Fermilab/BD/TEV
Beams-doc-510
March 15, 2003

Notes on Sextupole Fields in the Tevatron

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Introduction

This memo contains a collection of notes, formulas, and calculations regarding the sextupole fields in the Tevatron. The major sources of sextupole fields are the b_2 component of the Tevatron dipole magnets, the chromaticity correction sextupole circuits, T:SF and T:SD, and the feeddown sextupole circuits S1, S2, S3, S4, S5, S6 and S7.

Sextupole Field Definitions

We define the magnetic field in a normal (as compared to skew) sextupole magnet as

$$B_x(x,y) = B_2 xy$$
$$B_y(x,y) = \tfrac{1}{2} B_2 (x^2 - y^2)$$

where the x is the horizontal coordinate with positive x in the radial outward direction, and y is the vertical coordinate with positive y in the upward direction.

The MAD program uses the multipole coefficient K_2 to quantify the sextupole field strength as

$$K_2 = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2}$$

where $B\rho$ is the magnetic rigidity ($B\rho = 3335.66$ Tesla - meters / Tev) and the MAD program uses units of m^{-3} for K_2. Relating this to the magnetic field defined above gives $B_2 = B\rho \cdot K_2$.

We can relate the sextupole field to the magnetic multipole coefficients usually used in magnet measurements at Fermilab. The magnetic field is written as a Taylor expansion of the form

$$B_y(x,y) + iB_x(x,y) = B_0 \sum_{n=0} \frac{(x+iy)^n}{r_0^n}(b_n + ia_n) \times 10^{-4}$$

where the reference radius, r_0 , is one inch, B_0 is the strength of the dipole field which is nominally 4.44 Tesla @ 1000 Gev, and the multiple coefficients, b_n and a_n, are quoted in units of 10^{-4}. For a normal sextupole field this expansion gives

$$B_y(x,y) = B_0 \frac{(x^2 - y^2)}{r_0^2} b_2 \times 10^{-4} .$$

Relating this to the magnetic field defined above gives

$$B_2 = \frac{2B_0 b_2}{r_0^2} \times 10^{-4} \quad \text{or} \quad b_2 = \frac{\frac{1}{2} B_2 r_0^2}{B_0} \times 10^4$$

where the factor of 10^4 accounts for the fact that the multipole coefficients are usually quoted in units of 10^{-4}.

We can also give the relation between the MAD sextupole field coefficient K_2 and magnet measurement multipole coefficient b_2 as

$$K_2 = \frac{2B_0}{B\rho\, r_0^2} b_2 \times 10^{-4}$$

or

$$K_2 = 4.126 \times 10^{-4}\ b_2\ (\text{meter}^{-3}) .$$

Sextupole Correction Magnets

In the Tevatron there are sextupole corrector magnets located in the spool pieces along with other types of magnets such as dipole and quadrupole correctors. Depending on the type of spool piece, a sextupole corrector magnet may have either normal or skew field and may be one of two strengths.

Many of the sextupole correctors are of the "chromaticity corrector" type, which are normal sextupoles and the stronger of the two types of sextupole correctors. All of the magnets in the chromaticity correction circuits T:SF and T:SD are of these type and the magnets in the S6 and S7 feeddown circuits, (C:S6A4A, C:S6C4A, C:S7B1A, and C:S7D1A) are of this type. The other sextupole corrector type is usually located in the "downstream spool packages." These are the weaker of the two types of sextupole fields and may have normal or skew sextupole fields. The magnets in the feeddown circuits S1, S2, S3, S4, and S5 are of this type.

We get the strengths of the sextupole fields from "The Tevatron Energy Doubler: A Superconducting Accelerator", H. Edwards, (*Ann. Rev. Nucl. Part Sci.* 1985, 35:605-60). For the "chromaticity corrector" type magnet the above reference gives an integrated field strength of 57 kG-inch at 1 inch at 50 Amps. For the "downstream spool package" type magnet the reference gives an integrated field strength of 44 kG-inch at 1 inch at 50 Amps.

We can convert these strengths into the more familiar form of the sextupole fields such as those used in the MAD program.

Chromaticity Sextupoles

The magnets in the chromaticity sextupoles, T:SF and T:SD, have an integrated strength of

$$\int B_y(\text{at}\, x = 1\ \text{inch}) \cdot dl = 57\,\text{kG} - \text{inch} \ @\ 50\ \text{Amps}$$

which gives

$$\tfrac{1}{2} B_2 (1\ \text{inch})^2 L/I = 5.7\ \text{Tesla} - \text{inch} / 50\ \text{Amps}$$

or

$$\text{B}_2\text{L/I} = 2.0 \times 5.7\ \text{Tesla / 50Amp / inch} \times \text{(inch/0.0254meter)} = 8.98\ \text{T/m/A}$$

where L is the length of the sextupole and I is the current in the corrector magnet. If the sextupoles are treated as a thin sextupole magnet, then (in the MAD convention) the thin sextupole strength is given by

$$\text{K}_2\text{L} = (B_2 L / I) \frac{1}{|B\rho|} I$$

where I is the current in the sextupole, and $|B\rho|$ is the magnetic rigidity. At 150 Gev $B\rho = 500.35$ Tesla - meter so we get

$$\mathrm{K}_2\mathrm{L} = (B_2 L/I)\frac{1}{|B\rho|}I = (8.98\ \mathrm{T/m/A})\frac{1}{(500.35\ \mathrm{T\,m})}\mathrm{I}$$

or

$$\mathrm{K}_2\mathrm{L} = 0.01795\ \mathrm{m}^{-2}/\mathrm{Amp}\times\mathrm{I} \quad @150\ \mathrm{Gev}$$

for the chromaticity correction sextupoles.

Feeddown (or "Downstream Package") Sextupoles

The sextupole correctors in the downstream package are used for most of the feeddown circuits. Using an integrated strength of 44 kG-inch at 1 inch at 50 Amps we convert to more familiar notation,

$$B_2 L/I = 2.0\times 4.4\ \mathrm{Tesla}\ /\ 50\mathrm{Amp}\ /\ \mathrm{inch}\times(\mathrm{inch}/0.0254\mathrm{meter}) = 6.93\ \mathrm{T/m/A}$$

or

$$\mathrm{K}_2\mathrm{L} = 0.01386\ \mathrm{m}^{-2}/\mathrm{Amp}\times\mathrm{I} \quad @150\ \mathrm{Gev}$$

for the "downstream package" sextupole correctors.

Tune Shift and Coupling from Feeddown Effect

Next we look at the effective quadrupole field generated by the feeddown effect from an orbit offset through a sextupole field. If the closed orbit through a thin sextupole has the coordinates x_0 and y_0, then a thin sextupole with strength K_2L and a tilt angle ψ will give a normal quadrupole field with strength

$$(\mathrm{K}_1\mathrm{L})_{\mathrm{norm}} = \mathrm{K}_2\mathrm{L}\left(\mathrm{x}_0 \cos 3\psi + \mathrm{y}_0 \sin 3\psi\right)$$

and a skew quadrupole field with strength

$$\left(\mathrm{K}_1\mathrm{L}\right)_{\mathrm{skew}} = \mathrm{K}_2\mathrm{L}\left(\mathrm{x}_0 \sin 3\psi - \mathrm{y}_0 \cos 3\psi\right)$$

In the Tevatron the chromaticity sextupoles have zero tilt angle, $\psi = 0$, and then

$$(\mathrm{K}_1\mathrm{L})_{\mathrm{norm}} = \mathrm{K}_2\mathrm{L}\ \mathrm{x}_0$$
$$\left(\mathrm{K}_1\mathrm{L}\right)_{\mathrm{skew}} = -\ \mathrm{K}_2\mathrm{L}\mathrm{y}_0$$

Given the strengths of the sextupoles we can calculate the feeddown effect they have on the tune and coupling. The changes in horizontal and vertical tune are given by the simple equations

$$\Delta \nu_x = \frac{1}{4\pi} \sum \beta_{\mathrm{x,i}} \, (\mathrm{K_1L})_{\mathrm{norm,i}}$$

$$\Delta \nu_y = -\frac{1}{4\pi} \sum \beta_{y,i} \, (\mathrm{K_1L})_{\mathrm{norm,i}}$$

where β_x, β_y are the beta functions at the locations of the feeddown sextupoles, and $(\mathrm{K_1L})_{\mathrm{norm,i}}$ is the normal component of the quadrupole field created by the off-center closed orbit and the sextupole field.

To quantify the effect that the feeddowns have on the coupling we calculate a sine and cosine term of the coupling as

$$\Delta \mathrm{C_{coup}} = \frac{1}{2\pi} \sum \, (\mathrm{K_1L})_{\mathrm{skew,i}} \sqrt{\beta_{x,i}\beta_{y,i}} \, \cos(\phi_{\mathrm{y}} - \phi_{\mathrm{x}})$$

$$\Delta \mathrm{S_{coup}} = \frac{1}{2\pi} \sum \, (\mathrm{K_1L})_{\mathrm{skew,i}} \sqrt{\beta_{x,i}\beta_{y,i}} \, \sin(\phi_{\mathrm{y}} - \phi_{x})$$

where $(\mathrm{K_1L})_{\mathrm{skew,i}}$ is the skew component of the quadrupole field and $(\phi_{\mathrm{y}} - \phi_{\mathrm{x}})$ is the difference between the vertical and horizontal betatron phase advances.

To better understand the coupling constants we express the minimum tune split in terms of $\Delta C_{\mathrm{coup}}, \Delta S_{\mathrm{coup}}$. If the Tevatron is tuned up such that there is no coupling then in principle the horizontal and vertical tunes could be set equal to one another. However, if we add a small amount of coupling then the measured minimum tune split would be

$$\Delta \nu_{\min} = \sqrt{\Delta \mathrm{C_{coup}^2} + \Delta \mathrm{S_{coup}^2}} \; .$$

(We also note that if the multipole coefficients are used, then the feeddown relationship is given by $b_1 = \dfrac{2b_2\Delta x}{r_0}$ where Δx is the horizontal orbit offset and r_0 is the reference radius of one inch.)

Chromaticity from Sextupole Fields

The chromaticity, ξ, is defined as

$$\Delta\nu = \xi\left(\frac{\Delta p}{p}\right),$$

where $\frac{\Delta p}{p}$ is the momentum offset of a particle. From the above expressions above for the tune shift from a feeddown effect, we can calculate the change in chromaticity due to sextupole fields. In general the tune shift from a normal quadrupole field, $(\mathrm{K_1L})_{\mathrm{norm}}$, is given by

$$\Delta\nu_x = \frac{1}{4\pi}\sum \beta_{\mathrm{x,i}}\,(\mathrm{K_1L})_{\mathrm{norm,i}}.$$

We can then calculate the tune shift from a momentum offset by using the feeddown effect and an orbit offset, Δx, due to the dispersion D_x,

$$\Delta x = D_x\left(\frac{\Delta p}{p}\right)$$

giving the tune shift expected from a momentum change

$$\Delta\nu_x = \frac{1}{4\pi}\sum \beta_{\mathrm{x,i}}(\mathrm{K_2L\Delta x})_{\mathrm{i}}$$

$$\Delta\nu_x = \frac{1}{4\pi}\sum \beta_{\mathrm{x,i}}(\mathrm{K_2LD_x})_{\mathrm{i}}\left(\frac{\Delta p}{p}\right).$$

This results in the relationship between the sextupole field, $\mathrm{K_2}$, and the chromaticity

$$\xi_x = \frac{1}{4\pi}\sum \beta_{\mathrm{x,i}}(\mathrm{K_2LD_x})_{\mathrm{i}}$$

$$\xi_y = -\frac{1}{4\pi}\sum \beta_{\mathrm{y,i}}(\mathrm{K_2LD_x})_{\mathrm{i}}$$

If we are interested in expressing the chromaticity in terms of $\mathrm{b_2}$ then the expression for the chromaticity becomes

$$\xi_x = \frac{1}{4\pi}\sum \beta_{\mathrm{x}}\mathrm{D_x}\frac{2\mathrm{B_0}L}{B\rho\, r_0^2}b_2\times 10^{-4}$$

$$\xi_y = -\frac{1}{4\pi}\sum \beta_{\mathrm{y}}\mathrm{D_x}\frac{2\mathrm{B_0}L}{B\rho\, r_0^2}b_2\times 10^{-4}$$

which we can write as

$$\xi_x = 0.328 \times N \cdot \langle \beta_x D_x \rangle \cdot L \cdot b_2 \times 10^{-4}$$
$$\xi_y = -0.328 \times N \cdot \langle \beta_y D_x \rangle \cdot L \cdot b_2 \times 10^{-4}$$

where N is the number of magnetic elements in the circuit, L is the length of the magnets, and $\langle \beta_x D_x \rangle$ and $\langle \beta_y D_x \rangle$ are the average values of the product of the beta functions and dispersion at the locations of the magnetic elements in the circuit.

Lattice Functions at Locations of Sextupoles

In the following table we summarize average values for various lattice functions at the locations of the Tevatron dipoles and the T:SF and T:SD corrector magnets. For the T:SF and T:SD circuits the averages are calculated over the 88 elements in each circuit and the lattice functions are taken at the center of the corrector elements. For the 772 full length dipoles in the Tevatron we calculate the average values at three locations: At the upstream end of the magnet, in the center of the dipole, and at the downstream end of the magnet. These values were calculated using the MAD program with the design Tevatron injection lattice. The average beta functions, $\langle \beta_x \rangle, \langle \beta_y \rangle$, average dispersion, $\langle D_x \rangle$, and average products $\langle \beta_x D_x \rangle$, $\langle \beta_y D_x \rangle$ are listed. Note that in general $\langle \beta_x D_x \rangle \neq \langle \beta_x \rangle \langle D_x \rangle$

Location	N	$\langle \beta_x \rangle$ (meter)	$\langle \beta_y \rangle$ (meter)	$\langle D_x \rangle$ (meter)	$\langle \beta_x D_x \rangle$ (meter^2)	$\langle \beta_y D_x \rangle$ (meter^2)	$\langle \sqrt{\beta_x}\sqrt{\beta_y} \cos(\phi_y-\phi_x) \rangle$ (meter)	$\langle \sqrt{\beta_x}\sqrt{\beta_y} \sin(\phi_y-\phi_x) \rangle$ (meter)
Start of dipoles	772	57.227	58.544	2.825	170.321	156.316	51.237	-13.839
Center of dipoles	772	57.414	57.944	2.827	170.213	155.611	51.205	-13.897
End of dipoles	772	58.481	58.225	2.841	176.030	155.651	51.201	-13.923
Center ½ dipole	2	70.826	69.417	1.892	145.187	121.255	59.975	4.934
TSF	88	93.810	30.090	3.810	358.711	115.793	51.388	-12.054
TSD	88	30.295	93.285	2.301	70.837	214.546	50.182	-16.487

Table 1 The average values of lattice functions at the locations of the Tevatron dipoles, the Tevatron half-dipoles, and chromaticity sextupole magnets for the design values of the Tevatron injection lattice.

Chromaticity Correction from T:SF and T:SD

With the chromaticity sextupole strength of $K_2L = 0.01795\ m^{-2}$/Amp, using the average lattice functions listed in Table 1, and the formulas presented above, we can calculate the change in chromaticity from the T:SF and T:SD circuits at 150 Gev in the Tevatron.

$$\Delta\xi_x = \frac{88}{4\pi}\langle\beta_x D_x\rangle(\Delta K_2 L)$$
$$\Delta\xi_y = -\frac{88}{4\pi}\langle\beta_y D_x\rangle(\Delta K_2 L)$$

or

$$\Delta\xi_x = +\frac{88}{4\pi}\times 358.7\ \mathrm{m}^2 \times 0.01795\ \mathrm{m}^{-2}/\mathrm{Amp} = +45.1\ \mathrm{units/(Amp\ in\ T:SF)}$$
$$\Delta\xi_y = -\frac{88}{4\pi}\times 115.8\ \mathrm{m}^2 \times 0.01795\ \mathrm{m}^{-2}/\mathrm{Amp} = -14.6\ \mathrm{units/(Amp\ in\ T:SF)}$$

$$\Delta\xi_x = +\frac{88}{4\pi}\times 70.84\ \mathrm{m}^2 \times 0.01795\ \mathrm{m}^{-2}/\mathrm{Amp} = +8.90\ \mathrm{units/(Amp\ in\ T:SD)}$$
$$\Delta\xi_y = -\frac{88}{4\pi}\times 214.6\ \mathrm{m}^2 \times 0.01795\ \mathrm{m}^{-2}/\mathrm{Amp} = -27.0\ \mathrm{units/(Amp\ in\ T:SD)}$$

A MAD calculation with K_2L increased by $0.01795\ m^{-2}$ (corresponding to 1.0 Amp at 150 Gev) gives the following changes to the chromaticity

$$\begin{pmatrix}\Delta\xi_x \\ \Delta\xi_y\end{pmatrix} = \begin{pmatrix} 45 & 8.9 \\ -14.36 & -26.96\end{pmatrix}\begin{pmatrix}\Delta I_{SF}\ \mathrm{(amps)} \\ \Delta I_{SD}\ \mathrm{(amps)}\end{pmatrix}$$

which is consistent with the above calculations.

We note that a calibration[1] of the T:SF and T:SD circuits on 11/06/02 gave the following transfer coefficients

$$\begin{pmatrix} \Delta\xi_x \\ \Delta\xi_y \end{pmatrix} = \begin{pmatrix} 43.8 & 8.6 \\ -11.5 & -27.9 \end{pmatrix} \begin{pmatrix} \Delta \mathrm{I}_{\mathrm{SF}} \text{ (amps)} \\ \Delta \mathrm{I}_{\mathrm{SD}} \text{ (amps)} \end{pmatrix}$$

which differ from the values calculated using the design sextupole field strengths and the design Tevatron injection lattice.

Chromaticity from b2 Component of Dipoles

With the dipole magnet sextupole multipole b_2, using the average lattice functions listed in Table 1, and the formulas presented above, we can calculate the change in chromaticity from changes in the b_2 multipole component in the Tevatron dipoles at 150 Gev.

$$\Delta\xi_x = \frac{772}{4\pi}\left\langle \beta_{\mathrm{x}} \mathrm{D}_{\mathrm{x}} \right\rangle (\Delta \mathrm{K}_2 \mathrm{L})$$
$$\Delta\xi_y = -\frac{772}{4\pi}\left\langle \beta_{\mathrm{y}} \mathrm{D}_{\mathrm{x}} \right\rangle (\Delta \mathrm{K}_2 \mathrm{L})$$

or

$$\Delta\xi_x = 0.328 \times N \cdot \left\langle \beta_{\mathrm{x}} \mathrm{D}_{\mathrm{x}} \right\rangle \cdot L \cdot \Delta b_2 \times 10^{-4}$$
$$\Delta\xi_y = -0.328 \times N \cdot \left\langle \beta_{\mathrm{y}} \mathrm{D}_{\mathrm{x}} \right\rangle \cdot L \cdot \Delta b_2 \times 10^{-4}$$

where L = 6.124 m is the magnetic length of a Tevatron dipole. This gives

$$\Delta\xi_x = +0.328 \times 772 \times 170.2\mathrm{m}^2 \times 6.1214\mathrm{m} \times \Delta\, \mathrm{b}_2 \times 10^{-4} = +26.38 \times \Delta \mathrm{b}_2$$
$$\Delta\xi_y = -0.328 \times 772 \times 155.6\mathrm{m}^2 \times 6.1214\mathrm{m} \times \Delta \mathrm{b}_2 \times 10^{-4} = -24.12 \times \Delta \mathrm{b}_2$$

or

$$\Delta\xi_x = +26.38 \times \Delta \mathrm{b}_2$$
$$\Delta\xi_y = -24.12 \times \Delta \mathrm{b}_2$$

for the change in chromaticity per unit change of b_2 in the Tevatron dipoles.

[1] ***" Measurement of the Tevatron tune shift versus RF frequency and strength of the T:SF and T:SD circuits.", M. Martens, Beams-doc-436, http://beamdocs.fnal.gov/cgi-bin/public/DocDB/ShowDocument?docid=436.***

In a similar calculation, with $\Delta K_2 = 1\times10^{-4}\ \mathrm{m}^{-3}$

$$\Delta\xi_x = +\frac{772}{4\pi}\times 170.2\,\mathrm{m}^2\times(1\times10^{-4}\ \mathrm{m}^{-3})\times 6.1214\,\mathrm{m} = +6.40\,\mathrm{units}/(10^{-4}\ \mathrm{m}^{-3}\mathrm{of}\ K_2)$$
$$\Delta\xi_y = -\frac{772}{4\pi}\times 155.6\,\mathrm{m}^2\times(1\times10^{-4}\ \mathrm{m}^{-3})\times 6.1214\,\mathrm{m} = -5.85\,\mathrm{units}/(10^{-4}\ \mathrm{m}^{-3}\mathrm{of}\ K_2)$$

Changing K_2 in the BEND elements in the Tevatron injection lattice MAD file by $1\times10^{-4}\ \mathrm{m}^{-3}$ changes the horizontal chromaticity by 6.44 units and the vertical chromaticity by –5.86 units which is consistent with the above calculations.

As a summary of the calculated contributions to the chromaticity from the dipole magnets and the SF and SD chromaticity sextupoles at 150 Gev we have

$$\begin{pmatrix}\Delta\xi_x\\ \Delta\xi_y\end{pmatrix} = \begin{pmatrix}45 & 8.9\\ -14.36 & -26.96\end{pmatrix}\begin{pmatrix}\Delta I_{SF}\\ \Delta I_{SD}\end{pmatrix} + \begin{pmatrix}26.38\\ -24.12\end{pmatrix}\Delta b_2$$

where ΔI_{SF}, ΔI_{SD} are the changes in amps in the T:SF and T:SD circuits and Δb_2 is the change in sextupole field in the Tevatron dipoles in units of 10^{-4}.

Natural Chromaticity

In addition to the chromaticity contributions from sextupole fields the Tevatron also has a natural chromaticity from the momentum dependence of quadrupole magnet focusing strength. A MAD calculation using the design injection lattice gives a natural chromaticity of

$$\xi_x = -29.59$$
$$\xi_y = -28.96$$

Total Chromaticity

Using all of the above information we can estimate the total chromaticity in the Tevatron at 150 Gev from the sextupole fields in the dipoles, the SF and SD circuits[2], and the natural chromaticity. The total calculated chromaticity is

[2] Note that the total current in the SF magnets is actually the sum of T:SF and C:SFB2. The total current in the SD magnets is the sum of T:SD and C:SDB2.

$$\begin{pmatrix}\xi_x \\ \xi_y\end{pmatrix} = \begin{pmatrix} 45 & 8.9 \\ -14.36 & -26.96 \end{pmatrix}\begin{pmatrix} I_{SF} \\ I_{SD} \end{pmatrix} + \begin{pmatrix} 26.38 \\ -24.12 \end{pmatrix} b_2 + \begin{pmatrix} -29.59 \\ -28.96 \end{pmatrix}$$

where I_{SF} and I_{SD} is the total current (in amps) in the SF and SD chromaticity sextupoles and b_2 is the amount of sextupole field in the main dipoles in units of 10^{-4} at 1 inch.