Robinson stability criterion for double RF systems

C.Y. Tan

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Introduction

- The questions arose as to whether there is a simple inequality like that for a single RF system for checking beam stability
  - Robinson stability criterion for single RF system
  - Above transition: $1-\frac{I_b R_s}{V_c}\frac{\sin\psi\cos\psi}{\cos 2\phi_s}>0$
- It turns out that T.S. Wang (LANL) had done this in a two PAC papers in the 1990s
  - Problems:
    - Many steps were skipped in the derivation.
    - I found many typos (or really errors) in the derivation.

Stability criteria for double RF systems

- The method to get the solution is write out the differential equations that time evolve the system with 2 RF systems
  - Linearize the odes.
    - RLC circuit is basis for differential equations
  - Perturb the cavity voltages, phases of beam currents and cavity about their steady state values.
  - Create a 6x6 matrix that relates all the odes and set its determinant to zero for non-trivial solutions
  - Apply Routh-Hurwitz stability criterion to determine stability.
    - Technically do not need to do this anymore. Mathematica can find roots of 6th order polynomial easily.
- The algebra is ridiculously long.
  - Checked with Mathematica
  - Robyn double checked matrix entries
  - So, the results should be correct.
  - My results are **different** than Wang's although they are close.

Stability criterion for double RF systems (cont'd)

- Start with 6th order characteristic equation
  - $$s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0$$
- Where
  - $$b_5 = 2(\alpha_1 + \alpha_2) \qquad \text{matches} \qquad (49)$$
    $$b_4 = \rho_1 + \rho_2 + 4\alpha_1\alpha_2 + \Omega_s^2(\xi - 1) \qquad \text{matches} \qquad (50)$$
    $$b_3 = 2(\alpha_1\rho_2 + \alpha_2\rho_1) - 2\Omega_s^2(1 - \xi)(\alpha_1 + \alpha_2) \qquad \text{matches} \qquad (51)$$
    $$b_2 = \rho_1\rho_2 + \Omega_s^2(\xi - 1)(\rho_1 + \rho_2 + 4\alpha_1\alpha_2) - \lambda_1 + \xi\lambda_2 \qquad \text{no match} \qquad (52)$$
    $$b_1 = 2(\xi - 1)(\alpha_2\rho_1 + \alpha_1\rho_2)\Omega_s^2 - 2(\alpha_2\lambda_1 - \xi\alpha_1\lambda_2) \qquad \text{no match} \qquad (53)$$
    $$b_0 = \Omega_s^2(\xi - 1)\rho_1\rho_2 - \lambda_1\rho_2 + \xi\lambda_2\rho_1 \qquad \text{no match} \qquad (54)$$

$$\Omega_s = \sqrt{\frac{q\eta\omega_1^2 v_{1s}\cos\phi_{1s}}{2\pi h E_0 \beta^2}} \qquad \xi = -\frac{n v_{2s}\cos\phi_{2s}}{v_{1s}\cos\phi_{1s}}$$

$$\rho_k = \alpha_k^2 \sec^2\psi_k \qquad \lambda_k = \frac{\alpha_k^2\Omega_s^2 i_{\omega_k} R_k \tan\psi_k}{v_{ks}\cos\phi_{ks}}$$

Stability criterion for double RF systems (cont'd)

- Use Routh-Hurwitz criteria to determine stability

$$b_0, b_1, b_2, b_3, b_4, b_5 > 0$$

$$b_5 \det \begin{pmatrix} b_4 & b_5 \\ b_2 & b_3 \end{pmatrix} > \det \begin{pmatrix} b_3 & b_5 \\ b_1 & b_3 \end{pmatrix}$$

$$b_1 \det H_4 > b_0 \left[+b_0 b_5^3 + b_1 b_5 (b_3 - b_4 b_5) + b_3 \det H_3 \right]$$

where

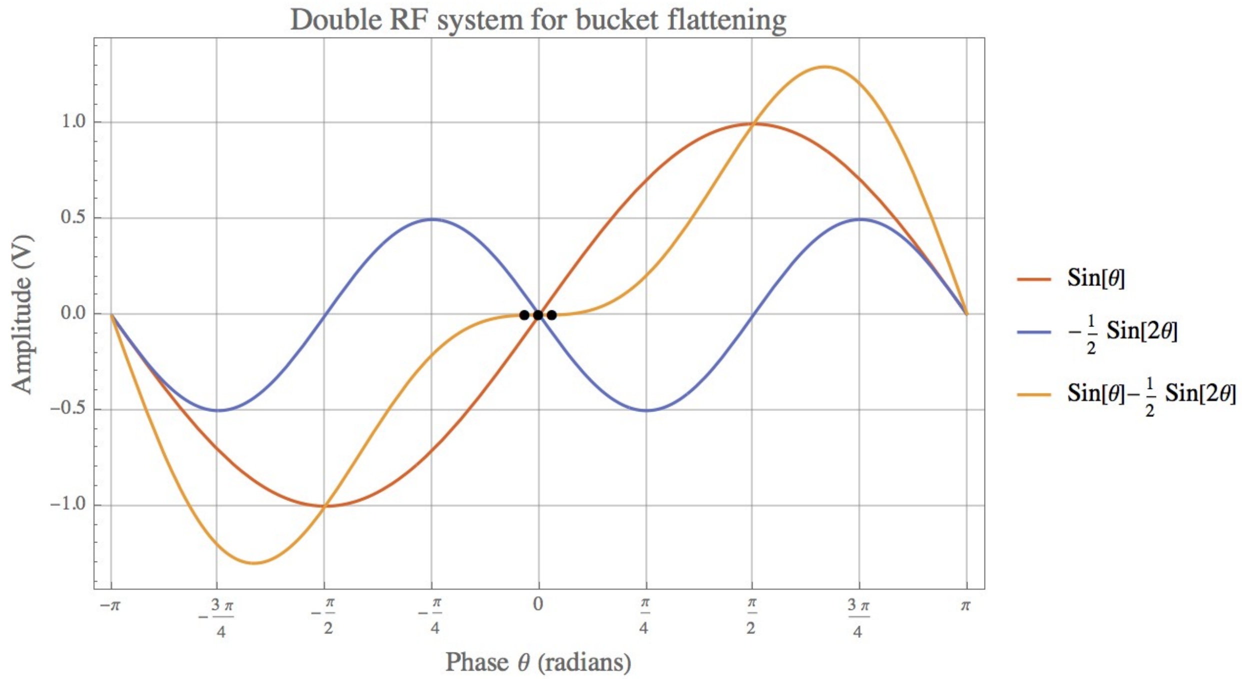
$$\det H_3 = \det \begin{pmatrix} b_5 & 1 & 0 \\ b_3 & b_4 & b_5 \\ b_1 & b_2 & b_3 \end{pmatrix} > 0 \quad \text{and} \quad \det H_4 = \det \begin{pmatrix} b_5 & 1 & 0 & 0 \\ b_3 & b_4 & b_5 & 1 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} > 0$$

All the above inequalities have to be satisfied for stability!

Check with single RF system

- Use condition $b_0 > 0$ requirement
  - $$\Omega_s^2(\xi - 1)\rho_1\rho_2 > \lambda_1\rho_2 - \xi\lambda_2\rho_1$$
  - $$\Rightarrow \quad \frac{i_{\omega_1} R_1}{v_{1s}\cos\phi_{1s}} < \frac{2(\xi - 1)}{\sin 2\psi_1} + \xi \frac{i_{\omega_2} R_2 \sin 2\psi_2}{v_{2s}\cos\phi_{2s}\sin 2\psi_1}$$
- Set $\xi = 0$, i.e. voltage in second cavity is off.
  - $$\left.\begin{array}{r} \dfrac{i_{\omega_1} R_1 \sin 2\psi_1}{v_{1s}\cos\phi_{1s}} < 2 \\ \Rightarrow \quad 1 - Y\dfrac{\sin\psi_1\cos\psi_1}{\cos\phi_{1s}} > 0 \end{array}\right\}$$
- The above is the same as that of the normal Robinson instability criterion.

Apply to our system at injection



- Note that below transition, if only the 2nd harmonic is on, it is **phase unstable.**
  - So stability has to be evaluated when both systems are ON and not individually.

Parameters and results

Parameter	Value	Units	Description
Fundamental cavity at 38 MHz			
Ω_s	69.68×10^3	rad/s	synchrotron frequency
Q_1	300	–	quality factor
N_c	20	–	number of fundamental cavities
R_1	$N_c \times (18 \times 10^3) = 360 \times 10^3$	Ω	total shunt impedance of N_c cavities
ϕ_{1s}	π	rad	synchronous phase. In normal convention, $\Phi_{1s} = 0$
v_{1s}	$N_c \times (10 \times 10^3) = 200 \times 10^3$	V	accelerating voltage from N_c cavities
2nd harmonic cavity at 76 MHz			
Q_2	3252	–	quality factor
R_2	92.75×10^3	Ω	shunt impedance
ϕ_{2s}	π	rad	synchronous phase. In normal convention, $\Phi_{2s} = 0$
v_{2s}	-100×10^3	V	accelerating voltage. Note negative sign.

TABLE III. The parameters of the Booster fundamental and 2nd harmonic cavities for maximum flattening of the bucket at injection.

Parameter	Value	Units	Description
ψ_1	0.663	rad	detuning angle of fundamental cavity
ψ_2	-0.048	rad	detuning angle of 2nd harmonic cavity
α_1	3.97×10^5	rad/s	damping factor of fundamental cavities
α_2	7.34×10^5	rad/s	damping factor of 2nd harmonic
ρ_1	2.55×10^{11}	$(\text{rad/s})^2$	See Eq. 55
ρ_2	5.40×10^9	$(\text{rad/s})^2$	See Eq. 55
λ_1	-4.70×10^{20}	$(\text{rad/s})^4$	See Eq. 55
λ_2	-6.11×10^{16}	$(\text{rad/s})^4$	See Eq. 55
ξ	1	–	See Eq. 55

TABLE IV. The derived quantities for 6×10^{12} protons using the parameters shown in Table III.

I am calculating the case when we have maximal flattening: When fundamental is at 200 kV and 2$^{\text{nd}}$ harmonic is at 100 kV

$$i_{\omega_1} = F_1 \times 2I_{\text{DC}} = I_{\text{DC}} = 0.43\ \text{A}$$

$$i_{\omega_2} = F_2 \times 2I_{\text{DC}} = 0.12 \times I_{\text{DC}} = 0.05\ \text{A}$$

$$\left.\begin{array}{ll} b_0 = 2.5 \times 10^{30} & b_1 = 6.9 \times 10^{25} \\ b_2 = 1.8 \times 10^{21} & b_3 = 4.2 \times 10^{16} \\ b_4 = 3.8 \times 10^{11} & b_5 = 9.4 \times 10^{5} \end{array}\right\} > 0$$

$$\det H_3 = 1.2 \times 10^{34} > 0$$

$$\det H_5 = -2.0 \times 10^{80} < 0$$

?? Really? Don't believe this!

Approximation is necessary

- The order of magnitude difference is astounding!
  - b0 = 2.5e30 while b6=1 and b5 = 9.4e5.
    - Note: I can solve for the roots of 6th order polynomial, and indeed there are roots in the positive half of the complex plane.
  - I suspect the disparity is sizes of the coefficients makes the H5 result suspicious.
  - Solution: Make an approximation to neglect *s* terms with ratios bn/b0 < 1e-10.
  - This gives me new characteristic equation:
  - $$s^2 + 86195s + 2.7 \times 10^9 = 0$$
  - With roots $s = -43 \times 10^3 \pm (29 \times 10^3)j$
  - Since the roots are in the left half of the complex plane, the system is stable!

Conclusion

- Analytic form of Robinson stability criterion for double RF systems may not be that useful if applied directly.
  - At this point I'm not even sure that the conclusion from the analytic formulas (which are linearizations) are correct!
    - Different scenarios give me reason to suspect that the criterion is bogus.
    - Need to understand why because the formalism doesn't look wrong. But its conclusions may be and it annoys me!
  - The coefficients have HUGE differences in magnitude
    - Incorrect results unless approximations are applied. (MAYBE, looks like *a posteriori*)
    - Need proof that the "approximations" are ok.
  - I'm leaving the solution here.
    - Somebody else can look at it to see if I screwed up in the analysis. (I don't think so :))
  - I have also did the analysis for bunch rotation at extraction and again, have to apply approximation to get sane results.