TRANSITION CROSSING TIME SHIFTS

(Following Faugeras, Faugier & Garcyte, SPS Imp. Rep. 130, '7

JOHNSON EFFECT

Consider infinitesimal variations in momentum $\frac{dp}{p}$ about a reference trajectory with finite momentum offset

$\delta = \frac{\Delta p}{p_0}$ (Note that the subscript from δ_p is now dropped

On the reference trajectory,

$$\boxed{\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0} = \alpha_0 \frac{\Delta p}{p_0} + \alpha_1 \left(\frac{\Delta p}{p_0}\right)^2 + \ldots}$$ (see previous page)

Transition for the reference particle is when neighboring orbits have the same revolution frequency. That is, when

$$\boxed{\frac{dL}{L} = \frac{d\beta}{\beta}}$$

about the reference orbit.

RHS: $\frac{d\beta}{\beta} = \frac{1}{\gamma_T^2} \frac{dp}{p}$

LHS: $L = L_0 + \Delta L = L_0\left(1 + \frac{\Delta L}{L_0}\right) = L_0\left(1 + \alpha_0 \frac{\Delta p}{p} + \alpha_1 \left(\frac{\Delta p}{p}\right)^2\right)$

$$\frac{dL}{L_0} = \alpha_0\left(\frac{\Delta p}{p_0} + \frac{dp}{p_0}\right) + \alpha_1\left(\frac{\Delta p}{p_0} + \frac{dp}{p_0}\right)^2 - \alpha_0 \frac{\Delta p}{p_0} - \alpha_1\left(\frac{\Delta p}{p_0}\right)^2$$

$$= \alpha_0 \frac{dp}{p_0} + 2\alpha_1 \frac{\Delta p}{p_0} \frac{dp}{p_0} + \ldots$$

AND $\frac{dL}{L} = \frac{dL/L_0}{L/L_0} = \frac{\alpha_0 + 2\alpha_1 \frac{\Delta p}{p_0}}{1 + \alpha_0 \frac{\Delta p}{p} + \alpha_1 \left(\frac{\Delta p}{p}\right)^2} \frac{dp}{p_0}$

Noting that $\frac{dp}{p_0} = \left(1 + \frac{\Delta p}{p_0}\right)\frac{dp}{p}$

and equating LHS = RHS,

$$\frac{1}{\gamma_T^2} = \frac{\left(\alpha_0 + 2\alpha_1 \frac{\Delta p}{p_0}\right)}{1 + \alpha_0 \frac{\Delta p}{p_0} + \alpha_1 \left(\frac{\Delta p}{p_0}\right)^2}\left(1 + \frac{\Delta p}{p_0}\right)$$

Keeping only 1st order terms in $\frac{\Delta p}{p_0}$,

$$\frac{1}{\gamma_T^2} = \left(\alpha_0 + 2\alpha_1 \frac{\Delta p}{p_0}\right)\left(1 - \alpha_0 \frac{\Delta p}{p_0}\right)\left(1 + \frac{\Delta p}{p_0}\right)$$

$$= \alpha_0 + \left(2\alpha_1 + \alpha_0 - \alpha_0^2\right)\frac{\Delta p}{p_0}$$

$$= \alpha_0\left[1 + \left(1 + \frac{2\alpha_1}{\alpha_0} - \alpha_0\right)\frac{\Delta p}{p_0}\right]$$

OR

$$\boxed{\gamma_T = \gamma_{T0}\left[1 - \left(\frac{1}{2} + \frac{\alpha_1}{\alpha_0} - \frac{\alpha_0}{2}\right)\frac{\Delta p}{p_0}\right]} \qquad \gamma_{T0} = \frac{1}{\alpha_0^{\frac{1}{2}}}$$

<u>But the important question is</u>, what is the energy of the <u>nominal</u> particle when the <u>reference</u> particle passes thru' transition

Since $\gamma_T^2 = \gamma_0^2\left\{1 + \left[\left(1 + \frac{\Delta p}{p_0}\right)^2 - 1\right]\beta_0^2\right\}$

$$\approx \gamma_0^2\left[1 + 2\beta_0 \frac{\Delta p}{p_0}\right]$$

$$\approx \gamma_0^2\left[1 + 2\frac{\Delta p}{p_0}\right]$$

Then $\gamma_0 = \gamma_T\left[1 - \frac{\Delta p}{p_0}\right]$

and $\boxed{\gamma_0 = \gamma_{T0}\left[1 - \left(\frac{3}{2} + \frac{\alpha_1}{\alpha_0} - \frac{\alpha_0}{2}\right)\frac{\Delta p}{p_0}\right]}$

The reference particle passes through transition Δt after the nominal particle, where

$$\Delta t = \frac{\gamma_0 - \gamma_{T0}}{\dot{\gamma}}$$

where $\dot{\gamma}$ is the ramp rate. The Main Injector will ra at 240 GeV per second, hence $\dot{\gamma} \simeq 240\ \text{sec}^{-1}$

Thus

$$\boxed{\Delta t = -\frac{\gamma_{T0}}{\dot{\gamma}}\left[\frac{3}{2} + \frac{\alpha_1}{\alpha_0} - \frac{\alpha_0}{2}\right]\frac{\Delta p}{p_0}}$$

Have previously shown that in a FODO cell lattice with thin quadrupoles and chromatic sextupoles, half cell length L, bending radius R, half cell phase advance $\phi_{\frac{1}{2}}$

p.132

$$\boxed{\alpha_0 = \frac{L^2}{R^2}\frac{1}{s^2}\left[1 - \frac{s^2}{12}\right]}$$

$$\boxed{\alpha_1 = \frac{L^2}{R^2}\frac{1}{s^2}\left[1 - f + \frac{s^2}{12}\right]}$$

where $s = \sin(\phi_{\frac{1}{2}})$, $\frac{1}{\sqrt{2}}$ for Main Injector

and f is the fractional strength of the chromaticity sextupoles, $f = 1$ to correct for natural FODO chromaticity only. Normally (especially with strong IR quads) $f \gtrsim 1$

Hence

$$\boxed{\Delta t = -\frac{\gamma_{T0}}{\dot{\gamma}}\left[\frac{3}{2} + \frac{1 - f + \frac{s^2}{12}}{1 - \frac{s^2}{12}} - \frac{L^2}{R^2}\frac{1}{2s^2}\left[1 - \frac{s^2}{12}\right]\right]\frac{\Delta p}{p_0}}$$

For example, Main Injector [MI-CDR, p. 4]

$L = 17.15$ meters

$R \approx \frac{C}{2\pi} = \frac{3319.4}{2\pi} = 528$ meters

$\gamma_{T0} = 20.4$

$\dot{\gamma} \cong 240 \ \text{sec}^{-1}$

$s = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f = 1$

$$\Delta t = -8.5\times10^{-2}\left[1.5 + 4.3\times10^{-2} - 1.0\times10^{-3}\right]\frac{\Delta p}{p_0}$$

$$\boxed{\Delta t = -0.13\,\frac{\Delta p}{p_0}} \quad \text{seconds}$$

Note that this is ONLY for a FODO lattice, and that the delay approaches zero for (more realistic) f values greater than 1.

This number, with a realistic $\frac{\Delta p}{p_0}$, is to be compared with the non-adiabatic time.

According to Bill Ng, $\left.\frac{\Delta p}{p_0}\right|_{max} \simeq 0.01$ at transition,

so $\boxed{|\Delta t_{max}| \simeq 1.3 \text{ milliseconds}}$ JOHNSON EFFECT

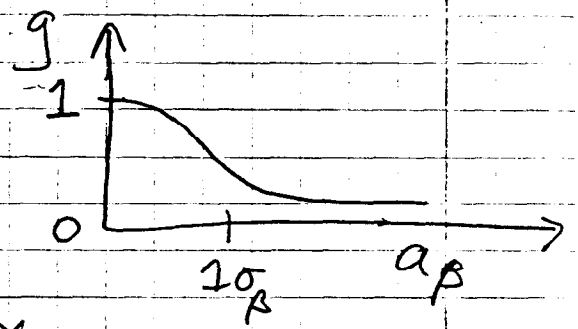
Transition jumps with a rise time much less than this do not make good sense!

UMSTATTER EFFECT

Consider a reference particle with $\frac{\Delta p}{p_0} = 0$, but with horizontal betatron amplitude a_β, and a tune shifted by space charge to be

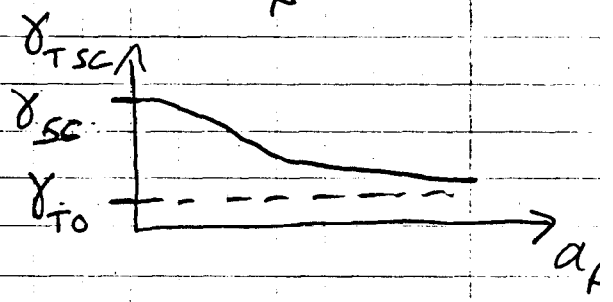
$$Q = Q_0 - \Delta Q_{sc}\, g(a_\beta)$$

(Note sign: ΔQ_{sc} is positive)



This causes a shift in γ_T

$$\gamma_T \to \gamma_{T\,sc}(a_\beta)$$



with a maximum shift at $a_\beta = 0$

The characteristic (maximum) time shift is

$$\boxed{\Delta t = \frac{\gamma_{T\,sc}(0) - \gamma_{T0}}{\dot{\gamma}} \equiv \frac{\gamma_{sc} - \gamma_{T0}}{\dot{\gamma}}}$$

What is $\gamma_{sc} = \gamma_{T\,sc}(0)$?

The effective (horizontal) space charge quadrupole field has strength

$$k = \frac{k_0}{\sigma_H(\sigma_H + \sigma_V)}$$

dispersion function

where $\sigma_H = \left(\frac{\epsilon_{HN}}{2\gamma}\cdot\beta_H + \left(\eta\,\frac{\sigma_p}{p_0}\right)^2\right)^{\frac{1}{2}}$, $\sigma_V = \sqrt{\epsilon_{VN}\beta_V}$

To a fair approximation,

$$\frac{\eta}{\beta_H^{\frac{1}{2}}} \simeq \text{constant}, \qquad \sigma_H + \sigma_V = \text{constant}$$

and therefore

$$k \simeq \frac{k_\eta}{\eta}$$

Assumes that the dispersion is only slightly modified by space charge.

The equation of motion for dispersion is

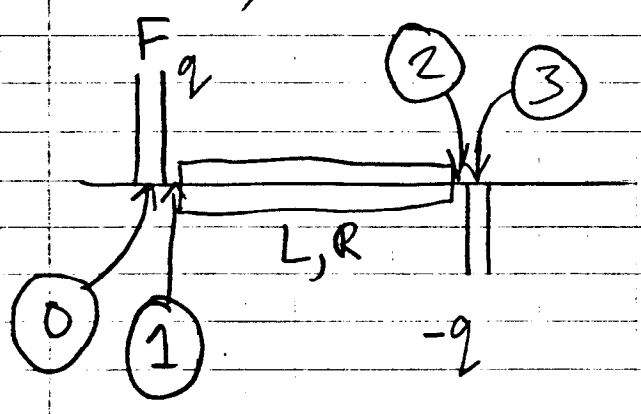
$$\eta'' + (K + k)\eta = \frac{1}{\rho} \qquad K \text{ +ve} \Rightarrow \text{horizontally focussing}$$

$K \to \infty$ in the zero length FODO quads, $\to 0$ in bend
$\rho \to 0$ " " " " " " , $\to R$ in bend

In the bend, then, a fair approximation is

$$\eta'' = \frac{1}{\tilde{R}} = \frac{1}{R} - k\eta$$

Thus, the motion through $\frac{1}{2}$ a FODO cell is



$$\begin{pmatrix}\eta_0\\ \eta_0'\end{pmatrix} = \begin{pmatrix}\hat{\eta}\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\eta_1\\ \eta_1'\end{pmatrix} = \begin{pmatrix}\eta_0\\ -q\eta_0\end{pmatrix}$$

$$\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix} = \begin{pmatrix}\eta_1 + \eta_1' L + \frac{1}{2}\frac{L^2}{\tilde{R}}\\ \eta_1' + \frac{L}{\tilde{R}}\end{pmatrix}$$

$$\begin{pmatrix}\eta_3\\ \eta_3'\end{pmatrix} = \begin{pmatrix}\eta_2\\ \eta_2' + q\eta_2\end{pmatrix} = \begin{pmatrix}\check{\eta}\\ 0\end{pmatrix}$$

Combining all this together gives

$$\eta_0 = \frac{(2+s)}{2s^2} \cdot \frac{L^2}{\tilde{R}} \qquad s = \sin(\phi_{\frac{1}{2}})$$

and the compaction factor is

$$\alpha_{sc} = \frac{1}{\gamma_{sc}^2} = \frac{1}{L}\int_0^L \frac{\eta}{R}\, ds$$

$$= \frac{1}{R}\left[\eta_0\left(1-\frac{s}{2}\right) + \frac{1}{6}\frac{L^2}{\tilde{R}}\right]$$

$$\alpha_{sc} = \alpha_0 \frac{R}{\tilde{R}} = \alpha_0\left[1 - k_\eta R\right]$$

so $$\boxed{\gamma_{sc} = \gamma_{TO}\left[1 + \tfrac{1}{2} k_\eta R\right]}$$

<u>But how is k_η related to ΔQ_{sc}?</u>

The tune shift due to a quad strength k of thickness d is

$$dQ = \frac{k\beta}{4\pi}\, ds$$

If there are N FODO cells,

$$-Q_{sc} = \frac{2N}{4\pi}\int_0^L \beta k\, ds$$

$$= \frac{N}{2\pi} k_\eta \int_0^L \frac{\beta}{\eta}\, ds$$

For 90° cell phase advance,

$$\frac{\beta_F}{\eta_F} = 1.261 \frac{R}{L}$$

$$\frac{\beta_D}{\eta_D} = 0.453 \frac{R}{L}$$

$$\frac{\beta_{MID}}{\eta_{MD}} = 0.453 \frac{R}{L}$$

and, preparing to use Simpsons rule,

$$\frac{1}{6}\left[\frac{\beta_F}{\eta_F} + 4\frac{\beta_{MID}}{\eta_{MID}} + \frac{\beta_D}{\eta_D}\right] = 0.646 \frac{R}{L}$$

so that

$$-Q_{SC} \simeq 0.65 \frac{N}{2\pi} k_\eta R$$

Now

$$Q_0 = \frac{2N}{2\pi} \cdot \phi_{1/2} = \frac{N}{2\pi} \cdot \frac{\pi}{2}$$

so, finally,

$$\boxed{k_\eta R \simeq -\frac{\pi}{2 \times 0.65} \frac{Q_{SC}}{Q_0} = -2.42 \frac{Q_{SC}}{Q_0}}$$

$$\boxed{\gamma_{SC} = \gamma_{T0}\left[1 - 1.21 \frac{Q_{SC}}{Q_0}\right]}$$

$$\boxed{\Delta t = -1.21 \frac{\gamma_{T0}}{\dot{\gamma}} \frac{Q_{SC}}{Q_0}}$$

For example, as before

$$\gamma_{TO} = 20.4 \,, \quad \dot{\gamma} = 240 \text{ sec}^{-1}, \quad Q = 22.4,$$

$$\boxed{\Delta t_{sc} \approx -0.46 \left[\frac{Q_{sc}}{0.1}\right] \text{ milliseconds}}$$

UMSTATTER EFFECT

which appears to be significantly smaller than with the Johnson effect!

Note that a naive treatment would say

$$\gamma_T \simeq Q$$

or $\gamma_{sc} = \gamma_{TO}\left[1 - \frac{Q_{sc}}{Q_0}\right]$!! Not bad !!