MI-0165

Evaluation of Resonance Widths from Recycler Magnetic Field Multipoles

S. D. Holmes
April 3, 1996

The Recycler Ring will be constructed from 344 combined function and 88 discrete quadrupole magnets. These magnets will show field non-uniformities due to both their intrinsic design and to magnet fabrication errors. These non-uniformities can be characterized in terms of field multipoles. This characterization is based on a description of the field in transverse coordinates via:

$$B_y + iB_x = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{1}$$

where b_0, b_1, b_2 . . . are the normal dipole, quadrupole, sextupole, . . . field components, and a_0, a_1, a_2 . . . are the corresponding skew components.

The higher-order field components shown in (1) can produce unstable motion. The nature of the motion can be analyzed semi-analytically within the vicinity of a resonance of the form

$$l\nu_x + m\nu_y = k \tag{2}$$

where l, m, and k are integers, or it can be analyzed through particle tracking. Both analyses need to be done to gain confidence that the Recycler will indeed store particles. The resonance analysis is useful for identifying field components that need to be suppressed or corrected, while only particle tracking can yield information on very long-term stability away from the lowest order resonances. The purpose of this note is to present a criterion for evaluation of allowed field multipole components in Recycler magnets within the framework of a standard resonance analysis. It is assumed that any such evaluation will be accompanied by long term particle tracking to assure desired performance in the Recycler.

Systematic Multipoles: One Dimensional Analysis

The easiest case to analyze is the one for which either l or m in (2) is equal to zero. As shown in the attached Appendix the location of unstable fixed points can be determined in terms of the difference between the bare tune of an accelerator and the resonant tune. For a horizontal resonance,

$$\delta = \nu_x - \frac{k}{l} \tag{3}$$

It is shown that as δ decreases the location of the fixed points moves toward the origin. We characterize the strength of the resonance, δ_R, as distance between the bare tune and the resonant tune leading to unstable fixed points at oscillation amplitudes corresponding the beam emittance. We would then expect to see stable beam behavior only for $\delta >$ a few times δ_R.

The resonance strength is given by (see Appendix)

$$\delta_R = \frac{\varepsilon^{(l-2)/2}}{\pi 2^l}\left(\frac{l!}{2\left(\frac{l}{2}\right)!\left(\frac{l}{2}\right)!}\sum s_i \beta_{x_i}^{l/2} + \left|\sum s_i \beta_{x_i}^{l/2} e^{i(l\phi_x)}\right| \right) \tag{4}$$

where the sum is over all non-linear elements, s_i is the element strength,

$$s_i = \frac{b_{l-1} B_0 L}{B\rho},$$

ε is the physical beam emittance, and the first term, representing a 0^{th} harmonic tune shift with amplitude, only applies if l is even. A similar expression applies for the vertical resonance.

At Fermilab we measure field multipole components, s_i, in terms of "units", where the definition of a "unit" is a field contribution of $1\text{x}10^{-4}$ of the main field as measured at 1". So, if B_0 is the main field (dipole in a dipole or combined function magnet, quadrupole in a quadrupole magnet, etc.) then the "units" are simply $1\text{x}10^4$ times the b_n and a_n as measured in $(\text{inches})^{-n}$.

Resonance widths as calculated using expression (4) are given in Tables 1 and Table 2 for a systematic multipole error of one unit in the combined function and quadrupole magnets respectively. The calculation assumes a normalized emittance of 40π mm-mr and is based on the lattice RRV6. It is notable that the 4ν=105 resonance width is the largest in the table. Unfortunately, this is also the second nearest resonance to the nominal tune, (ν_x,ν_y)=(26.225, 26.215). The large width for this resonance is a result both of the 0^{th} harmonic tune shift with amplitude, and of the coherent addition of systematic terms in the 90^o cells chosen for the Recycler. As discussed below, this resonance will restrict the tolerable systematic octupole component of the combined function magnets.

The scale of allowable resonance widths is set by the distance from the nominal tunes to the resonant tunes. The nominal Recycler tune lies close to and between the 4ν=105 and 9ν=236 resonances and we would expect that these would potentially prove most disruptive. Figure 1 displays the ratio of the resonance width to resonance separation for the 16 resonances listed in

Table 1. In producing this figure we make the (worst case) assumption that the long and short dipoles contribute coherently to the total resonance width. With the exception of the 4ν=105 resonance, all separations are more than ten times the resonance width for a one unit multipole component.

In addition to the sensitivities listed in Table 1, the third order resonance width has been explicitly calculated for the designed systematic sextupole (about 8 units) in the long combined function magnets. This field component is provided for chromaticity correction in the Recycler and produces a resonance width of 0.0043--a factor of 25 less than the separation between the nominal and resonant tunes.

Table 1: Resonance widths as evaluated using expression (4) for a one unit (relative to the dipole) systematic multipole component in the long and short combined function magnets, assuming a beam emittance of 40π mm-mr (normalized).

Resonance Index	l	m	k	δ_R (Long)	δ_R (Short)
1	3	0	79	0.521E-03	0.727E-03
2	0	3	79	0.680E-03	0.777E-03
3	4	0	105	0.129E-01	0.337E-02
4	0	4	105	-0.651E-02	-0.204E-02
5	5	0	131	0.794E-04	0.256E-04
6	0	5	131	0.105E-03	0.360E-04
7	6	0	157	0.296E-02	0.721E-03
8	0	6	157	-0.294E-02	-0.676E-03
9	7	0	184	0.512E-05	0.419E-05
10	0	7	184	0.596E-05	0.224E-05
11	8	0	210	0.941E-03	0.203E-03
12	0	8	210	-0.899E-03	-0.190E-03
13	9	0	236	0.911E-06	0.304E-06
14	0	9	236	0.104E-05	0.445E-06
15	10	0	262	0.293E-03	0.597E-04
16	0	10	262	-0.292E-03	-0.565E-04

Table 2: Resonance widths as evaluated using expression (4) for a one unit (relative to the quadrupole) systematic multipole component in the quadrupole magnets, assuming a beam emittance of 40π mm-mr (normalized).

Resonance Index	l	m	k	δ_R
1	3	0	79	0.585E-04
2	0	3	79	0.296E-04
3	4	0	105	0.220E-03
4	0	4	105	-0.150E-03
5	5	0	131	0.593E-05
6	0	5	131	0.413E-05
7	6	0	157	0.531E-04
8	0	6	157	-0.576E-04
9	7	0	184	0.418E-07
10	0	7	184	0.364E-06
11	8	0	210	0.164E-04
12	0	8	210	-0.178E-04
13	9	0	236	0.530E-07
14	0	9	236	0.789E-07
15	10	0	262	0.505E-05
16	0	10	262	-0.575E-05

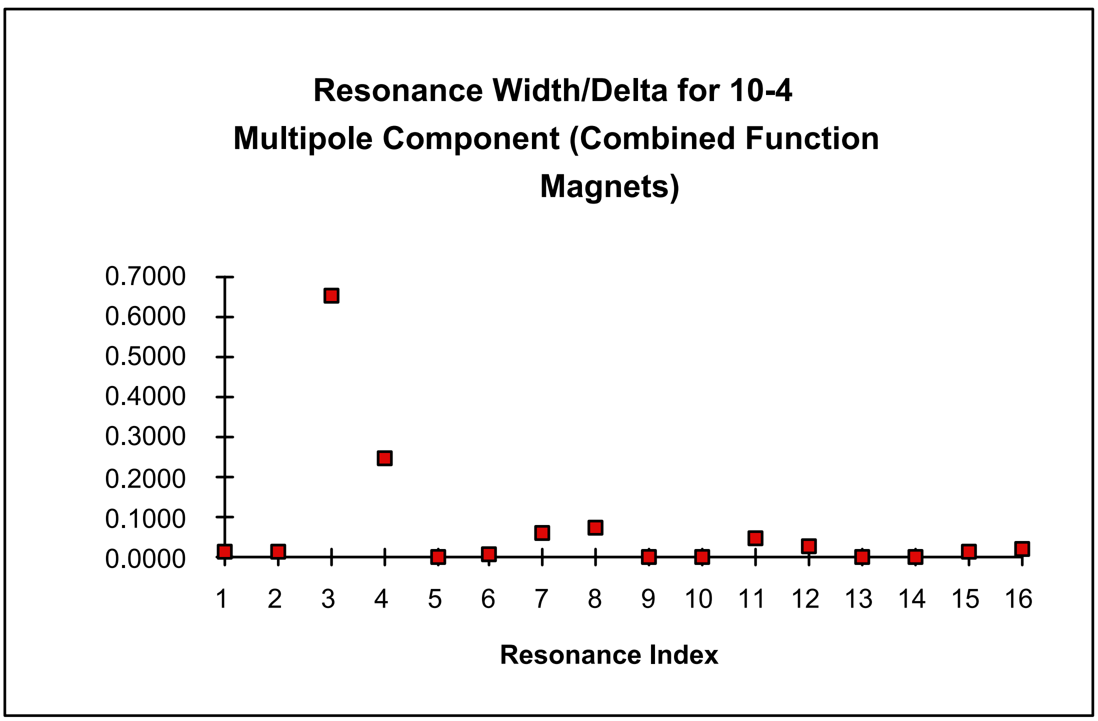

Figure 1: The ratio of the calculated resonance width to the separation of the nominal tune from the resonant tune for a one unit systematic multipole component in the combined function

magnets. Resonances from third to tenth order are displayed with indices as indicated in Table 1. An emittance of 40π mm-mr (normalized) is used in this calculation.

Random Multipoles: One Dimensional Analysis

Resonance behavior is also potentially driven by randomly distributed multipole errors. An expected resonance width due to uncorrelated multipoles can be derived using the standard statistical analysis of expression (4):

$$\delta_R = \frac{\varepsilon^{(l-2)/2}}{\pi 2^l}\left(1+\left\{\frac{l!}{2\left(\frac{l}{2}\right)!\left(\frac{l}{2}\right)!}\right\}^2\right)^{\frac{1}{2}}\left(\sum s_i^2 \beta_{x_i}^l\right)^{\frac{1}{2}} \tag{5}$$

where again the 0^{th} harmonic term applies only if l is even and a similar expression applies for the vertical resonance.

Resonance widths as calculated using expression (5) are given in Tables 3 and Table 4 for a rms random multipole error of one unit in the combined function and quadrupole magnets respectively. The calculation again assumes a normalized emittance of 40π mm-mr and is based on the lattice RRV6.

Table 3: Resonance widths as evaluated using expression (4) for a one unit (relative to the dipole) random multipole component in the long and short combined function magnets, assuming a beam emittance of 40π mm-mr (normalized).

Resonance Index	l	m	k	δ_R (Long)	δ_R (Short)
1	3	0	79	0.110E-02	0.437E-03
2	0	3	79	0.110E-02	0.431E-03
3	4	0	105	0.103E-02	0.391E-03
4	0	4	105	0.103E-02	0.383E-03
5	5	0	131	0.966E-04	0.355E-04
6	0	5	131	0.965E-04	0.346E-04
7	6	0	157	0.288E-03	0.104E-03
8	0	6	157	0.288E-03	0.101E-03
9	7	0	184	0.853E-05	0.302E-05
10	0	7	184	0.852E-05	0.294E-05
11	8	0	210	0.887E-04	0.312E-04
12	0	8	210	0.886E-04	0.305E-04
13	9	0	236	0.753E-06	0.264E-06
14	0	9	236	0.752E-06	0.262E-06
15	10	0	262	0.282E-04	0.993E-05

16	0	10	262	0.282E-04	0.998E-05

Table 4: Resonance widths as evaluated using expression (4) for a one unit (relative to the quadrupole) random multipole component in the quadrupole magnets, assuming a beam emittance of 40π mm-mr (normalized).

Resonance Index	l	m	k	δ_R
1	3	0	79	0.360E-04
2	0	3	79	0.376E-04
3	4	0	105	0.332E-04
4	0	4	105	0.350E-04
5	5	0	131	0.308E-05
6	0	5	131	0.328E-05
7	6	0	157	0.910E-05
8	0	6	157	0.976E-05
9	7	0	184	0.267E-06
10	0	7	184	0.288E-06
11	8	0	210	0.276E-05
12	0	8	210	0.299E-05
13	9	0	236	0.232E-07
14	0	9	236	0.254E-07
15	10	0	262	0.866E-06
16	0	10	262	0.949E-06

Resonance widths from random mulitpoles are seen to be comparable to those generated by systematic multipoles for odd-order resonances, and are typically an order of magnitude lower for even-order resonances. This leads us to expect that magnets produced with rms multipole distributions at, or a few times, the average systematic multipole should be acceptable.

Conclusions

The primary conclusion of this analysis is that the systematic octupole component of the combined function magnets needs to be kept under one unit in order to allow operation at the design tune point. Other systematic multipoles should be tolerable at the few unit level. In particular the design sextupole component of the combined function magnets, used for chromaticity correction, appears to be benign. Random multipoles of several units also appear to be tolerable.

Non-linear coupling resonances are not included in this analysis. However, given the equality of horizontal and vertical lattice functions in the Recycler we would expect senstitivities very similar to those shown here for one dimensional motion.

The resonance analysis in-and-of itself is not capable of assuring good storage characteristics in the Recycler. It is expected that this analysis will be accompanied by long-term tracking simulations and that an overall field flatness criterion could be extracted from such a study.

Appendix

The motion of an individual particle in a storage ring can be parameterized by,

$$\begin{cases} x(s) = A\sqrt{\beta(s)}\cos(\phi(s)) \\ x'(s) = -\dfrac{\alpha x(s)}{\beta(s)} - \dfrac{A}{\sqrt{\beta(s)}}\sin(\phi(s)) \end{cases} \tag{A1}$$

where A is the oscillation amplitude and β, ϕ are the optical functions associated with the linear lattice. We note that if the slope changes discontinuously at a certain point in the lattice, as it will if it traverses a short non-linear element, then the subsequent motion behaves as if A and ϕ have changed by an amount:

$$\begin{cases} \Delta A = -\sqrt{\beta_i}\sin(\phi_i)\Delta x'_i \\ \Delta\phi = -\dfrac{\sqrt{\beta_i}}{A}\cos(\phi_i)\Delta x'_i \end{cases} \tag{A2}$$

where the index, i, refers to the particular element responsible for the extra kick Δx'.

We are interested in behavior when the extra kick is caused by a non-linear field component:

$$\Delta x'_i = s_i x_i^n = s_i A^n \sqrt{\beta_i^n}\cos^n\phi_i \tag{A3}$$

Note that equation (A3) ignores coupling terms and so this analysis is only applicable to one-dimensional resonances. We adopt a model in which we paramaterize the motion through A and ϕ, the changes in which we calculate over one revolution of the accelerator:

$$\begin{cases} \dfrac{dA}{dn} = -\sum\limits_i \sqrt{\beta_i}\sin(\phi_i)\Delta x'_i = -A^n \sum\limits_i s_i \beta_i^{\frac{n+1}{2}} \sin(\phi_i)\cos^n(\phi_i) \\ \dfrac{d\phi}{dn} = -\sum\limits_i \dfrac{\sqrt{\beta_i}}{A}\cos(\phi_i)\Delta x'_i + 2\pi\nu = -A^{n-1}\sum\limits_i s_i \beta_i^{\frac{n+1}{2}} \cos^{n+1}(\phi_i) + 2\pi\nu \end{cases} \tag{A4}$$

We will assume that the non-linear perturbations, s_i, are sufficiently weak that A can be regarded as constant over one revolution of the ring and hence taken outside the summation in (A4), and that the sumations can be evaluated over one revolution using the unperturbed phase advances. The resonant behavior of the motion can be most clearly seen by substituting the following expressions for the trigonometric terms:

$$\cos^{n+1}\phi = \frac{1}{2^n}\left[\cos(n+1)\phi + (n+1)\cos(n-1)\phi + \frac{(n+1)(n)}{2!}\cos(n-3)\phi + \ldots \frac{(n+1)!}{2(\frac{n+1}{2})!^2}\right]$$

$$\sin\phi\cos^n\phi = \frac{1}{2^n}\left[\sin(n+1)\phi + (n-1)\sin(n-1)\phi + \frac{n(n-3)}{2!}\sin(n-3)\phi + \ldots\right]$$

Resonance occurs when (n+1)ν is close to an integer, leading to a coherent addition of the applied non-linear perturbation over many revolutions of the accelerator. We will define δ as the difference between the linear tune and the resonant tune:

$$\delta = \nu - \frac{k}{n+1} = \nu - \frac{k}{l} \tag{A5}$$

where we have made the identification of n+1 with l in equation (2). We can then rewrite (A4) including only the highest order resonant term as,

$$\begin{cases} \dfrac{dA}{dn} = -\dfrac{A^{l-1}}{2^{l-1}}\sum_i s_i\beta_i^{l/2}\sin(l\phi_i) \\ \dfrac{d(\phi - 2\pi nk/l)}{dn} = -\dfrac{A^{l-2}}{2^{l-1}}\sum_i s_i\beta_i^{l/2}\left(\dfrac{l!}{2(\frac{l}{2})!(\frac{l}{2})!} + \cos(l\phi_i)\right) + 2\pi\delta \end{cases}$$

where the first term represents a direct tune shift with amplitude and is only present if l is even. The sums can formally be written as,

$$\begin{cases} \dfrac{dA}{dn} = -\dfrac{A^{l-1}}{2^{l-1}}\sqrt{\alpha^2+\beta^2}\sin(l\phi_0 - \psi) \\ \dfrac{d(\phi_0 - (2\pi nk/l))}{dn} = -\dfrac{A^{l-2}}{2^{l-1}}\left(\gamma + \sqrt{\alpha^2+\beta^2}\cos(l\phi_0 - \psi)\right) + 2\pi\delta \end{cases} \tag{A6}$$

where ϕ_0 is the phase measured at an arbitrary, but fixed, point in the ring, and α and β are the sine-like and cosine-like Fourier components of the non-linear perturbation at the resonant frequency,

$$\begin{cases} \alpha = \sum_i s_i \beta_i^{l/2} \sin(l\phi_i - \phi_0) \\ \beta = \sum_i s_i \beta_i^{l/2} \cos(l\phi_i - \phi_0) \\ \gamma = \dfrac{l!}{2(\frac{l}{2})!^2} \sum_i s_i \beta_i^{l/2} \end{cases} \tag{A7}$$

The unstable fixed points can be easily identified in (A6):

$$\begin{cases} \phi_0 = \dfrac{2\pi k + \psi}{l} \\ A^{l-2} = \dfrac{2^l \pi\delta}{\gamma + \sqrt{\alpha^2 + \beta^2}} \end{cases} \tag{A8}$$

We will regard the value of A at the unstable fixed point as characterizing the scale of oscillation amplitudes that could be expected to lead to stable motion under the influence of a distribution of non-linear perturbations. Turning this around, we can define a minimum separation between the linear tune and the resonant tune, δ_R, that can be tolerated if all particles with oscillation amplitudes up to A are expected to remain stable. Finally, making the identification of $A=\sqrt{\varepsilon}$ we are left with an estimate of how close the linear tune can approach the resonant tune while maintaining stability for all particles within a beam of emittance ε:

$$\delta_R = \frac{(\gamma + \sqrt{\alpha^2 + \beta^2})\varepsilon^{\frac{l-2}{2}}}{2^l \pi} \tag{A9}$$