MI NOTES
08/90

Burning up the Beampipe

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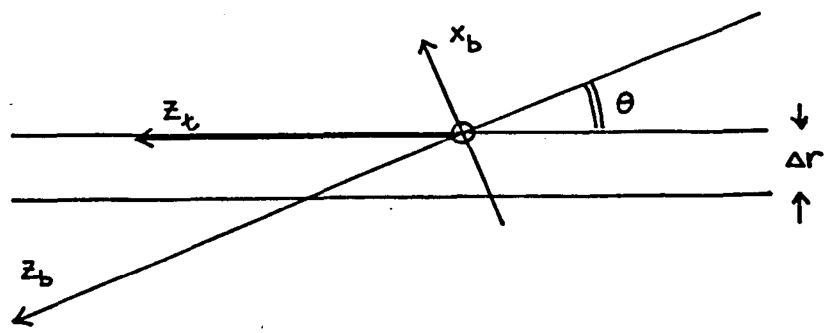
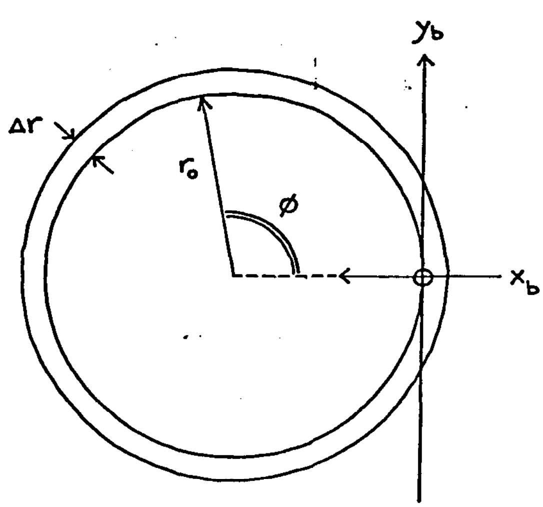
The question is; if the beam runs into the beampipe wall, after what length of time, if ever, does the pipe melt and thereby lose vacuum. The geometry of the situation and relationship between the reference frames of the beam and wall are illustrated below.



a) View along the y_{beam} axis.



b) View along the z_{target} axis.

1. Approximate Energy Deposition in the Pipe

The energy lost by a single proton hitting a wall of thickness Δr at a small angle θ is:

$$\Delta E \approx \frac{\Delta r}{\theta} \frac{\delta E}{\delta x} \tag{1}$$

$\delta E / \delta x$ for an ultra-relativistic proton incident on a composite target of partial densities ρ_n, electronic charges Z_n, and mass numbers A_n is given by:

$$\frac{\delta E}{\delta x} \approx D \sum_n \frac{\rho_n Z_n}{A_n} \left[\ln \left(\frac{2 m_e \gamma}{I_n} \right) - 1 \right] \tag{2}$$

with $I_n \approx 16\, Z_n^{0.9}$ eV and $D = .3070$ MeV/cm^2/gm. For 150 GeV protons incident on 15% chrome steel, (2) predicts $\delta E / \delta x$ to be ≈ 13.5 MeV/cm.

If the small increase in beam size due to multiple scattering is ignored, then for N incident protons/sec characterized by a Gaussian cross-sectional density:

$$\rho(x_b, y_b) = \frac{N}{2\pi\sigma^2}\, e^{-(x_b^2 + y_b^2)/2\sigma^2} \tag{3}$$

the energy deposited at some point (ϕ, r, z) in the pipe is simply proportional to the beam density at that point. Furthermore, since the pipe radius r_0 is by necessity several times larger than σ of the beam, the curvature of the pipe over the region of energy deposition can be neglected. For this case, in the reference frame of the pipe, the energy deposition density becomes:

$$\hat{\rho}_E(\phi, r, z) \approx \frac{N}{2\pi\sigma^2} \left[\frac{\delta E}{\delta x} \right] e^{-r^2\phi^2/2\sigma^2}\, e^{-(\theta z - (r - r_0))^2/2\sigma^2} \tag{4}$$

For a thin wall, $\Delta r / r_0 \ll 1$, the weak dependence of $\hat{\rho}_E$ on r can be averaged to produce:

$$\hat{\rho}_E(\phi, z) \approx \frac{N}{2\pi\sigma^2} \left[\frac{\delta E}{\delta x} \right] e^{-r_0^2\phi^2/2\sigma^2}\, e^{-(\theta z - \Delta r/2)^2/2\sigma^2} \tag{5}$$

with $-\pi < \phi < \pi$ and $-\infty < z < \infty$.

2. Diffusion of Heat in the Beampipe

If energy losses due to radiative and convective processes are ignored, the temperature of the pipe changes with time according to the diffusion equation:

$$\frac{\partial T}{\partial t} = \frac{\hat{\rho}_E}{\rho c} + \frac{\kappa}{\rho c}\left[\frac{1}{r_0^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] \tag{6}$$

where ρ is the density, c the specific heat, and κ the thermal conductivity. In general, κ can vary significantly with temperature, but for steels with chrome content in the vicinity of 10→20%, κ is roughly constant from room temperature up to the melting point.

From (5) it can be seen that $\hat{\rho}_E$ is a very slowly varying function of z relative to ϕ for incident angles $\theta \approx$ mrads. As a result, in the neighborhood of the peak in the energy distribution, at $z \approx \Delta r/2\theta$, it can be concluded that:

$$\frac{\partial^2 T}{\partial z^2} \ll \frac{1}{r_0^2}\frac{\partial^2 T}{\partial \phi^2} \tag{7}$$

In other words, near the maximum in the temperature distribution heat diffuses predominantly around the pipe rather than along its length, as one expects intuitively. Near the center therefore the temperature T can be approximately separated into:

$$T(\phi, z; t) \approx T_0 + \tau(\phi; t)\, e^{-(\theta z - \Delta r/2)^2/2\sigma^2} \tag{8}$$

where T_0 is the temperature at time $t = 0$, and corrections to (8) are of $O[\theta^2]$.

It follows from (6) and (8) that τ is the solution of the approximate equation:

$$\frac{\partial \tau}{\partial t} \approx \rho_0 \frac{e^{-r_0^2\phi^2/2\sigma^2}}{\rho c} + \frac{\kappa}{\rho c r_0^2}\frac{\partial^2 \tau}{\partial \phi^2} \tag{9}$$

where $\rho_0 \equiv N\ [\delta E/\delta x]\ /2\pi\sigma^2$.

Employing a Fourier series transform in ϕ, τ is solved from (9) to be:

$$\tau(\phi;t) = \frac{\rho_0\sigma}{\sqrt{2\pi}\, r_0\, \rho c}\; t\; erf\left[\frac{\pi r_0}{\sqrt{2}\sigma}\right] \tag{10}$$

$$+\ \frac{\rho_0 r_0^2}{\pi\kappa} \sum_{n=1}^{\infty} \frac{cos n\phi}{n^2} \left[1 - e^{-\kappa n^2 t / r_0^2}\right] \int_{-\pi}^{\pi} dx\ cos nx\ e^{-r_0^2 x^2/2\sigma^2}$$

The result (10) can be simplified considerably. Replacement of the error function by 1 in the first line of (10) introduces an error $\varepsilon < 1 \bullet 10^{-6}$ for $r_0 >$ about 1.5σ. To the same level of accuracy the $-\pi \rightarrow \pi$ integration limits can be extended to $\pm\infty$ and the integration performed analytically. The resulting series in n can then be summed using the Euler-Maclaurin approximation, with the final form for τ becoming:

$$\tau(\phi;t) \approx \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\left[\alpha\, e^{-\Phi^2/\alpha^2} - e^{-\Phi^2} + \sqrt{\pi}\,\Phi\left(erf\,[\Phi/\alpha] - erf\,[\Phi]\right)\right] \tag{11}$$

with

$$\Phi^2 \equiv \frac{\phi^2 r_0^2}{2\sigma^2} \quad ; \quad \alpha^2 \equiv 1 + \frac{2\kappa}{\rho c\, \sigma^2} t$$

Again, with $r_0 >$ about 1.5σ, the expression (11) for τ approximates (10) to an accuracy of much better than 1%.

The hottest point in the pipe obviously occurs at the maximum in the energy deposition density (5); that is, at $\phi = 0$, $z = \Delta r / 2\theta$. From (8) and (11) the variation of temperature with time at this point is:

$$T_{\max}(t) = T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right][\,\alpha - 1\,] \tag{12}$$

The melting point of the steel is therefore reached after a time given approximately by:

$$t_{melt} \approx \frac{2\pi^2 \rho c\, \kappa\sigma^2}{N^2}\left[\frac{\delta E}{\delta x}\right]^{-2}[\,T_{melt} - T_0\,]^2 \tag{13}$$

In this form, several simple scaling laws become apparent. The time required to melt the beam pipe is found to increase linearly with the cross-sectional area of the beam, but decreases as the square of the beam current. A particularly interesting prediction of (13) is that, at least for the small incident angles θ considered here, the melting time is expected to be largely independent of θ.

Solving the diffusion equation (3) exactly, rather than employing the separable approximation (8) for the temperature distribution, does not alter these conclusions significantly. It is found in this case that the maximum temperature varies with time according to:

$$T_{max}(t) = T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\frac{1}{2\theta}\left[\ln\left[\frac{\beta+\theta\alpha}{\beta-\theta\alpha}\right] - \ln\left[\frac{1+\theta}{1-\theta}\right]\right] \qquad (14)$$

with:

$$\alpha^2 \equiv 1 + \frac{2\kappa}{\rho c\,\sigma^2}t \quad ; \quad \beta^2 \equiv 1 + \theta^2\frac{2\kappa}{\rho c\,\sigma^2}t$$

The approximation (12) reproduces the exact result (14) up to corrections of $O\,[\theta^2]$.

The thermal properties of steels with chrome content in the range 10→20% are fairly similar. A representative example is provided by 304 steel, which is a common stainless steel with 17→19% chrome content. The relevant parameters for this material are:

Melting point	1415 ^oC
density	8.03 g/cm^3
specific heat	.502 $J/g/^oC$
thermal conductivity	.215 $W/cm/^oC$ [@500 ^oC]

With $N = 2\bullet 10^{13}$ protons/sec, $\delta E/\delta x$ = 13.5 MeV/cm, σ = .15 cm, and θ = 5 mrads, eqn.(12) predicts that the pipe will melt in 7 minutes, whereas the exact result (14) gives about 7 minutes 10 seconds.

3. Energy Losses Through Diffusion + Convection

For a thin-walled pipe, *i.e.* $\Delta r \ll 2\pi r_0$, the loss of heat through convection to the atmosphere can not be justifiably ignored. If, near the maximum in the temperature distribution, the small diffusion of heat along the length of the pipe is once again neglected, then the diffusion equation becomes:

$$\frac{\partial T}{\partial t} = \frac{\hat{\rho}_E}{\rho c} + \frac{\kappa}{\rho c}\left[\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right]\right] \tag{15}$$

where $\hat{\rho}_E$ is now defined to be:

$$\hat{\rho}_E(\phi) = \frac{N}{2\pi\sigma^2}\left[\frac{\delta E}{\delta x}\right] e^{-r_0^2\phi^2/2\sigma^2} \tag{16}$$

Convective losses are most easily incorporated by rewriting (15) in terms of the average temperature $\bar{T}$ from r_0 to $r_0+\Delta r$. Integrating (15) gives:

$$\frac{\partial \bar{T}}{\partial t} \approx \frac{\hat{\rho}_E}{\rho c} + \frac{\kappa}{\rho c}\left[\frac{1}{r_0^2}\frac{\partial^2 \bar{T}}{\partial \phi^2} + \frac{1}{\Delta r}\left(\frac{\partial T}{\partial r}\bigg|_{r_0+\Delta r} - \frac{\partial T}{\partial r}\bigg|_{r_0}\right)\right] \tag{17}$$

At the inner wall of the pipe $\partial T/\partial r = 0$. At the outer surface the gradient of the temperature is related to the heat loss via:

$$\kappa\frac{\partial T}{\partial r}\bigg|_{r_0+\Delta r} = -h_c\,[\,T - T_0\,] \tag{18}$$

where h_c is the convection heat transfer coefficient, and T_0 is room temperature.

Since, for a thin wall, the average and surface temperatures must be very nearly the same, it is a good approximation to write the variation of $\bar{T}$ with time as:

$$\frac{\partial \bar{T}}{\partial t} \approx \frac{\hat{\rho}_E}{\rho c} + \frac{\kappa}{\rho c}\frac{1}{r_0^2}\frac{\partial^2 \bar{T}}{\partial \phi^2} - \frac{h_c}{\Delta r\,\rho c}\,[\,\bar{T} - T_0\,] \tag{19}$$

Solutions to (19) are fundamentally different from those discussed in the preceding section, where it was shown that asymptotically the temperature increase with time was proportional to $t^{1/2}$. Here, because convective losses are proportional to $\bar{T}$, it is clear that there *always* exists a maximum attainable temperature, at which point the energy being dispersed through diffusion and convection equals the energy deposited by $\hat{\rho}_E$. Whether this maximum temperature is above or below the steel's melting point for a particular $\hat{\rho}_E$ depends upon the thickness of the pipe Δr and the heat transfer coefficient h_c.

With $r_0 \gg \sigma$ the diffusion equation (19) can be solved analytically using Fourier transforms, with the result:

$$\bar{T} = T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\frac{\sqrt{\pi}}{4}\frac{e^{\eta^2}}{\eta}\Big(e^{\eta\Phi}\left[\, erf\,(\eta\alpha+\Phi/\alpha) - erf\,(\eta+\Phi)\,\right] + e^{-\eta\Phi}\left[\, erf\,(\eta\alpha-\Phi/\alpha) - erf\,(\eta-\Phi)\,\right]\Big) \qquad (20)$$

and

$$\eta^2 \equiv \frac{h_c\sigma^2}{2\Delta r\,\kappa} \quad ; \quad \Phi^2 \equiv \frac{\phi^2 r_0^2}{2\sigma^2} \quad ; \quad \alpha^2 \equiv 1 + \frac{2\kappa}{\rho c\,\sigma^2}t$$

At $\phi = 0$, the hottest point in the pipe, eqn.(20) reduces to:

$$\bar{T}(\phi{=}0) = T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\frac{\sqrt{\pi}}{2}\frac{e^{\eta^2}}{\eta}\Big(erf\,[\eta\alpha] - erf\,[\eta]\Big) \qquad (21)$$

In the limit that $h_c \to 0$, eqn.(21) reduces to (12). For $h_c \neq 0$, $\bar{T}$ does not increase indefinitely with time, but reaches a maximum value determined by the limit $\alpha \to \infty$:

$$T_{\max} \equiv \lim_{\alpha\to\infty} \bar{T} = T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\frac{\sqrt{\pi}}{2}\frac{e^{\eta^2}}{\eta}\Big(1 - erf\,[\eta]\Big) \qquad (22)$$

For small values of η (h_c small), $T_{\max}$ is approximately given by:

$$T_{\max} \approx T_0 + \frac{N}{2\pi\kappa}\left[\frac{\delta E}{\delta x}\right]\frac{\sqrt{\pi}}{2\eta} = T_0 + N\left[\frac{\delta E}{\delta x}\right]\left[\, 8\pi\kappa\sigma^2 h_c/\Delta r\,\right]^{-1/2} \qquad (23)$$

A completely reliable value for h_c is diffult to obtain. In principle h_c can be calculated, but this is very complicated and depends upon the geometry, whether the air exhibits laminar or turbulent flow, and the temperature distribution in the pipe. Since this temperature profile is determined in part by h_c there is the further difficulty of acheiving a self-consistent solution. However, a reasonable approximation to h_c can be obtained by considering the simplified geometry in which the temperature distribution described by (20) is replaced by a step function, and the wall of the pipe is replaced by a vertical sheet. In this case, for a sheet of vertical height L (cm) and uniform temperature T_{sf} (^oC) the heat transfer coefficient to air is given approximately by:

$$h_c \approx 4.5\times10^{-4} \left[\frac{T_{sf}-T_0}{L}\right]^{1/4} \frac{W}{cm^2\,{}^oC} \tag{24}$$

The height T_{sf} and width L of the fictitious temperature step function approximating $\overline{T}$ can be defined by the conditions that at any time the total heat content and mean square width of the distribution agree with those of eqn.(20). That is:

$$T_{sf} \equiv T_0 + \frac{1}{L}\int_{-\infty}^{\infty} dy\,(\overline{T}-T_0) \quad ; \quad \frac{L^2}{12} \equiv \langle y^2 \rangle = \frac{\int_{-\infty}^{\infty} dy\, y^2\,(\overline{T}-T_0)}{\int_{-\infty}^{\infty} dy\,(\overline{T}-T_0)} \tag{25}$$

where $y \equiv \phi r_0$. For small values of h_c, eqns.(25) can be solved to give:

$$\frac{T_{sf}-T_0}{L} \approx \frac{N}{\sqrt{2\pi}\sigma 24\kappa}\left[\frac{\delta E}{\delta x}\right] F(t) \tag{26}$$

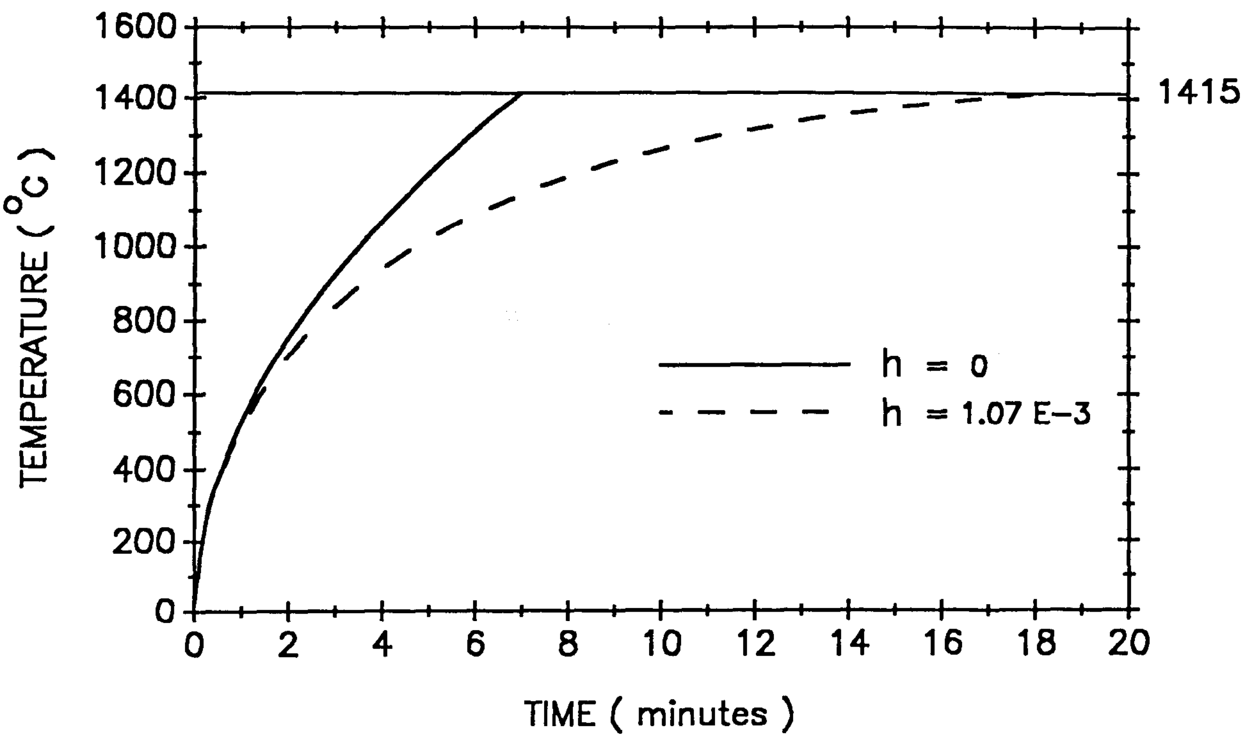
with

$$F(t) \equiv \frac{\left[1-e^{-\gamma^2}\right]^2}{1-(1+\gamma^2)\,e^{-\gamma^2}} \tag{27}$$

and $\gamma^2 \equiv \eta^2(\alpha^2-1)$. The function $F(t)$ is a slowly varying function of t, ranging from 2 at t=0 to 1 at $t\rightarrow\infty$ independent of h_c. Inserting eqns.(26) and (27) into (25), this variation of $F(t)$ with t translates into less than a 20% change in h_c. With the steel and beam parameters listed at the end of the last section, h_c is found to range from $1.16 \bullet 10^{-3}$ W/cm^2/°C at $\bar{T}=T_0$ to $0.98 \bullet 10^{-3}$ W/cm^2/°C at $\bar{T} \to T_{max}$.

The graph shown below gives the temperature variation with time of the hottest point in a pipe of thickness Δr=1/16 inch [eqn.(21)]. The curve h=0 is the analytic result obtained in the preceding section [eqn.(15)]. The h=$1.07 \bullet 10^{-3}$ curve is eqn.(21) with h_c given by its average value from t=0 to ∞. It can be seen in this case that the amount of heat lost to the air by convection increases the melting time of the pipe from 7 minutes to slightly more than 18 minutes.

Since the value derived above for h_c is only approximate, it is worth remarking that a modest 12% increase in h_c to $1.20 \bullet 10^{-3}$ W/cm^2/°C would result in the pipe never reaching the melting point.



4. Energy Losses Through Diffusion + Convection + Thermal Radiation

As the pipe reaches high temperatures the energy lost through thermal radiation can become a significant factor. To include this effect explicitly the analysis proceeds as in the development of eqn.(17). Now, however, at the outer surface of the pipe the gradient of the temperature is related to the heat loss by:

$$\kappa \frac{\partial T}{\partial r}\Big|_{r_0+\Delta r} = - h_c \, [\, T - T_0 \,] - \varepsilon \sigma_{SB} \, [\, T^4 - T_0^4 \,] \qquad (28)$$

where σ_{SB} is the Stefan-Boltzmann constant, ε is the emissivity of the steel, and T_0 is room temperature.

The interior surface also radiates, but to some extent the emission and absorption processes tend to cancel in this case. Heat radiated from one section of the surface must either be reflected or absorbed and partially re-emitted by other areas of the pipe. Computationally, it would be difficult, and probably not very rewarding, to incorporate this subtle interplay of mechanisms exactly. It is not difficult, however, to bracket the correct answer. The two extremes to be considered are: 1) the interior surface does not radiate or, equivalently, emission and absorption at every point cancel, and; 2) as much heat is lost at the inner wall through radiation as at the outer surface.

Again employing the approximation that the surface temperature and average temperature of the steel must be nearly the same, the variation of $\overline{T}$ with time becomes:

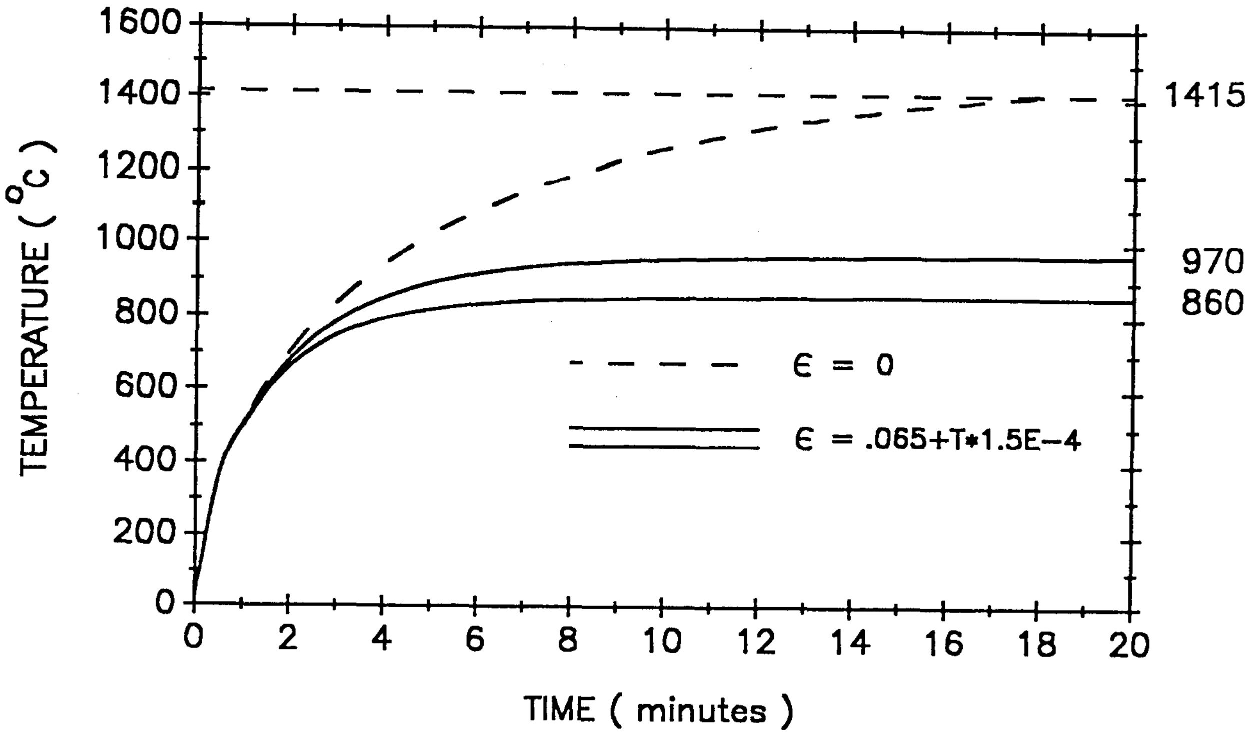
$$\frac{\partial \overline{T}}{\partial t} \approx \frac{\hat{\rho}_E}{\rho c} + \frac{\kappa}{\rho c} \frac{1}{r_0^2} \frac{\partial^2 \overline{T}}{\partial \phi^2} - \frac{h_c}{\Delta r \rho c} \, [\, \overline{T} - T_0 \,] - \frac{\varepsilon \sigma_{SB}}{\Delta r \rho c} \, [\, \overline{T}^4 - T_0^4 \,] \qquad (29)$$

The above equation describes the case in which the interior wall is assumed not to radiate. For the other extreme, in which interior radiation losses equal those of the exterior surface, ε is replaced by 2ε in eqn.(29).

Emissivity data are scarce and unreliable. The few available values of ε vary widely, and seem to depend at the very least upon the type of steel, condition of the surface, temperature, and author. A fairly representative set of values for unoxidized steel gives the two points $\varepsilon = .08$ at 100 ^{o}C, and $\varepsilon = .28$ as a liquid.

The two graphs on the following page show the variation with time of the maximum temperature in a pipe of thickness $\Delta r = 1/16$ inch which result from slightly different assumptions about the value of ε. Otherwise the steel and beam parameters are those listed at the end of section 2, and the convection heat transfer coefficient $h_c = 1.07 \bullet 10^{-3}$ W/cm^2/^{o}C. The $\varepsilon = 0$ curves are the analytic result obtained in the preceding section [eqn.(21)]. The upper and lower solid lines in both graphs correspond to the two extremes that the inner wall does not radiate, and that the interior radiation losses equal those of the exterior, respectively.

In the first graph the temperature dependence of ε is taken approximately into account. For want of any better guidance, it is assumed that ε varies linearly with T, so that $\varepsilon = .065 + T\ 1.5 \bullet 10^{-4}$ (with T in ^{o}C) in agreement with the two points given earlier. In the second graph temperatures have been calculated assuming that ε is constant, with the value $\varepsilon = .15$ being chosen as roughly the average value between $T = T_0$ and $T = 1000$ ^{o}C. With either assumption it is found that the pipe will never melt. After $\approx$ 10 minutes the temperature has essentially converged to its asymptotic value, with this maximum temperature being roughly in the range 850 $\rightarrow$ 1000 ^{o}C.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600
400
200
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TEMPERATURE (°C)
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2
4
6
8
10
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14
16
18
20
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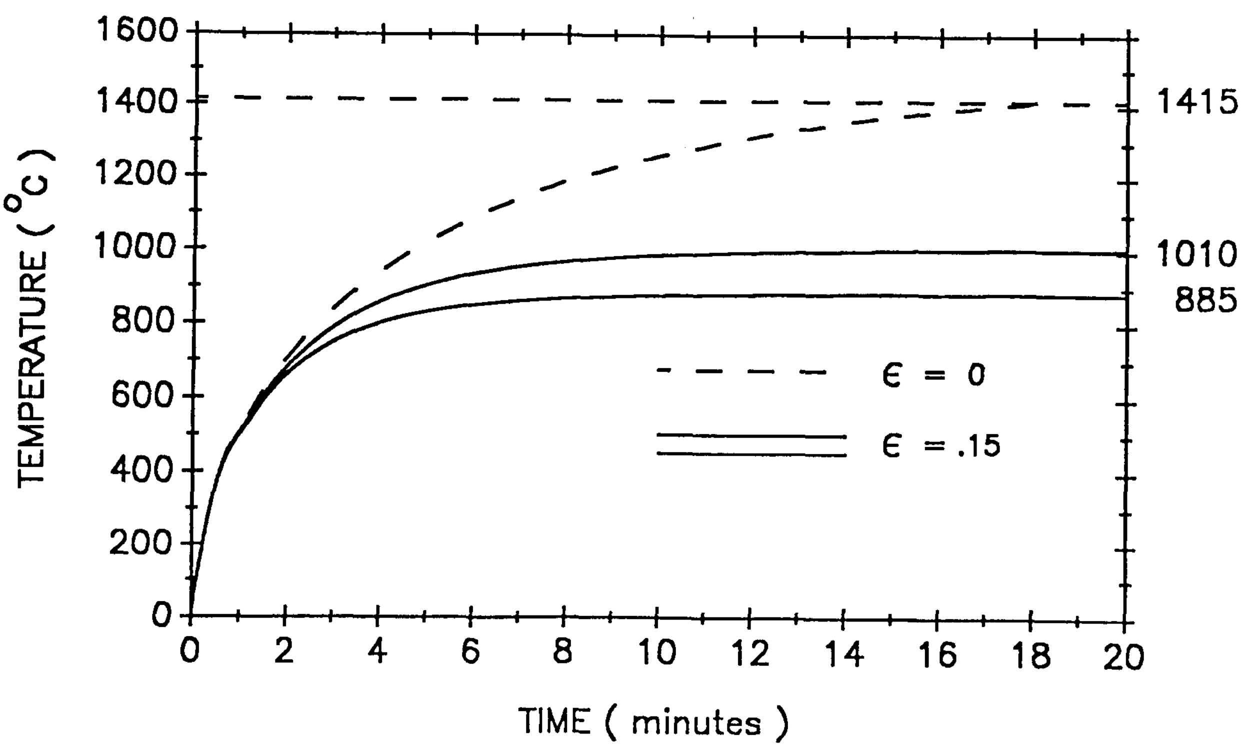
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TIME (minutes)
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ϵ = 0
ϵ = .15

5. Summary

In short, it has been found that the beampipe will never melt as a consequence of the beam striking it at some small incident angle. With reasonable parameters for the energy deposited by the beam and the thermal characteristics of the steel pipe, a maximum temperature is reached which is approximately 400 $\rightarrow$ 550 $^{\circ}$C below the melting point of the steel. At this point the energy dispersed through diffusion, convection, and radiation cancels the energy deposited by the beam.