MICROWAVE INSTABILITY AT TRANSITION – STABILITY DIAGRAM APPROACH

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Here, a question of microwave stability at transition is addressed. A simple model of a beam at transition driven by a storage ring impedance is formulated in the framework of the nonlinear Vlasov equation. Nonperturbative analytic treatment via Vlasov equation yields a set of coupled equations of motion describing time evolution of a single coherent mode (an azimuthal harmonics of the density function) and the overall equilibrium density distribution function. This pair of equations together with the dispersion relation fully describe the longitudinal beam dynamics. At transition, contour integration in the dispersion relation can be carried out analytically and a simple closed formula for the coherent frequency is obtained. From the resulting stability diagram further conclusions about the growth time of the microwave instability and the evolution of the equilibrium distribution function are derived. Finally, the longitudinal emittance blowup at transition is discussed.

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INTRODUCTION

A purpose of this paper is to investigate microwave stability at transition. A question of accompanying it longitudinal emittance blowup is also addressed. One would like to sort out purely kinematic contribution to the emittance blowup (due to the Johnsen and Umstätter effects)[1] from the intensity dependent one caused by the microwave instability, plausibly building up at transition. The longitudinal phase space has a very peculiar structure; at transition the rf bucket does not exist in the usual sense and a beam can be considered a coasting beam to a good approximation. When the influence of the external restoring force disappears the beam is very susceptible to any fast growing instability which may in turn (through nonlinear driving terms) reshape the overall longitudinal phase space of the beam.

Here we present a systematic description of the microwave instability at transition derived from Vlasov equation. The resulting nonperturbative formalism allows us to study the long-time behavior of driven coherent modes, their saturation[2] due to the increasing Landau damping, and finally, how they modify the uniform part of the density distribution function and contribute to the longitudinal emittance blowup.

LONGITUDINAL BEAM DYNAMICS AT TRANSITION

Consider initially uniform distribution of particles inside a storage ring modeled by a statistical density distribution function defined in a classical phase space as

$$f(\varepsilon,\theta,t) = f^{o}(\varepsilon,t) + \sum_{n\neq 0} h_n(\varepsilon,t)\, e^{i\theta n} \quad , \tag{1}$$

where θ is the azimuthal angle around the ring circumference and ε represents the energy deviation from its synchronous value, E_o. Here $f(\varepsilon,\theta,t)$ is normalized to the total number of particles in the storage ring, N.

$$\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\varepsilon \; f(\varepsilon,\theta,t) = N \quad . \tag{2}$$

Fourier series representation of the nonuniform part guarantees periodicity of the distribution, while the condition

$$h_{-n}(\varepsilon,t) = h_n^*(\varepsilon,t) \, , \tag{3}$$

assures that the distribution function defined by Eq.(1) is a real quantity. The phase space continuity equation which governs $f(\varepsilon,\theta,t)$ can be written as follows

$$\frac{\partial}{\partial t} f(\varepsilon,\theta,t) + \omega \frac{\partial}{\partial \theta} f(\varepsilon,\theta,t) + \dot{\varepsilon} \frac{\partial}{\partial \varepsilon} f(\varepsilon,\theta,t) = 0 \quad . \tag{4}$$

Revolution frequency, ω, of a given particle depends on its momentum offset, Δp, via the momentum compaction factor, α. The fractional frequency shift is given by

$$\frac{\Delta \omega}{\omega_o} = -\left(\alpha - \frac{1}{\gamma^2}\right)\delta \quad ,$$

where (5)

$$\delta = \frac{\Delta p}{p_o} = \frac{1}{\beta^2} \frac{\varepsilon}{E_o} \quad .$$

In principle the momentum compaction factor also exhibits some chromatic dispersion according to general expansion

$$\alpha = \frac{p}{C}\frac{dC}{dp} = \alpha_o + \alpha_p \, \delta + O(\delta^2) \tag{6}$$

Right at transition, $\gamma = \gamma_t$, the linear term in Eq.(5) disappears, since

$$\alpha_o - \frac{1}{\gamma^2} = 0 \tag{7}$$

and the leading term in $\omega(\varepsilon)$ happens to be quadratic in ε. One can summarize Eqs.(5)–(7) to the lowest leading term in ε as follows

$$\omega(\varepsilon) = \omega_o - K\,\varepsilon^2 \;, \tag{8}$$

where the coefficient K is given by the following expression

$$K = \frac{\alpha_p \omega_o}{\beta^4 E_o{}^2} \;. \tag{9}$$

Here α_p is a purely lattice dependent parameter, which is easy to express in terms of azimuthal averages of the lattice dispersion function[3].

The beam environment in Eq.(4) is modeled by the wake-field impedance of a storage ring represented in frequency domain by $Z(\omega)$. In turn, coupling through the nonuniform current induces additional potential,[4] changing the energy of the beam by

$$\dot{\varepsilon} = -\,e\omega_o \sum_{n\neq 0} Z_n \phi_n(t)\, e^{i\theta n} \quad , \qquad Z_n = Z(n\omega_o). \tag{9}$$

where one introduces a coherent mode amplitude defined as follows

$$\phi_n(t) = -\,e\omega_o\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon\; h_n(\varepsilon,t) \tag{9}$$

We notice in passing, that $Z_n{}^* = Z_{-n}$, since the wake function is real. This, together with Eq.(2), assures that $\dot{\varepsilon}$ is also a real quantity. Substituting Eqs.(1) and (9) into Eq.(4) and using orthogonality of azimuthal plane waves, one can rewrite Vlasov equation as a set of coupled equations of motion for individual azimuthal harmonics of the distribution function. The resulting equations fully describing the dynamics of the beam-storage ring system, are given by

$$\frac{\partial}{\partial t} f^{o}(\varepsilon,t) - e\omega_o \sum_{n\neq 0} Z_n{}^{*} \phi_n{}^{*}(t) \frac{\partial}{\partial \varepsilon} h_n(\varepsilon,t) = 0 \quad , \tag{10}$$

and

$$\frac{\partial}{\partial t} h_n(\varepsilon,t) + in\omega\, h_n(\varepsilon,t) - e\omega_o\, Z_n\, \phi_n(t) \frac{\partial}{\partial \varepsilon} f^{o}(\varepsilon,t)$$

$$- e\omega_o \sum_{m\neq 0} Z_{n-m}\, \phi_{n-m}(t) \frac{\partial}{\partial \varepsilon} h_m(\varepsilon,t) = 0 \quad . \tag{11}$$

One can notice that for a sharply centered impedance, $Q >> 1$, the real part of $Z(\omega)$ is peaked around a single harmonics, $n \approx 10^4$, with the imaginary part extending over several neighboring amplitudes; $n - m,\ldots, n,\ldots, n + m$; $m \approx 10$, (even in the $Q \rightarrow \infty$ limit, the imaginary part of $Z(\omega)$ still retains a hyperbolic, $1/\omega$, tail). This implies that the last term in Eq.(11) would couple pairs of modes h_{n+k} and h_{-k} , where $k = 1,2,\ldots m$. However, for our impedance Z_k is vanishingly small, therefore modes with low k will not be excited by the impedance, which justifies why the coupling term in Eq.(11) can be neglected for a peaked impedance.

When the initial amplitude of the coherent mode is small and the instability does not develop too rapidly, one can assume that the nonlinearity modifies the particle distribution at a rate much smaller than the linear response of the system. Under this adiabaticity assumption one can formulate instantaneous dispersion relation, similar to the one employed in the linear theories.

Here we introduce instantaneous coherent frequency, $\Omega_n(t)$, describing evolution of the n-th mode within a small time interval (t, t') according to the formula

$$h_n(\varepsilon,t') = e^{-i\,\Omega_n(t)(t-t')}\,h_n(\varepsilon,t), \qquad t \approx t' . \tag{12}$$

We also require that $f^o(\varepsilon,t)$ is a slowly varying function of time compared to rapidly oscillating coherent modes, $h_n(\varepsilon,t)$. Therefore a simple adiabatic approximation is made

$$\frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t) = \frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t'), \qquad t \approx t' . \tag{13}$$

Including both assumptions, Eqs.(12) and (13), one can rewrite Eq.(11) as follows

$$h_n(\varepsilon) = (e\omega_o)^2 \frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t)\ \frac{Z_n}{n\omega - \Omega_n(t)}\ \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon'\ h_n(\varepsilon') \ . \tag{14}$$

As was pointed out by Landau,[5] an appropriate integration of Eq.(14) over the entire range of ε (including a detour contour extending into the complex ε-plane) leads to the following dispersion relationship defining the coherent frequency, $\Omega_n(t)$

$$1 = \left(\frac{e\omega_o}{2\pi}\right)^2 N\, Z_n \int_C d\varepsilon\ \frac{\frac{\partial}{\partial\varepsilon}\psi(\varepsilon,t)}{i[n\omega(\varepsilon) - \Omega_n]} \ . \tag{15}$$

Here, Ω_n and $\omega(\varepsilon)$, given explicitly by Eq.(8), define configuration of poles in the complex ε-plane, while C is the Landau contour[5] of integration chosen so that Ω_n is continuous while crossing the real axis. For convenience, we also replaced $f^o(\varepsilon,t)$ with a normalized to unity distribution function;

$$\psi(\varepsilon,t) = \frac{2\pi}{N} f^o(\varepsilon,t) \ . \tag{16}$$

From here on in, we will confine our discussion to a single harmonic mode, h_n, therefore index n will be suppressed throughout the rest of the paper. Furthermore, the summation over all modes in Eq.(5) reduces to two terms only (n and –n). This, combined with the symmetry condition given by Eq.(3), yields the following formula[2]

$$\frac{\partial}{\partial t}\psi(\varepsilon,t) - \frac{2\pi}{N} e\omega_o\, 2\mathrm{Re}\{Z^*\phi^*(t)\, \frac{\partial}{\partial \varepsilon} h(\varepsilon,t)\} = 0 \quad . \tag{17}$$

We can easily generalize the above result to the case of coupling impedance extending over several, ΔN, azimuthal harmonics. Simply, replacing summation over n in Eq.(11) by integration, carrying it out and retaining only the leading, $\Delta N/N$, order one obtains Eq.(17) with Z replaced by Z ΔN. The last expression is obviously proportional to the area under the coupling impedance peak which assures the correct scaling of our result.

STABILITY DIAGRAM

The dispersion relation, given by Eq.(15), can by solved with respect to the coherent frequency, Ω, then contours of constant growth rate, $\mathrm{Im}(\Omega)$, can be composed in complex impedance plane. The curve of zero growth rate (the stability curve) divides the impedance plane into the stable and unstable regions producing the so called stability diagram. In order to find the stability diagram at transition one has to carry out the integration in Eq.(15) explicitly assuming specific form of the equilibrium distribution function, $\psi(\varepsilon)$. Here we assume a simple Gaussian distribution parametrized by $\sigma = \langle\varepsilon\rangle_{rms}$ as follows

$$\psi(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{\varepsilon^2}{2\sigma^2} \right) \tag{18}$$

Assuming quadratic in ε frequency dispersion, given by Eq.(8), one can express the integrant in Eq.(15) in terms of a pair of simple poles in extended complex ε–plane. Now the dispersion equation can be rewritten in the following form

$$1 = \left(\frac{e\omega_o}{2\pi}\right)^2 N\, Z_n \frac{1}{2\sigma^2 niK} \int_C d\varepsilon \, \frac{2\varepsilon\psi(\varepsilon,t)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} \quad . \tag{19}$$

where

$$\Xi^2 = \frac{n\omega_o - \Omega}{nK} \, .$$

Here the contour of integration, C, is chosen so that the coherent frequency, Ω, remains continuous while the poles cross the real axis. It therefore includes the real axis plus a detour piece around any pole lying outside the upper half-plane and it is closed by an infinite semi-circle in the upper half-plane. This integral will be treated in detail in the Appendix A. As was shown in detail in the Appendix A the contour integral in Eq.(19) can be easily evaluated applying Cauchy's integral theorem. The resulting simple expression is given by A(10) as follows

$$\int_C d\varepsilon \, \frac{2\varepsilon\psi(\varepsilon,t)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} = 2\pi i \, \psi(\Xi,t) \quad . \tag{20}$$

One can notice from the symmetry of the above integrant that only the detour piece of the contour gives a non-zero contribution to the overall integral. Therefore, right at transition, the microwave stability is strictly governed by the Landau damping residuum term given by the right-hand side of Eq.(20). The last statement would translate to a global microwave stability at transition. This will indeed be demonstrated explicitly at the end of this section.

Substituting Eq.(20) into Eq.(19) allows one to express the coherent frequency of a given mode in the following form

$$\Omega_n = n\omega_o - 2nK\sigma^2 \ln\left(\frac{(e\omega_o)^2}{(2\pi)^{3/2}}\frac{NZ_n}{2nK\sigma^3}\right) \tag{21}$$

Taking into account multivaluedness of the above expression one can still obtain continuous analytic solution by introducing appropriate cut-lines (connecting different Riemann sheets of the general solution) at the branch points. The growth rate, introduced as the imaginary part of the coherent frequency, is given by the following expression

$$\frac{1}{\tau_n} = -2\alpha_p n\omega_o \left(\frac{\sigma}{E}\right)^2 F_n(\omega)$$

with (22)

$$F_n(\omega) = \arctan\left(\frac{Y_n}{X_n}\right), \quad Z_n = X_n + iY_n$$

Here, $F_n(\omega)$, is a phase factor illustrated in Fig. 1. One can notice in passing that the growth rate does not depend on intensity and it is quadratic in relative energy spread, both features being characteristic for Landau damping rather than usual coherent gain.

For the purpose of this calculation we assume a resonant impedance given by

$$Z(\omega) = \frac{R}{1 + iQ(\omega/\omega_r - \omega_r/\omega)} \tag{23}$$

where R is a shunt impedance, Q is a quality factor and ω_r is a resonant frequency. One can see from Fig. 1 that the general solution for the phase factor, $F_n(x)$, spans an infinite family of curves joined by the vertical

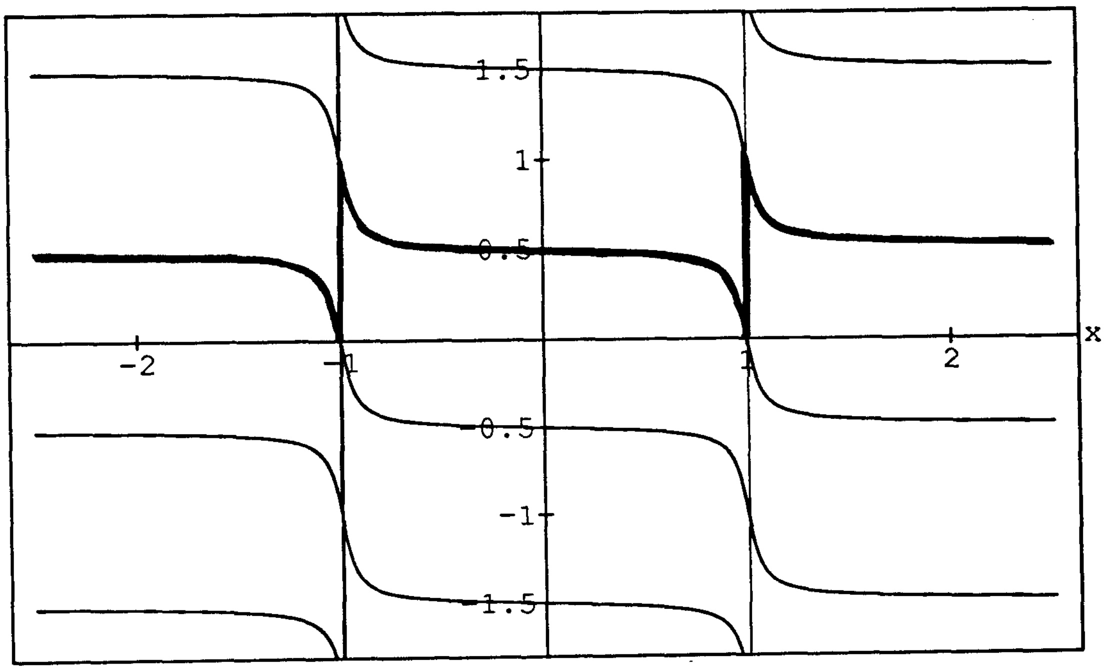

Fig. 1 Microwave stability diagram at transition, illustrated by a family of curves representing multivaluedness phase factor, $F_n(\omega)$, plotted in units of π. The impedance defined by Eq.(23) is used, where $x = \omega/\omega_r$ is a dimensionless frequency. The physical solution is highlighted in bold curve. It is fully confined to the stable region – the upper half-plane.

cut-lines at $x = \pm 1$. To select a physical solution we impose a following condition; the $x = 0$ point has to be equivalent to the $x \rightarrow \pm \infty$ asymptotics and it must be included in the stable region (zero impedance). This narrows down the allowed solutions to the one highlighted in Fig.1. A whole frequency spectrum is contained in the stable region (upper half-plane) with the resonant points, $x = \pm 1$, touching the stability curve.

CONCLUSIONS

We have shown that the self-consistent Vlasov equation formalism allows one to separate adiabatically equations of motion describing evolutions of a coherent mode and the overall equilibrium distribution function, as well as it yields the dispersion relation for the instantaneous coherent frequency, Ω_n. Right at transition, the dispersion integral was carried out analytically and it turned out to include only the Landau damping term making a beam globally stable against the microwave instability. This last statement trivially closes discussion about the emittance blowup due to the microwave instability. No instability develops, therefore the longitudinal emittance is not increased by the microwave instability.

APPENDIX A

Here we evaluate the dispersion integral at transition, which is introduced as follows

$$I = \int_C d\varepsilon \frac{2\varepsilon\psi(\varepsilon)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} \tag{A1}$$

where $\psi(\Xi,t)$ is a Gaussian parametrized by

$$\psi(\varepsilon) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha\varepsilon^2} . \tag{A2}$$

Using simple algebra one can rewrite our integral, I, in terms of the plasma dispersion integrals as follows

$$I = \sqrt{\frac{\alpha}{\pi}}\, [\, D(\sqrt{\alpha}\,\Xi) + D(-\sqrt{\alpha}\,\Xi)\,] . \tag{A3}$$

where the plasma dispersion integral is defined by the following formula

$$D(\xi) = \int_C d\varepsilon \frac{e^{-\alpha\varepsilon^2}}{(\varepsilon - \xi)} \tag{A4}$$

The contour of integration, C, is chosen so that $D(\xi)$ is contiguous when the pole, ξ, crosses the real axis. As it is shown in Fig.4, the contour C contains the real axis, an infinite semi-circle closed in the upper half-plane and a detour piece enclosing any singularity in the lower half-plane (if there is any). Contributions along the first two pieces of the contour is given by the principle value integral, while the integral along the detour piece is given by the residuum of the integrant at the singularity. Both contributions are summarized below

$$D(\xi) = W(\xi) \; + 2\pi i\theta_\xi \, e^{-\xi^2} \tag{A5}$$

where

$$W(\xi) = P\int_{-\infty}^{\infty} d\varepsilon \, \frac{e^{-\alpha\varepsilon^2}}{(\varepsilon - \xi)} \tag{A6}$$

and θ_ξ is defined as follows

$$\theta_\xi = \begin{cases} 0 & \text{if} \quad \mathrm{Im}(\xi) > 0 \\ \frac{1}{2} & \text{if} \quad \mathrm{Im}(\xi) = 0 \\ 1 & \text{if} \quad \mathrm{Im}(\xi) < 0 \end{cases} \tag{A7}$$

Substituting a sequence of above equations into Eq.(A3) one gets the following expression

$$I = \sqrt{\frac{\alpha}{\pi}} \, [\, W(\sqrt{\alpha}\,\Xi) + W(-\sqrt{\alpha}\,\Xi) + 2\pi i \, (\theta_\xi + \theta_{-\xi}) \, e^{-\alpha\Xi^2}] \, . \tag{A8}$$

One can easily check from the definition that $W(\xi)$ is an odd function of ξ and the following simple identity for θ_ξ holds

$$\theta_{-\xi} = 1 - \theta_\xi \tag{A9}$$

Applying these identities to Eq.(A8) reduces it to the following final expression

$$I = 2\pi i \; \psi(\Xi). \tag{A10}$$