\beta\lambda = 329$ mm at 23 MHz, so this is roughly the 36° desired.)

3. Strong x-y coupling due to axial field



$$F_{\mathrm{axial}B} = \begin{pmatrix} 0 & 1 & \frac{-1}{2\rho} & 0 & 0 & 0 \\ \frac{-1}{4\rho^2} & 0 & 0 & \frac{-1}{2\rho} & 0 & 0 \\ \frac{1}{2\rho} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2\rho} & \frac{-1}{4\rho^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

which arises from the solenoid Hamiltonian

$$H_{\mathrm{axial}B} = \frac{1}{2}\left(P_x - \frac{y}{2\rho}\right)^2 + \frac{1}{2}\left(P_y + \frac{x}{2\rho}\right)^2 + \frac{1}{2}P_z^2, \tag{15}$$

where $1/\rho = B(s)/(B\rho)$, is a function of the independent variable s. Interpolate it using cubic spline.

4. Strong x-y-z coupling in the inflector

(See *A Canonical Treatment of the Spiral Inflector for Cyclotrons* Baartman and Kleeven, Part. Acc. **41** (1993).)

$$
\begin{aligned}
&H(x, y, z, P_x, P_y, P_z; s) = \\
&\frac{1}{2}\left[\left(P_x + \frac{TC}{A}y\right)^2 + \left(P_y - \frac{TC}{A}x\right)^2 + \left(P_z + \frac{2TS}{A}y + \frac{2}{A}x\right)^2\right] \\
&-\frac{1}{2A^2}\left[\xi(x + k'Sy)^2 + x^2 + kk'(C^2x^2 + y^2) + 2TSxy\right].
\end{aligned} \tag{16}
$$

where

$$\xi = \frac{1 + kk'S^2}{1 + k'^2S^2},\ S = \sin(s/A),\ C = \cos(s/A),\ T = \frac{k + k'}{2},\ k = \frac{A}{\rho} + k',$$

A is electric radius, $\rho = \rho(s)$ is magnetic radius, k' is tilt parameter.

inflector matrix

$$
F_{\text{inflector}} = \begin{pmatrix}
0 & 1 & \frac{TC}{A} & 0 & 0 & 0 \\
\frac{3-\xi+(T^2-kk')C^2}{-A^2} & 0 & \frac{3TS-k'\xi S}{-A^2} & \frac{TC}{A} & 0 & \frac{-2}{A} \\
\frac{-TC}{A} & 0 & 0 & 1 & 0 & 0 \\
\frac{3TS-k'\xi S}{-A^2} & \frac{-TC}{A} & \frac{(1+3S^2)T^2-kk'-k'^2\xi S^2}{-A^2} & 0 & 0 & \frac{-2TS}{A} \\
\frac{2}{A} & 0 & \frac{2TS}{A} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \qquad (17)
$$

BTW, if integrated with no space charge, this gives matrix that agrees with other codes (`CASINO, AXORB`).

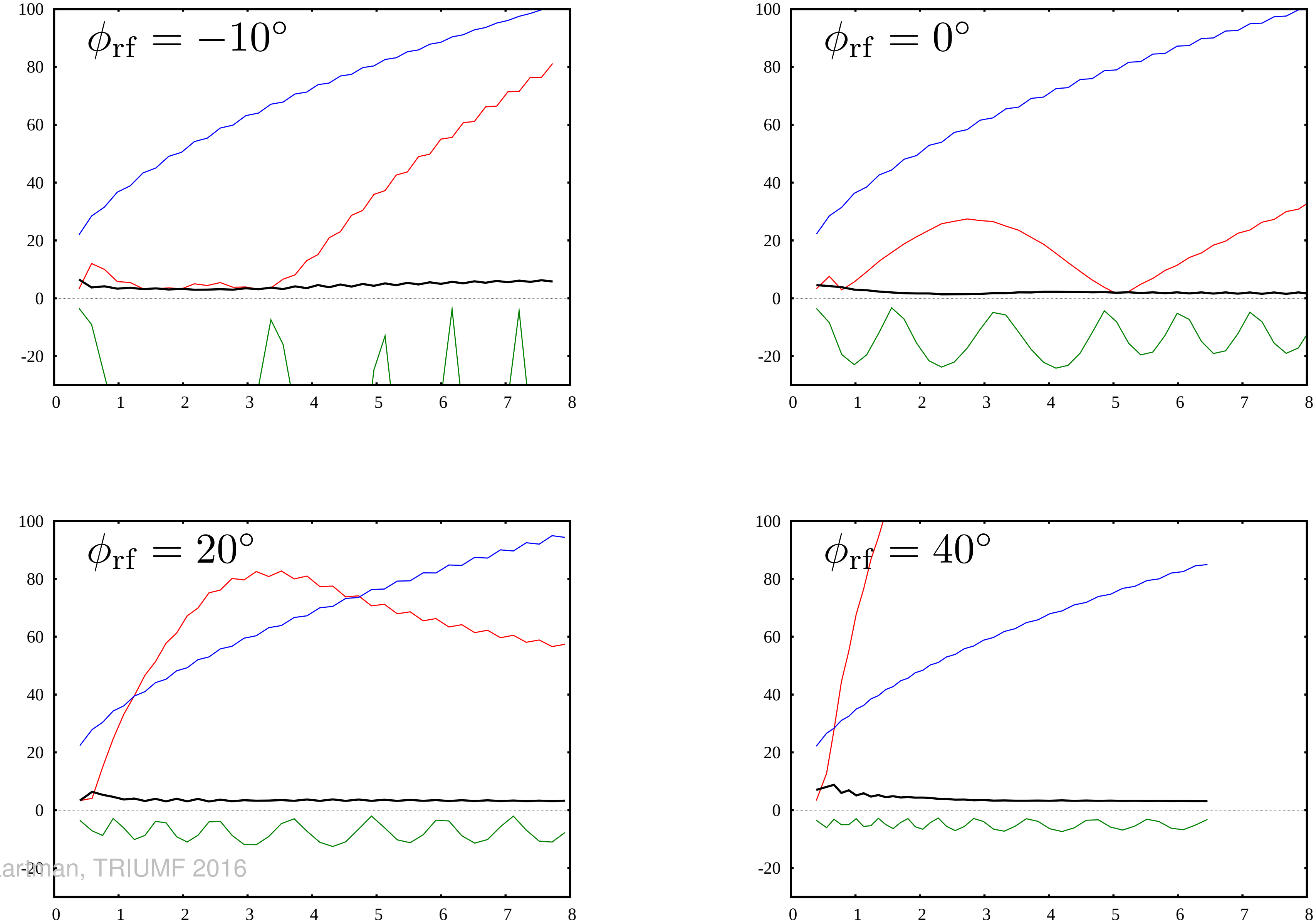
The inflector is followed by a deflector: crossed E and B fields so looks like a Wien filter.

5. Vertical acceptance depends upon particle's phase

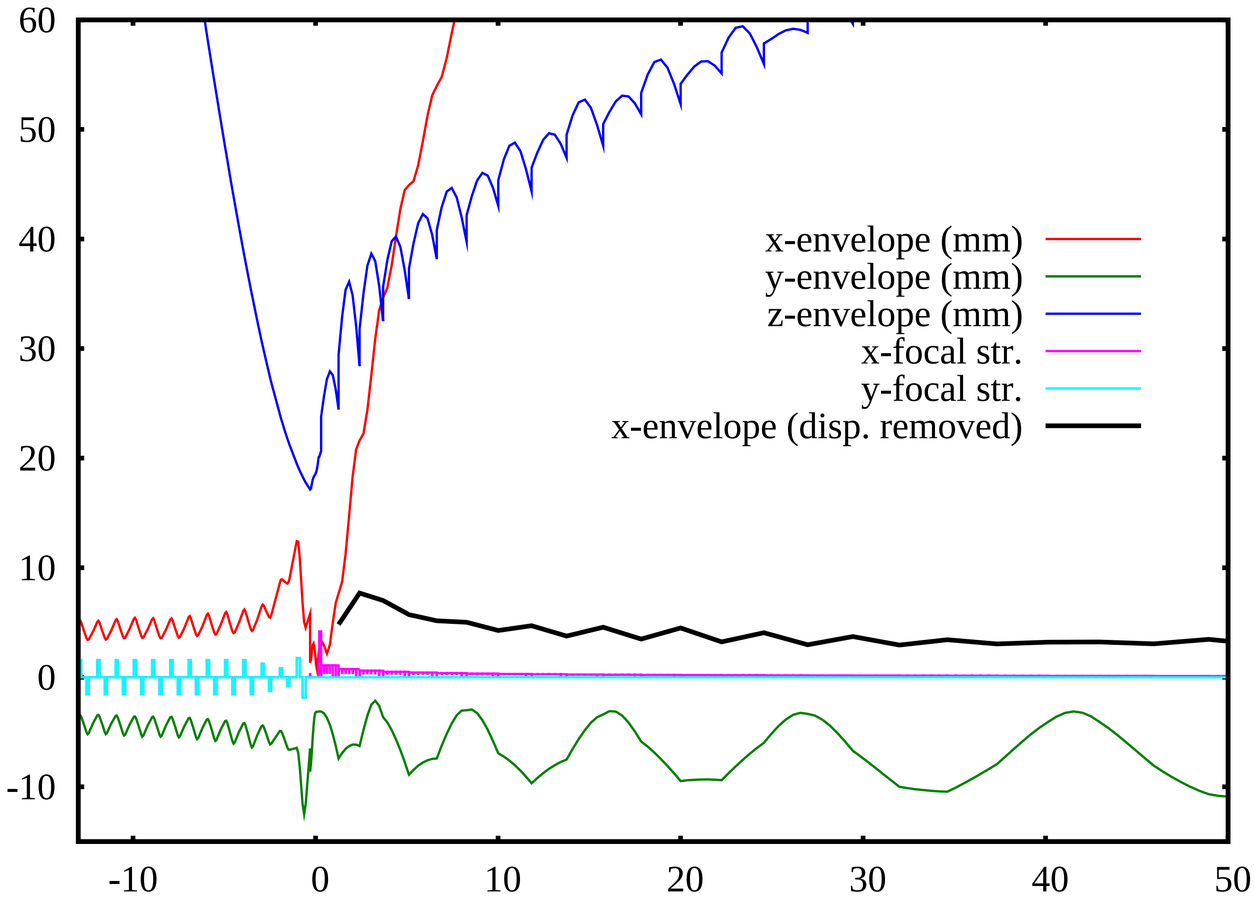
A simple cyclotron model is a flat field with thin lenses at the dee gaps. The focal length depends upon rf phase (i.e. it is an inherently nonlinear coupling), so one must choose an appropriate central phase. As the bunch charge is raised, the weakest-focused phases (leading the crest and near crest) are lost first. This requires some fiddling: place bunch phase too late and at will not gain sufficient energy.

Below: Blue is bunch length, red is radial, black is radial with dispersion removed, green is vertical.

some results vs. phase

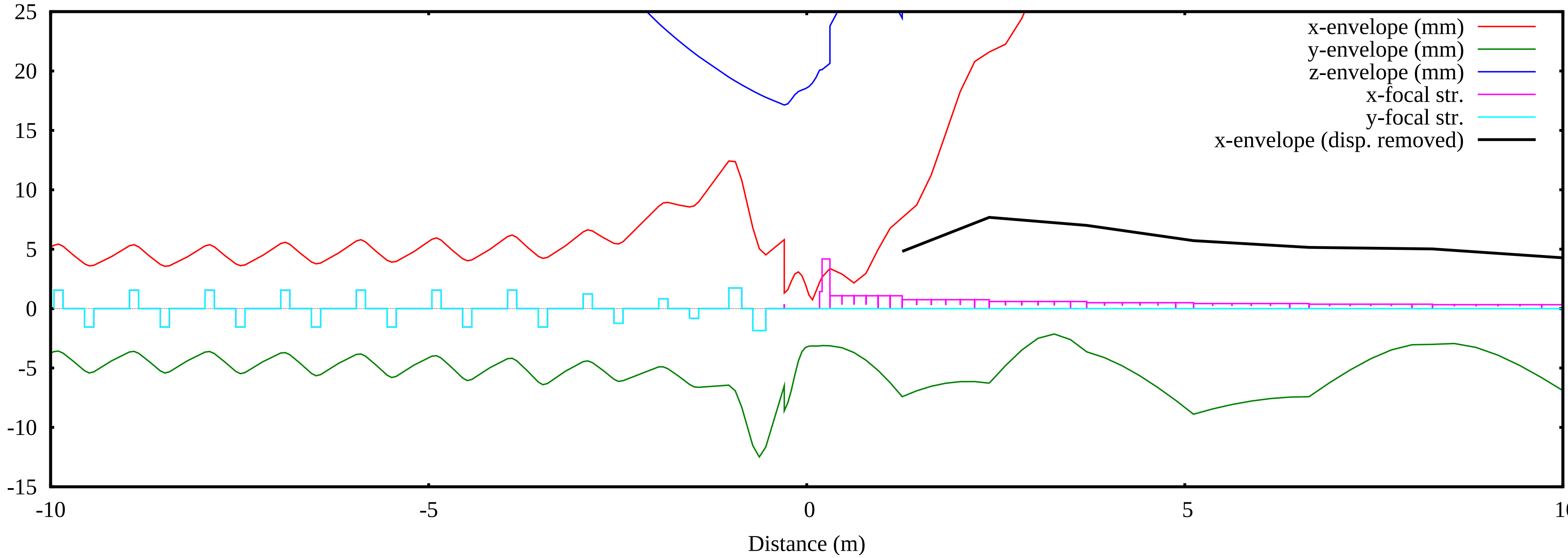


TRANSOPTR Calculation of Injection into TRIUMF Cyclotron and First Turns



Beam envelopes through the injection line and into the cyclotron versus distance in metres. Charge per bunch is $22\,p$C for a time average current $0.50\,$mA. This is for bunch injection phase $28°$.

Injection Matching Detail



New line in detail is very different from the old.

Notice the strong vertical mismatch. Causes factor ~ 5 increase in emittance.

6. How to optimize?

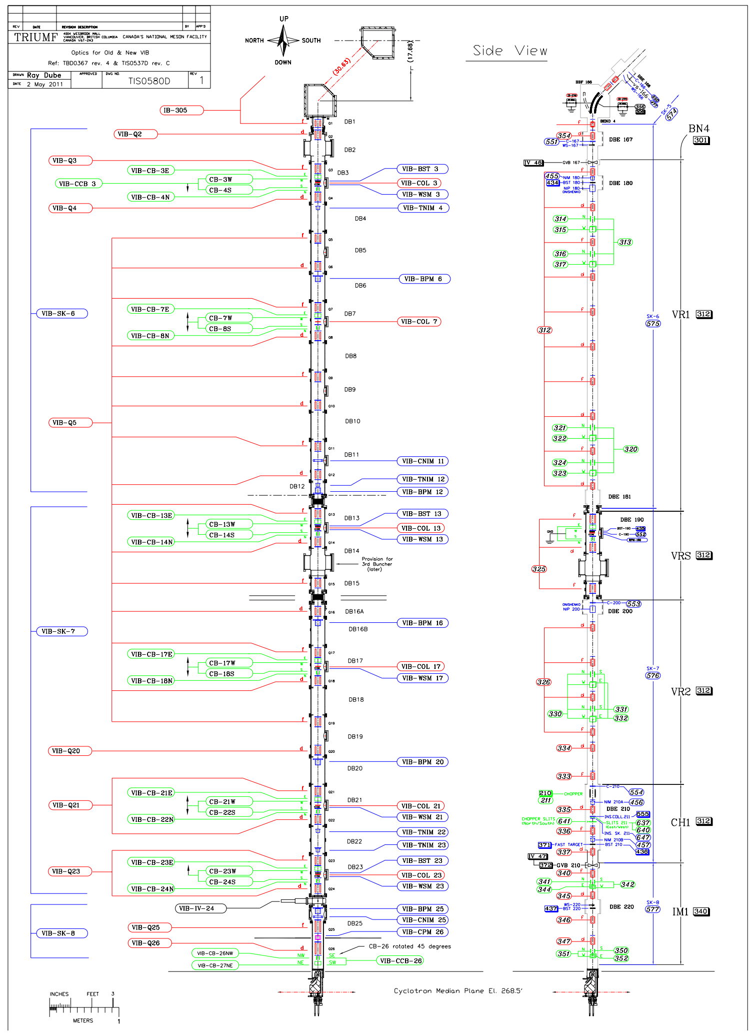
Three additional constraints:

- Keep the maximum quadrupole voltage below 5 kV.
- Accommodate anywhere from 0 to 500 μA (5 mA peak) with little change to quad settings.
- Minimize number of matching quad knobs.

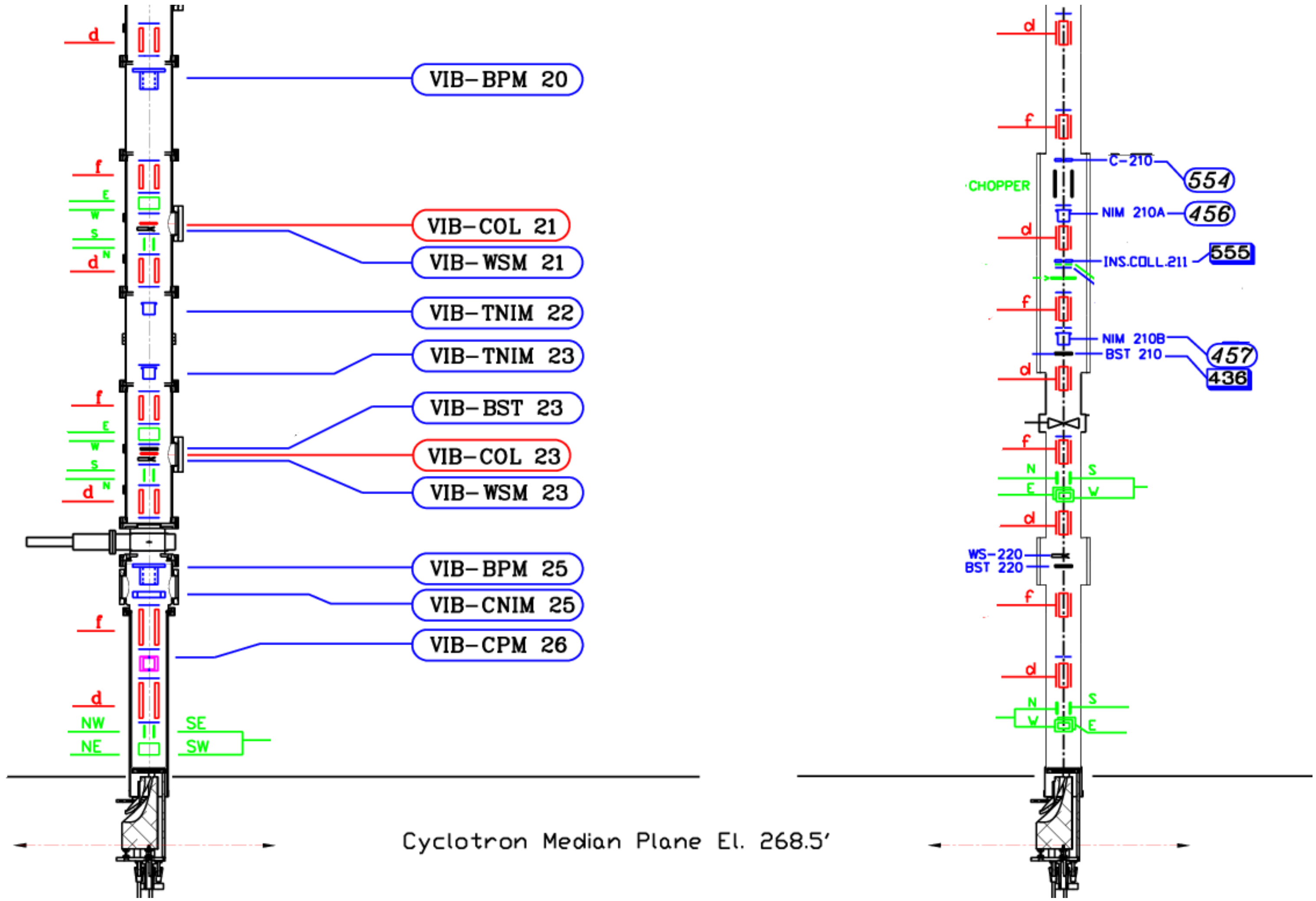
The calculation was run with a **simulated annealing optimizer** that varied the placement, strength and orientation of the final matching quadrupoles.

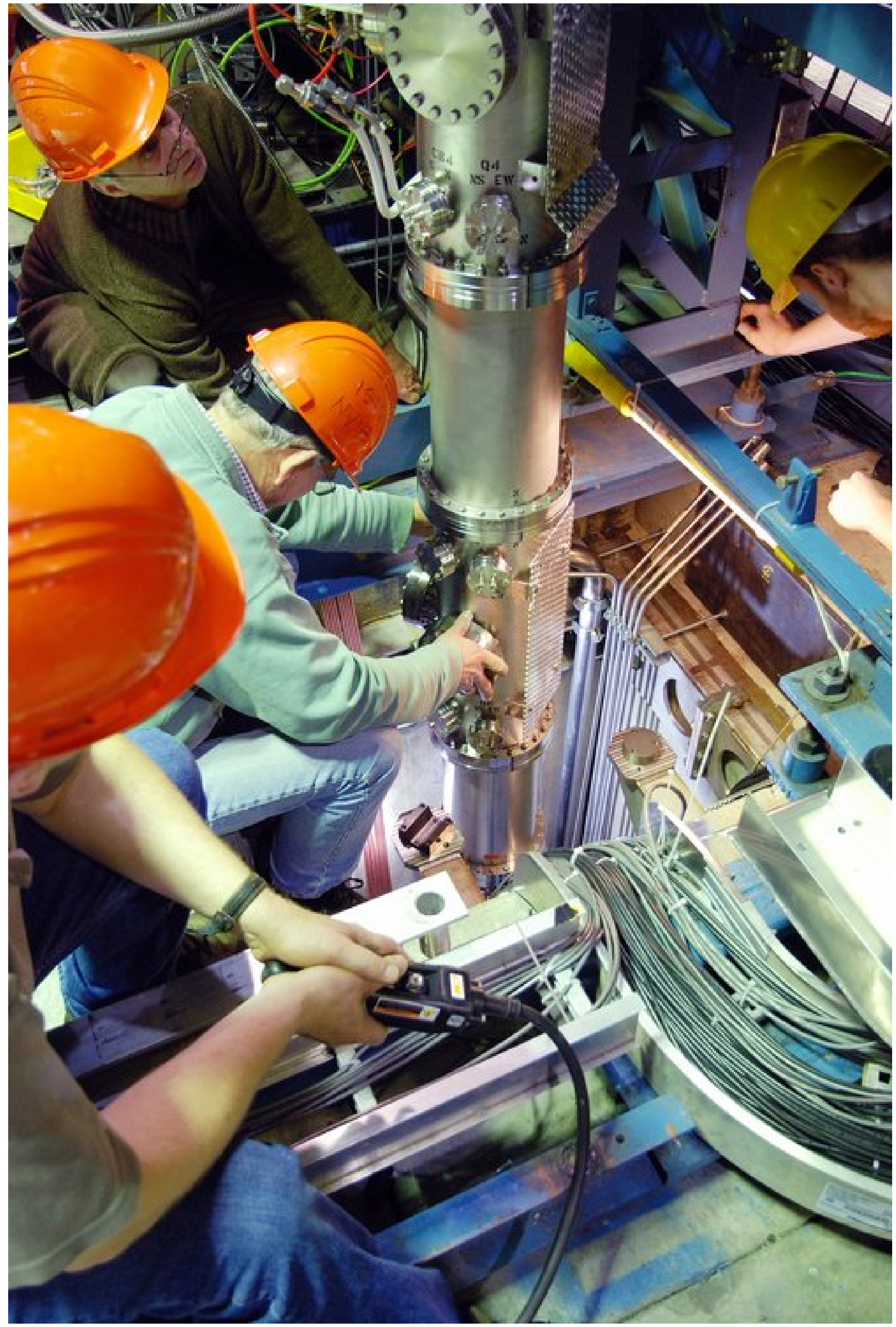
The minimization penalty parameter was the vertical and horizontal beam sizes weighted by their tunes. Sizes are calculated every half turn. Importantly, the radial size used is not the apparent size, but the **size with dispersion removed**, which is considerably smaller. The reason for this is we do not care about radial turn width as long as turn width and energy width are correlated, because we do not need separate turns.

New (left) vs. Old (right)



Detail: New (left) vs. Old (right)





Results

Selected e-log entries...

2011-04-14 14:00:00	Injecting, seeing a few nA on Q2 VF.	- Bob Scheepmaker
2011-04-15 08:18:32	Roman has been tuning the cyclotron. Transmission is 12%. 73nA to HE3.	- Angela Hoiem
[intervening time]	[Bunchers inoperative; attempts to fix.]	
2011-04-16 15:00:57	With Iouri's help, we have found the other end of the cable from the RF and connected it up. We now have bunching.	- Jaswinder Uppal
2011-04-16 15:04:55	We now have ISIS Bunchers working. We have 24% tx after about 5 minutes of tuning.	- Jaswinder Uppal
2011-04-16 15:20:00	Roman is here tuning ISIS.	- David Prevost
2011-04-16 15:45:00	Cyc at 62% transmission.	- David Prevost

IOW, **theoretical tune worked right out of the box**.

N.B. 12% unbunched is about as good as we ever get at 90 kV rf dee voltage. Somewhat later, we achieved 70% transmission bunched, which is about as good as we ever get.

Example 3: Linear Accelerator

Hamiltonian

With the distance along the reference trajectory s as the independent variable, the Hamiltonian is

$$\begin{aligned} & H(x, P_x, y, P_y, t, E; s) = \\ = \quad & -qA_s - \sqrt{\left(\frac{E - q\Phi}{c}\right)^2 - m^2c^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2} \end{aligned} \tag{18}$$

Potentials

The case of RF axially-symmetric electric field can be handled entirely with no electric potential ($\Phi = 0$), and time-varying vector potential. This has been presented a number of times in the past (e.g. E.E. Chambers;1968), but we are interested in the following more experimentally-useful case: The electric field along the axis $\mathcal{E}(s)$ has been measured and is therefore known, and the geometry is exactly axially symmetric.

Rob Ryne(1991) has treated this case, and we use his vector potential $\vec{A}(x, y, s, t)$ directly.

$$A_x = \frac{\mathcal{E}'(s)}{2}\frac{\sin(\omega t + \theta)}{\omega}\, x \tag{19}$$

$$A_y = \frac{\mathcal{E}'(s)}{2}\frac{\sin(\omega t + \theta)}{\omega}\, y \tag{20}$$

$$A_s = \left(-\mathcal{E}(s) + \frac{x^2 + y^2}{4}\left[\mathcal{E}''(s) + \frac{\omega^2}{c^2}\mathcal{E}(s)\right]\right)\frac{\sin(\omega t + \theta)}{\omega} \tag{21}$$

This is Coulomb/Lorenz gauge, satisfies Maxwell equations to second order in transverse coordinates, gives correct on-axis $\vec{\mathcal{E}} = -\partial\vec{A}/\partial t = \mathcal{E}\cos(\omega t + \theta)$.

A Word About Coordinates 5 and 6

SLAC-91 (Karl Brown) mentions "At any position in the system... ", so time t is **NOT** the independent variable. "...particle represented by a vector":

$$(x, \theta, y, \phi, l, \delta)$$

(where $\delta \equiv \Delta P/P$)

Last two should be $(t - t_0, E - E_0)$ or $(\Delta t, \Delta E)$, not $(l, \Delta P/P)$.

If we scale by βc, we can make them match (sort of), since $\beta c \Delta t = z$, $\Delta E/(\beta c) = \Delta P$, but **ONLY TRUE OF MAGNETIC ELEMENTS** $\Phi = 0$.

Mathematical Formulation of TRANSPORT(*)

The following of a charged particle through a system of magnetic lenses may be reduced to a process of matrix multiplication. At any specified position in the system an arbitrary charged particle is represented by a vector (single column matrix), X, whose components are the positions, angles, and momentum of the particle with respect to a specified reference trajectory.

$$\text{i. e.} \quad X = \begin{bmatrix} x \\ \theta \\ y \\ \varphi \\ \ell \\ \delta \end{bmatrix}$$

Definitions:

x = the radial displacement of the arbitrary ray with respect to the assumed central trajectory.

θ = the angle this ray makes in the radial plane with respect to the assumed central trajectory.

y = the transverse displacement of the ray with respect to the assumed central trajectory.

φ = the transverse angle of the ray with respect to the assumed central trajectory.

(*) For a more complete description of the mathematical basis of TRANSPORT, refer to SLAC Report 75, the Appendix of this report and to other References listed at the end of this report.

ℓ = the path length difference between the arbitrary ray and the central trajectory

δ = $\Delta P/P$ is the fractional momentum deviation of the ray from the assumed central trajectory.

The magnetic lens is represented by a square matrix, R, which describes the action of the magnet on the particle coordinates. Thus the passage of a charged particle through the system may be represented by the equation:

$$X(1) = R\,X(0) \qquad (1)$$

where $X(0)$ is the initial coordinate vector and $X(1)$ is the final coordinate vector of the particle under consideration; R is the transformation matrix for all such particles traversing the system (one particle differing from another only by its initial coordinate vector $X(0)$).

The traversing of several magnets and interspersing drift spaces is described by the same basic equation but with R now being the product matrix $R = R(n)\ldots R(3)R(2)R(1)$ of the individual matrices of the system elements. The following of a charged particle via TRANSPORT through a system of magnets is thus analogous to tracing rays through a system of optical lenses except that TRANSPORT is a matrix calculation which truncates the problem to either first or second-order in a Taylor's expansion about a central trajectory. For studying beam optics to greater precision than a second-order TRANSPORT calculation permits,

But: Don't know t_0 and E_0

A priori, we do not know the reference particle's energy and time coordinates. We need these in order to expand about them. They can be found from the equations of motion for $x = y = P_x = P_y = 0$:

$$\frac{dE}{ds} = \frac{\partial H}{\partial t} = q\mathcal{E}\cos(\omega t + \theta) \tag{22}$$

$$\frac{dt}{ds} = -\frac{\partial H}{\partial E} = \frac{E}{P} = \frac{1}{\beta_0 c} \tag{23}$$

These 2 are added to the 21 mentioned previously; 23 solved together.

(From here on, I drop the 0 subscript: β and γ are implicitly assumed to be the relativistic parameters of the reference particle.)

These give the functions $E_0(s)$ and $t_0(s)$ about which t and E are expanded: $E = E_0 + \Delta E$, $t = t_0 + \Delta t$. So we transform the canonical variables t and $-E$ to $(\Delta t, -\Delta E)$, using as generating function

$$G = -\left(t - \int \frac{ds}{\beta(s)c}\right)(\Delta E + E_0) \tag{24}$$

(Check: $\frac{\partial G}{\partial t} = -E$, $\frac{\partial G}{\partial(-\Delta E)} = \Delta t$.) The Hamiltonian gets the added terms

$$\frac{\partial G}{\partial s} = \frac{\Delta E + E_0(s)}{\beta(s)c} - \Delta t E_0'(s).$$

Then expanding the square root, we get:

$$H_{\Delta t} = \left(\frac{E_0}{\beta c} - P_0\right) - qA_s - \Delta t E_0'(s) + \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} + \frac{(P_x - qA_x)^2 + (P_y - qA_y)^2}{2P} \tag{25}$$

In expanding $P_x - qA_x$, $P_y - qA_y$, the time dependence disappears because it is higher order:

$$(P_x - qA_x)^2 = P_x^2 - q\mathcal{E}'\frac{\sin(\omega t_0 + \theta)}{\omega}\, xP_x + \left(\frac{q\mathcal{E}'}{2}\frac{\sin(\omega t_0 + \theta)}{\omega}\right)^2 x^2, \tag{26}$$

and similary for y. The term linear in Δt in the expansion of A_s about t_0 cancels the $-\Delta t E_0'(s)$ term, as it should but there is a remaining term quadratic in Δt, the bunching effect. This leaves

$$-qA_s - \Delta t E_0'(s) = q\mathcal{E}\frac{\sin(\omega t_0+\theta)}{\omega}\left(1-\frac{\omega^2(\Delta t)^2}{2}\right) - \frac{r^2 q}{4}\left(\mathcal{E}''+\frac{\omega^2}{c^2}\mathcal{E}\right)\frac{\sin(\omega t_0+\theta)}{\omega} \tag{27}$$

Notice the first term here and the first term in eqn. 25 depend only on the independent variable and not on the 6 dependent ones. Thus these do not affect the equations of motion and we ignore them. We have:

$$\begin{aligned} H_{\Delta t} \quad = \quad & -\frac{q\mathcal{E}}{2}\omega^2 T(\Delta t)^2 + \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} - \frac{r^2 q}{4}\left(\mathcal{E}''+\frac{\omega^2}{c^2}\mathcal{E}\right)T \\ & + \frac{P_x^2}{2P} - q\mathcal{E}'T\,\frac{xP_x}{2P} + \left(\frac{q\mathcal{E}'}{2}T\right)^2\frac{x^2}{2P} \\ & + \frac{P_y^2}{2P} - q\mathcal{E}'T\,\frac{yP_y}{2P} + \left(\frac{q\mathcal{E}'}{2}T\right)^2\frac{y^2}{2P} \end{aligned} \tag{28}$$

We defined here $T(s) = \sin[\omega t_0(s)+\theta]/\omega$ to clean up the notation a bit.

Finally, we wish to transform from $(\Delta t, -\Delta E)$ to $(z, P_z) = (-\beta c\Delta t, \Delta E/(\beta c))$. (The reason for the sign change is as follows: an early arrival implies $\Delta t < 0$, but this means the particle is **ahead** so $z > 0$.) The generating function is

$$G = -\beta c \Delta t P_z \tag{29}$$

(Check: $\frac{\partial G}{\partial \Delta t} = -\Delta E$, $\frac{\partial G}{\partial (P_z)} = z$.) The term to be added to the Hamiltonian is

$$\frac{\partial G}{\partial s} = \frac{\beta'}{\beta}\, zP_z = \frac{\gamma'}{\beta^2\gamma^3}\, zP_z = \frac{q\mathcal{E}C}{\beta c P\gamma^2}\, zP_z,$$

where $C \equiv \cos(\omega t_0 + \theta)$.

$$\begin{aligned} H_z \quad = \quad & \frac{P_x^2}{2P} - q\mathcal{E}'T\,\frac{xP_x}{2P} + \left[\frac{1}{P}\left(\frac{q\mathcal{E}'T}{2}\right)^2 - \frac{T}{2}\left(q\mathcal{E}'' + \frac{\omega^2}{c^2}q\mathcal{E}\right)\right]\frac{x^2}{2} + \\ & \frac{P_y^2}{2P} - q\mathcal{E}'T\,\frac{yP_y}{2P} + \left[\frac{1}{P}\left(\frac{q\mathcal{E}'T}{2}\right)^2 - \frac{T}{2}\left(q\mathcal{E}'' + \frac{\omega^2}{c^2}q\mathcal{E}\right)\right]\frac{y^2}{2} + \\ & \frac{P_z^2}{2\gamma^2 P} + \frac{2q\mathcal{E}C}{\beta c}\,\frac{zP_z}{2\gamma^2 P} - \frac{q\mathcal{E}}{\beta^2 c^2}\omega^2 T\,\frac{z^2}{2} \end{aligned} \tag{30}$$

Hamiltonian 2

Ryne(1991) has a transformation that gets rid of the second derivative of the on-axis elecric field. It's complicated. At the same time he transforms away the adiabatic damping; it's a neat and didactic trick but not strictly necessary for computational purposes. It is simple to just use $P_{x,y,z}$ directly and then just rescale by final P at the end.

But there's an easy way to get rid of the second derivative: it turns out that the vector potential can be simplified if we use a different Gauge.

I propose the following function

$$\Psi(x,y,s,t) = -\frac{\mathcal{E}'}{2}\frac{\sin(\omega t+\theta)}{\omega}\frac{x^2+y^2}{2} \tag{31}$$

Add the gradient of this function to the previous vector potential (19,20,21). This zeroes both A_x and A_y, leaving

$$A_s = -\mathcal{E}(s)\left(1-\frac{\omega^2}{c^2}\frac{x^2+y^2}{4}\right)\frac{\sin(\omega t+\theta)}{\omega} \tag{32}$$

This is considerably simpler, but now there is a scalar potential:

$$\Phi = -\frac{\partial \Psi}{\partial t} = \mathcal{E}' \cos(\omega t + \theta) \frac{x^2 + y^2}{4} \tag{33}$$

Now if we expand the Hamiltonian, we get a different result:

$$\begin{aligned} H_z \quad = \quad & \frac{P_x^2}{2P} + \frac{P_y^2}{2P} + \frac{q}{2\beta c} \left(\mathcal{E}' C - \mathcal{E} S \frac{\omega \beta}{c} \right) \frac{r^2}{2} + \\ & \frac{P_z^2}{2\gamma^2 P} + \frac{2q\mathcal{E}C}{\beta c} \frac{z P_z}{2\gamma^2 P} - \frac{q\mathcal{E}\omega S}{\beta^2 c^2} \frac{z^2}{2} \end{aligned} \tag{34}$$

$(C \equiv \cos(\omega t_0(s) + \theta)$, $S \equiv \sin(\omega t_0(s) + \theta))$

This is not only much simpler (P_x and P_y have their usual definitions, no transverse cross terms, no $\mathcal{E}''$), but has nice intuitive explanations for the individual terms. (1) The factor in parentheses represents usual the focal power of an RF gap, e.g. a buncher. (2) Taking the limit as $\omega \to 0$ reproduces precisely the Hamiltonian of the DC accelerator. Note that in that case, $\mathcal{E}' = -\phi''$.

Infinitesimal Transfer Matrix $\mathbf{F}$

Now that the Hamiltonian for linear motion (eqn. 34) has been obtained, it is a simple matter to find the infinitesimal transfer matrix F. Writing the equations of motion ($x' = \partial H/\partial P_x$, $P_x' = -\partial H/\partial x$, etc.), the following F-matrix is found for the axially symmetric linear accelerator:

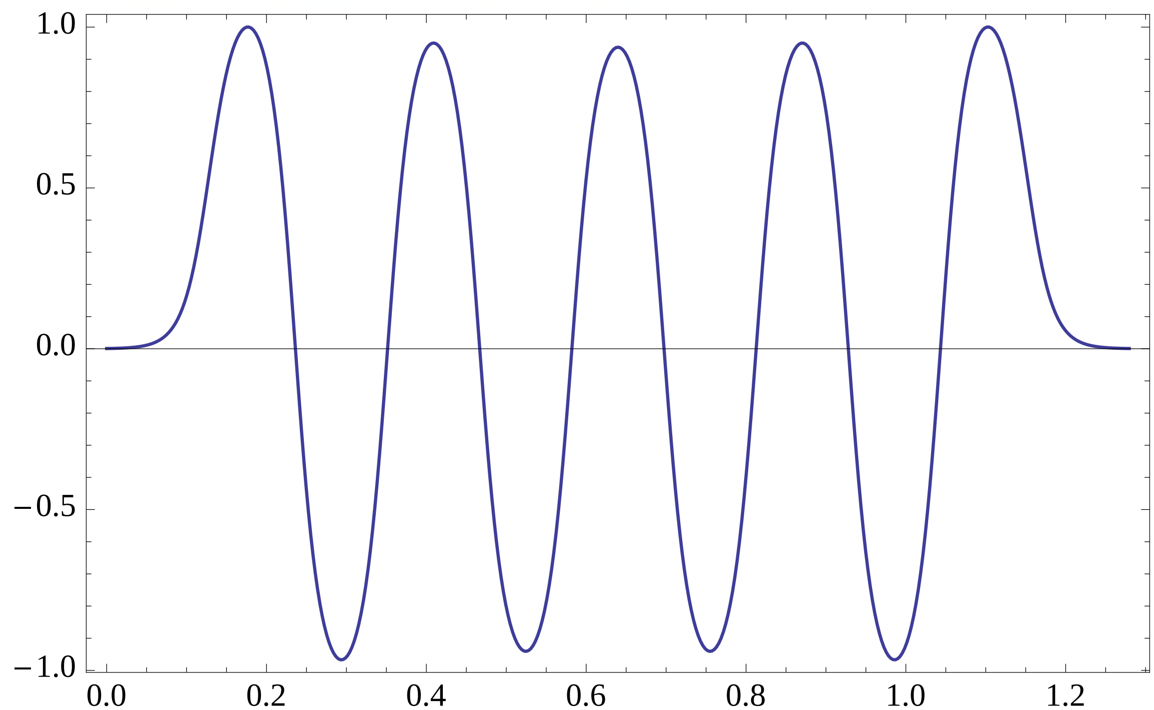
$$F = \begin{pmatrix} 0 & \frac{1}{P} & 0 & 0 & 0 & 0 \\ \mathcal{A}(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{P} & 0 & 0 \\ 0 & 0 & \mathcal{A}(s) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta'}{\beta} & \frac{1}{\gamma^2 P} \\ 0 & 0 & 0 & 0 & \mathcal{B}(s) & -\frac{\beta'}{\beta} \end{pmatrix}. \tag{35}$$

where we have defined:

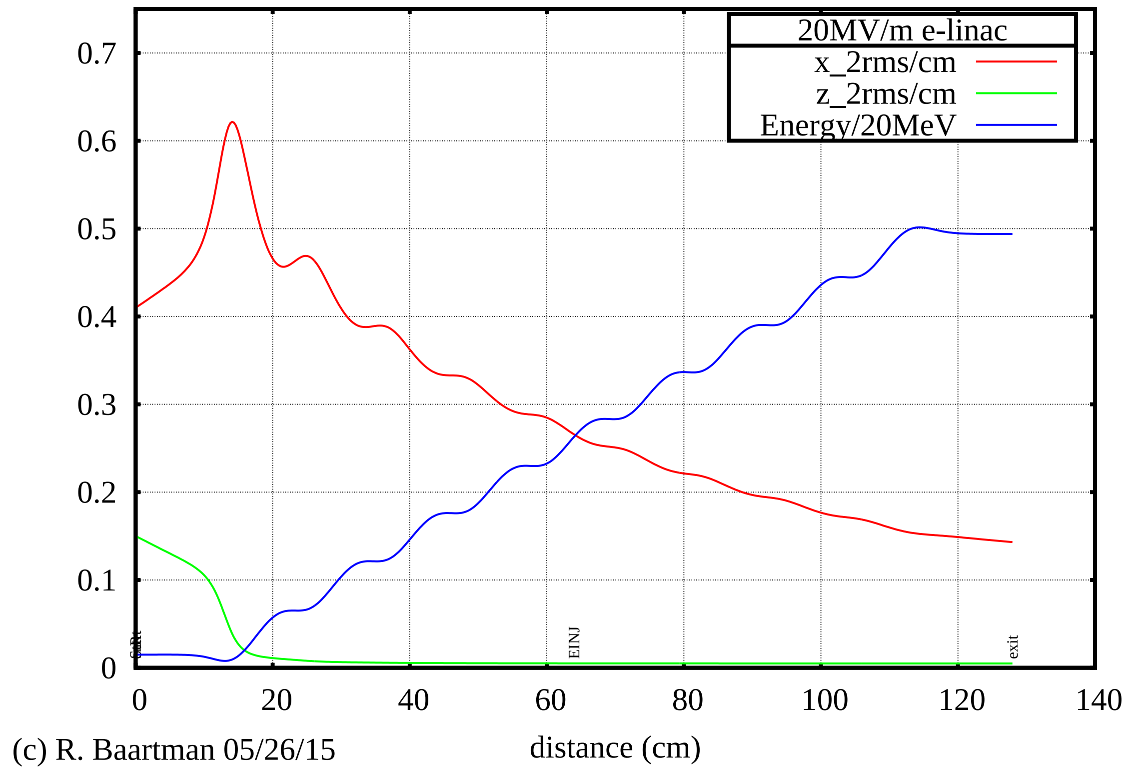
$$\mathcal{A}(s) = \frac{-q}{2\beta c}\left(\mathcal{E}'C - \mathcal{E}S\frac{\omega\beta}{c}\right),\ \mathcal{B}(s) = \frac{q\mathcal{E}\omega S}{\beta^2 c^2}. \tag{36}$$

Example Calculations

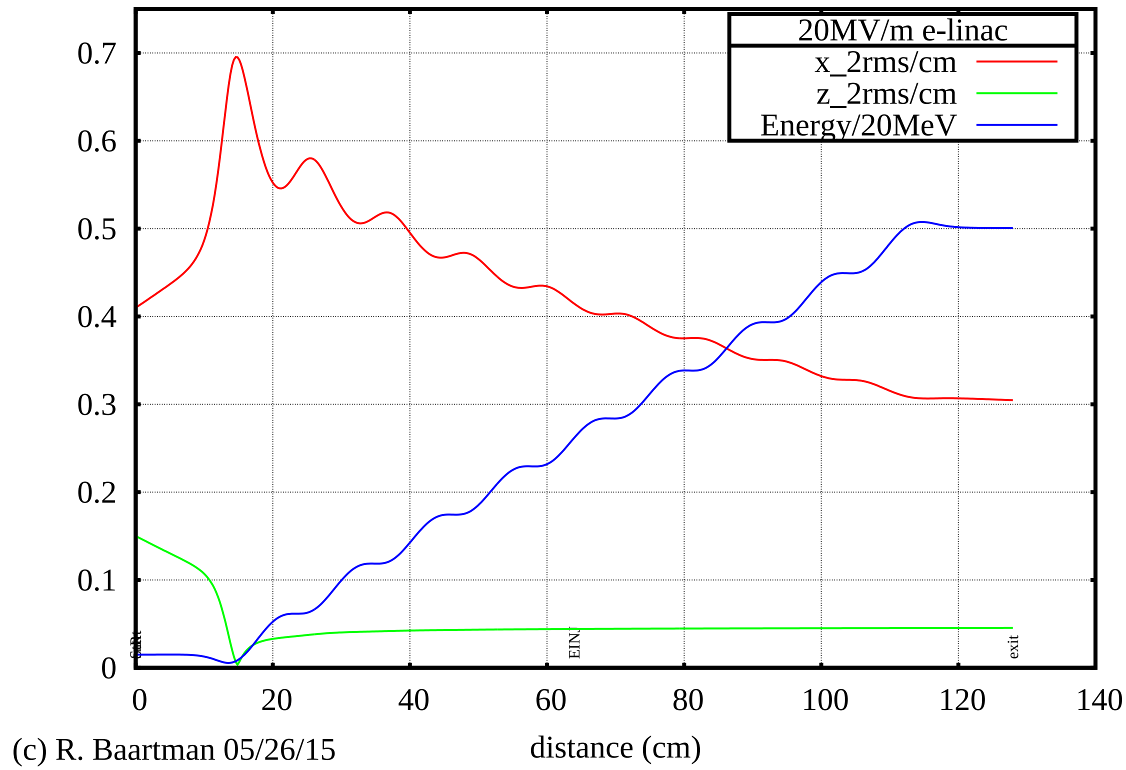
The TRIUMF injector electron linac, EINJ, takes bunches at 300 keV to ~ 10 MeV if properly phased and the peak gradient is 20 MV/m. Here is the input $\mathcal{E}(s)$. TRANSOPTR interpolates with cubic splines.



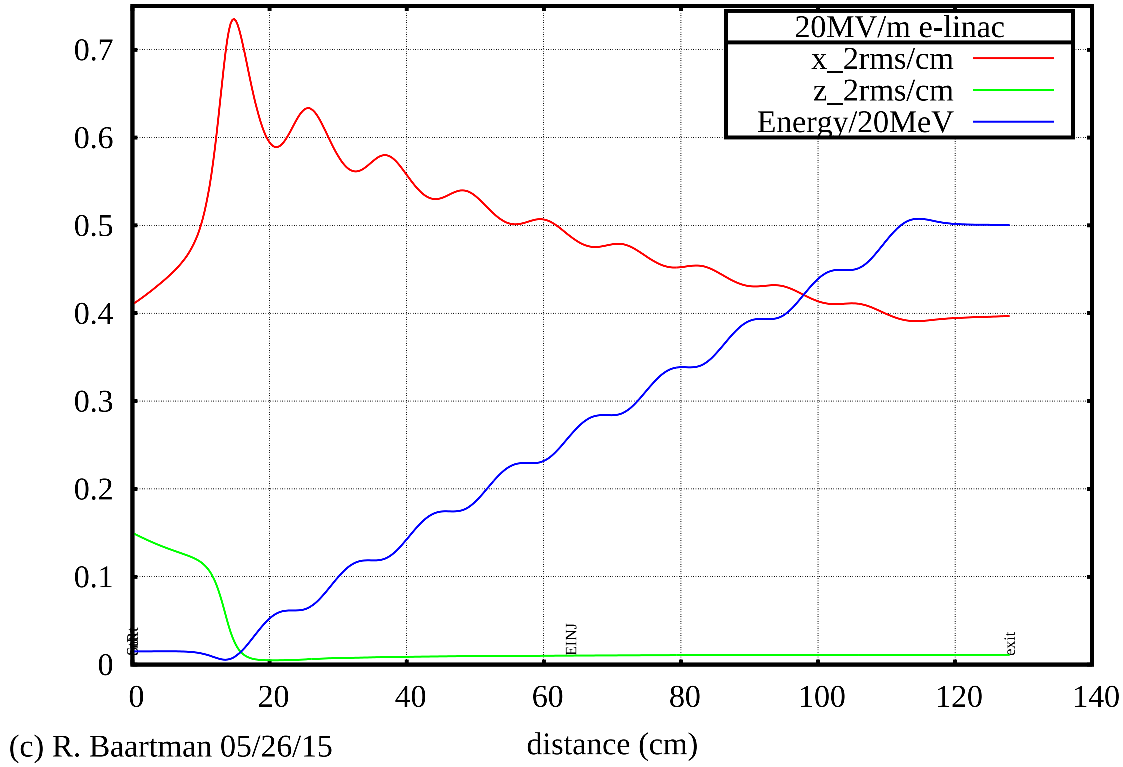
This is example for phase $\theta = 0$ at the start of the calculation. Red is the 2rms transverse size, and green is the 2rms longitudinal (bunch length). The input bunch parameters are somewhat arbitrary, roughly the condition for a minimum beam size at exit. This particular case has zero bunch charge.



In this second example, $\mathtt{TRANSOPTR}$ is instructed to fit the 65 matrix element to zero. This makes energy insensitive to input phase, thus finding the peak energy gain phase. This phase turns out to be $\theta = -15.46°$.



In the third example, bunch charge has been raised to $30\,$pC.



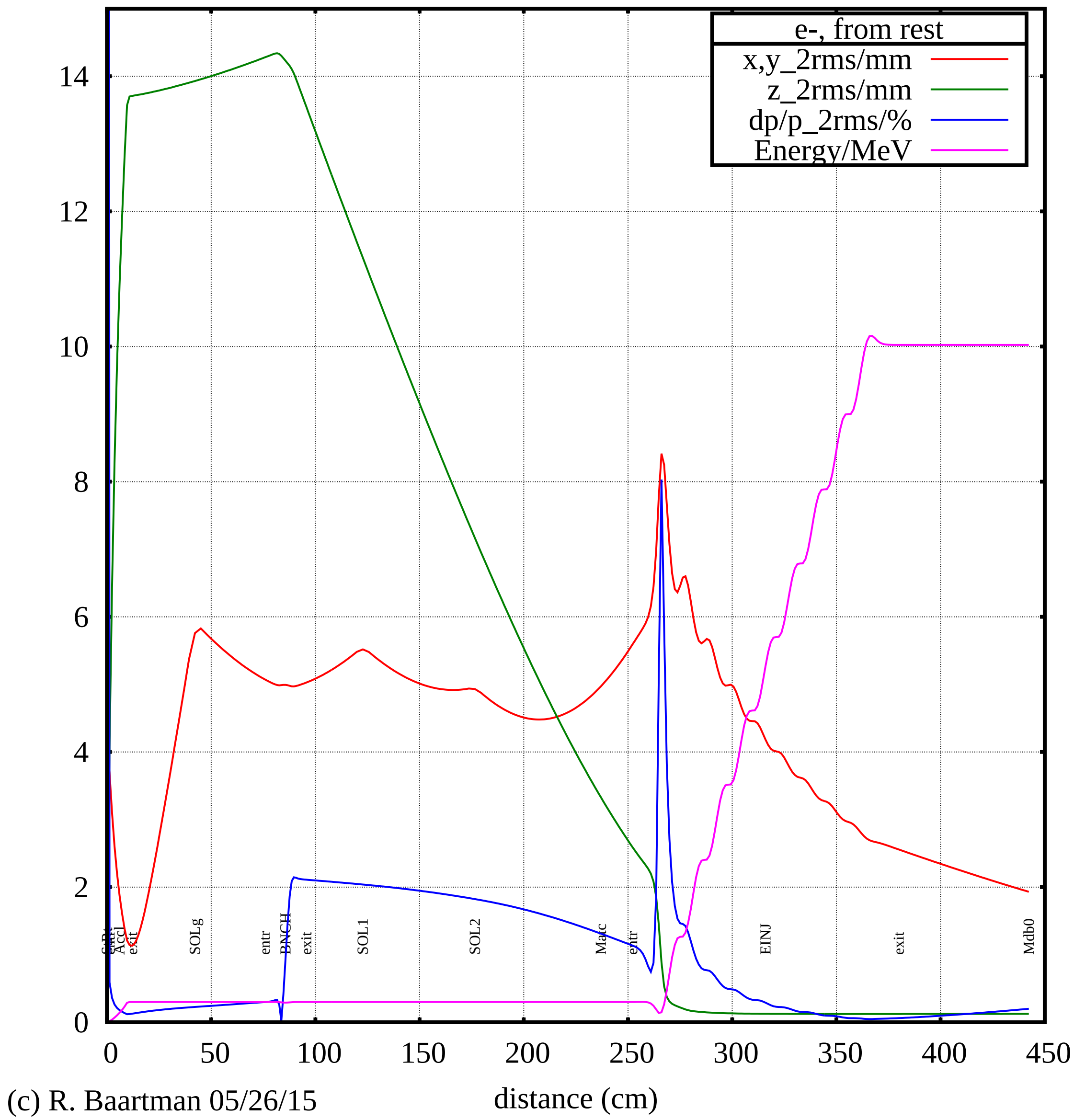
Timing

Each calculation above takes roughly 400 Runge-Kutta steps for 2400 calls to the `SCLINAC` routine. This gives 5-figure accuracy to the transfer matrix and the σ-matrix, and is easily enough for describing reality considering that the on-axis field is only known to 2 or 3 significant figures. The extra accuracy is useful, however for fitting matrix or beam matching, which is done with a downhill simplex method, or simulated annealing for cases of more than 3 fitting parameters.

On my unremarkable, circa 2006 Intel desktop, each run through the linac takes about 17 milliseconds with zero bunch charge and 25 milliseconds with space charge. The difference is due to the Carlson elliptic integrals needed for the space charge case.

On a typical optics matching case, one varies 2 solenoids, the buncher amplitude, and the linac phase, to minimize the bunch size and energy spread at the linac output. A calculation with such a fit requires typically a half million total calls to `SC` (the space charge routine for no-linac case) and `SCLINAC`, and so takes about 5 seconds CPU time. The result is shown below.

The bunch charge is 15 pC. Each calculation starts from the cathode whereas it would have been more efficient to store the beam parameter set at the buncher entrance and start it from there.
The Buncher itself, located at $s = 85$ cm, is calculated as just another linac, phased to give no energy gain.



Conclusions

Envelope calculations (`TRANSOPTR`) are most efficient for linear optics with space charge and/or any time the focal parameters vary with s and no closed-form matrix is possible.

- Beamline design (including minimizing aberrations, but that's another talk...)
- On-the-fly tuning/optimization.

Not good for

- Designing higher order corrections
- Collimation, High intensity beam losses due to halo.