Phase Space Tomography with Dispersion

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A key element in the design & tuning of beamline optics is the assumed initial beam parameters. The common assumption of elliptical initial phase space distributions characterized by the Courant-Snyder parameters is generally expected to be a valid description of beams extracted from circular machines. However, there are exceptions. One example is resonantly extracted beam, wherein the non-linear nature of the extraction process guarantees that the transported beam can not be described by conical functions.

This note provides a brief prescription for determining an arbitray transverse phase space distribution. The procedures assumes that the initial distribution $\rho(x, x', \delta)$ of interest is separated from a profile monitor downstream by some configuration of mangetic elements. Beam profiles on the monitor and tuning of the magnets are the only diagnostic tools.

A particle co-ordinate (x,x',δ) in the phase space density ρ(x,x',δ) will appear on a wire located transversely by an amount x_0 in the detector according to the transfer matrix M:

(1):

$$\begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}$$

The matrix elements m_{ij} are determined entirely by the hardware configuration and shouldn't make any reference to Courant-Snyder parameters (that would defeat the whole point of this exercise).

The paramaters x', m_{12} and δ, m_{13} can be tranformed to new variables $\hat{x}', \widehat{m}_{12}$ and $\hat{\delta}, \widehat{m}_{13}$ via:

(2):

$$\widehat{x'} = x' \cdot 1m \quad : \quad \widehat{m}_{12} = {}^{m_{12}}/_{1m}$$

$$\hat{\delta} = \delta \cdot 1m \quad : \quad \widehat{m}_{13} = {}^{m_{13}}/_{1m}$$

Now $\hat{x}'$ has dimensions of length, $\widehat{m}_{12}$is dimensionless, and likewise for $\hat{\delta}, \widehat{m}_{13}$. This is an utterly pedantic and very annoying, but necessary, piece of bookkeeping to define dimensionless phase space orientation angles θ,φ later.

The intensity on the wire at position x_0 is the projection of $\rho\left(x, \hat{x}', \hat{\delta}\right)$ determined from the transform:

(3):

$$P(x_0) = \iiint dx\, d\widehat{x'}\, d\widehat{\delta}\, \rho\left(x, \hat{x}', \hat{\delta}\right) \delta(m_{11}x + \widehat{m}_{12}\hat{x}' + \widehat{m}_{13}\,\hat{\delta} - x_0)$$

This expression can be simplified somewhat by introducing a scaling paramater "s" :

(4):

$$s = \sqrt{m_{11}^2 + \hat{m}_{12}^2 + \hat{m}_{13}^2}$$

From which phase 'oriention' angles θ, φ can be defined:

(5):

$$cos\theta \cdot cos\varphi = {}^{m_{11}}/_{s} \;\; ; \;\; sin\theta \cdot cos\varphi = {}^{\hat{m}_{12}}/_{s} \; ; \;\; sin\varphi = {}^{\hat{m}_{13}}/_{s}$$

with $0 \le \theta \le 2\pi, -{}^{\pi}/_{2} \le \varphi \le +{}^{\pi}/_{2}$. The angles θ, φ are ***not*** related to betatron phase advance.

The profile integral now simplifies to:

(6):

$$P({}^{x_0}/_{s}, \theta, \varphi) = \iiint dx\, d\hat{x'}\, d\hat{\delta}\, \rho(x, \hat{x}', \hat{\delta})\, \delta[s(cos\theta \cdot cos\varphi\, x + sin\theta \cdot cos\varphi\, \hat{x'} + sin\varphi\, \hat{\delta} - {}^{x_0}/_{s})]$$

or:

(7):

$$s \cdot P({}^{x_0}/_{s}, \theta, \varphi) = \iiint dx\, d\hat{x}' d\hat{\delta}\, \rho(x, \hat{x}', \hat{\delta})\, \delta(cos\theta \cdot cos\varphi\, x + sin\theta \cdot cos\varphi\, \hat{x'} + sin\varphi\, \hat{\delta} - {}^{x_0}/_{s})$$

With $\hat{x}_0 = {}^{x_0}/_{s}$, this corresponds to a modified profile function :

(8): *

$$\widehat{P_s}(\hat{x}_0, \theta, \varphi) = s \cdot P({}^{x_0}/_{s}, \theta, \varphi)$$

The modified profile $\widehat{P_s}(\hat{x}_0, \theta, \varphi)$ is related to the measured profile $P(x_0, \theta, \varphi)$ by being:

- scaled vertically (intensity) by "s", and;
- scaled horizontally (position) by "1/s".

Taking its Fourier transform $\widehat{T_s}$:

(9): *

$$\begin{aligned} \hat{T}_s(w, \theta, \varphi) &= \int d\hat{x}_0\, e^{-2\pi i w \hat{x}_0}\, \widehat{P_s}(\hat{x}_0, \theta, \varphi) \\ &= \iiint dx\, d\hat{x}'\, d\hat{\delta}\, e^{-2\pi i(wx\, cos\theta cos\varphi + w\hat{x}' sin\theta cos\varphi + w\hat{\delta}\, sin\varphi)}\, \rho(x, \hat{x}', \hat{\delta}) \end{aligned}$$

Where the 2nd line follows from eqns (7) & (8).

The initial phase space distribution $\rho(x, \hat{x}', \hat{\delta})$ is the inverse Fourier transform of $\widehat{T}_s(w, \theta, \varphi)$, which in spherical co-ordinates produces the final result:

(10): *

$$\rho(x, \hat{x}', \hat{\delta}) = \iiint w^2 dw \ \ cos\varphi \ d\varphi \ d\theta \ e^{+2\pi i(wx\ cos\theta cos\varphi + w\hat{x}' sin\theta cos\varphi + w\hat{\delta}\ sin\varphi)}\, \widehat{T}_s(w, \theta, \varphi)$$

$$\left[\text{more thoroughly annoying pedantry} \ : \ \rho(x, x', \delta) = \rho(x, {}^{\hat{x}'}\!/_{m}\, , {}^{\hat{\delta}}\!/_{m})\right]$$

Using eqns. (8), (9) & (10) the initial transverse phase space distribution $\rho(x, x', \delta)$ can be determined unambiguously just from measured profiles.

Summarizing the steps involved in extracting $\rho(x, x', \delta)$:

1) for a given tune of the magnets [i.e; a given θ, φ eqn(5)] the beam profile is measured as a function of transverse position x_0.
2) the profile is rescaled to a new function $\widehat{P}$ [eqn(8)].
3) the function $\widehat{P}$ is Fourier transformed w.r.t. $\hat{x}_0$ [$\widehat{T}$ eqn(9)].
4) retune the magnet string to obtain a new θ, φ pair.
5) repeat steps 1 → 4 until 'sufficient' values of θ, φ have been collected to characterize $\widehat{T}_s(w, \theta, \varphi)$.
6) $\rho(x, x', \delta)$ is then the inverse Fourier transform of $\widehat{T}_s(w, \theta, \varphi)$ [eqn(10)].

Multi-dimensional numerical integration is tough to do accurately even with analytic functions. Here, with discrete data points it's got to be harder. Accuracy in integration over profile positions is limited by the number and spacing of wires in the detectors. Limits on the accuracy of the angular θ, φ integrations are determined solely by the resolve, tenacity, perseverance and fortitude of the investigator.

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