Energy Spread and Emittance of a Beam in the Recycler

(how calculate longitudinal emittance by measurement of energy spread)

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1. Hamiltonian of longitudinal motion is:

$$H = \frac{\Delta E^2}{2M} + U(\tau) \tag{1}$$

where ΔE and τ are energy and time split between current and reference (central) particles,

$$M = \frac{\beta^2 E}{1/\gamma^2 - 1/\gamma_{tr}^2} = 1039 \text{ GeV}, \quad U(\tau) = \frac{1}{T}\int eV(\tau)\, d\tau \tag{2}$$

Here $E = 8.938$ GeV and $T = 11.13$ μsec are total energy and revolution frequency of the reference particle, $\beta = 0.9945$ and $\gamma = 9.526$ – its reduced energy and velocity, $\gamma_{tr} = 19.968$ – reduced transition energy, $V(\tau)$ – accelerating voltage. At stationary conditions, $H =$ const what is an equation of a phase trajectory.

2. The waveform $V(\tau)$ and potential energy $U(\tau)$ considered in this note are presented in Fig.1 by red and green lines. They are periodical functions of τ with period T. Equation of phase trajectory (blue line) is:

$$W(\tau, \Delta E) = \Delta E_m \tag{3}$$

where

$$W = \sqrt{\Delta E^2 + 2eV_0 M \frac{|\tau| - \tau_0/2}{T} \times \begin{pmatrix} 1 \text{ at } 0 < |\tau| - \tau_0/2 < \Delta\tau_0 \\ 0 \text{ in other cases} \end{pmatrix}} \tag{4}$$

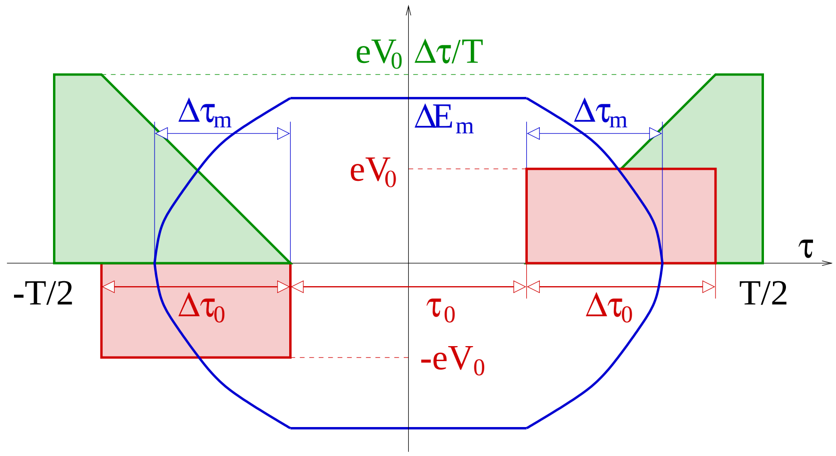

Figure 1: Waveform, potential wall, and phase trajectory (schematically)

and $V_0 = 2$ kV. We will consider only closed trajectories satisfying the inequality:

$$\Delta E_m^2 < \frac{2eV_0 M \Delta\tau_0}{T} \tag{5}$$

3. Stationary distribution function F can depend on W only. If a beam is subjected to some cooling and diffusion processes during enough long time, the distribution is probably Gaussian:

$$F = \exp\left(-\frac{W^2}{2\sigma_E^2}\right) \tag{6}$$

We will assume that

$$3\sigma_E < \sqrt{\frac{2eV_0 M \Delta\tau_0}{T}} \simeq 18 \text{ MeV} \tag{7}$$

to neglect the restriction following by Eq.(5). Then σ_E is r.m.s. energy spread.

4. Now we calculate area S enveloped by any phase trajectory, and relative number of particles N in this area:

$$S = 2\tau_0\sigma_E\left(x + \frac{x^3}{3x_0^2}\right) \tag{8}$$

$$N = \sqrt{\frac{2}{\pi}} \Big(\int_0^x e^{-\xi^2/2}\, d\xi - \frac{x e^{-x^2/2}}{1 + x_0^2} \Big) \tag{9}$$

where

$$x = \frac{\Delta E_m}{\sigma_E} \qquad \text{and} \qquad x_0^2 = \frac{eV_0 M \tau_0}{2\sigma_E^2 T} = \frac{\tau_0}{\sigma_E^2} \times 93.35 \frac{\mathrm{MeV}^2}{\mu\mathrm{sec}} \tag{10}$$

Dependence of $S/(2\tau_0\sigma_E)$ on $\sigma_E^{/}\tau_0 2$ is plotted in Fig.2 where parameter is relative number of particles in the area S. It is reasonable to refer this area as the bunch emittance for given part of the beam. For instance, at $\sigma_E^2/\tau_0 = 10\ MeV^2/\mu$sec we have: $S_{.8} = 3\tau_0\sigma_E$, and $S_{.9} = 4\tau_0\sigma_E$.

The following fit is possible for 90% phase area:

$$\frac{S_{.9}}{2\tau_0\sigma_E} \simeq 1.645 + 0.032 \left(\frac{\sigma_E^2}{\tau_0}\right) + 0.00172 \left(\frac{\sigma_E^2}{\tau_0}\right)^{3/2} \tag{11}$$

that is

$$S_{.9} \simeq 3.29\ \tau_0\sigma_E + 0.064\ \sigma_E^3 + 0.00344\ \sigma_e^4/\sqrt{\tau_0} \tag{12}$$

where units μsec, MeV are used. Relative accuracy of this fit is plotted in Fig.3. The error lies in a range $\pm 0.4\%$ at $\sigma_E^2/\tau_0 < 100$ MeV$^2/\mu$sec.

The phase area and emittance are adiabatic invariants, i.e. they do not change at any slow transformation of accelerating voltage. However, in non-adiabatic regime some increase of the emittance is unavoidable depending both on mismatching of initial and final phase trajectories, and velocity of the transformation.

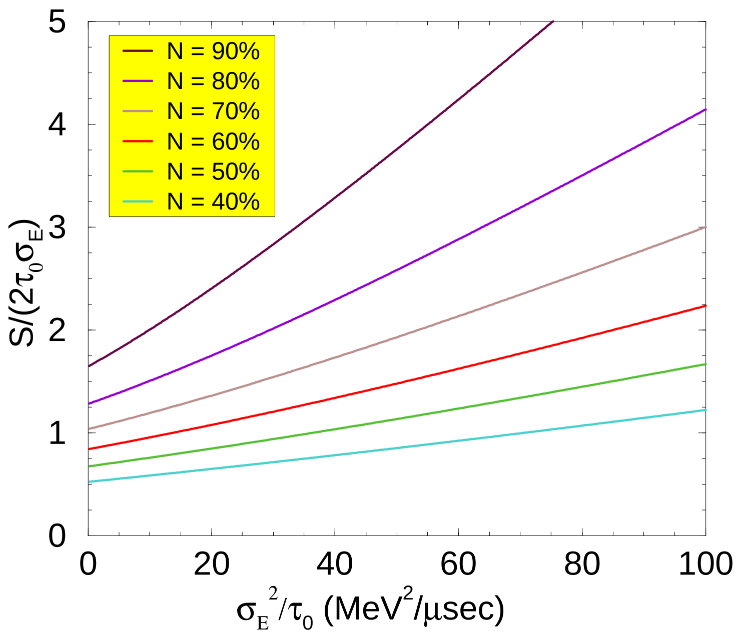

Figure 2: Phase area of Gaussian beam

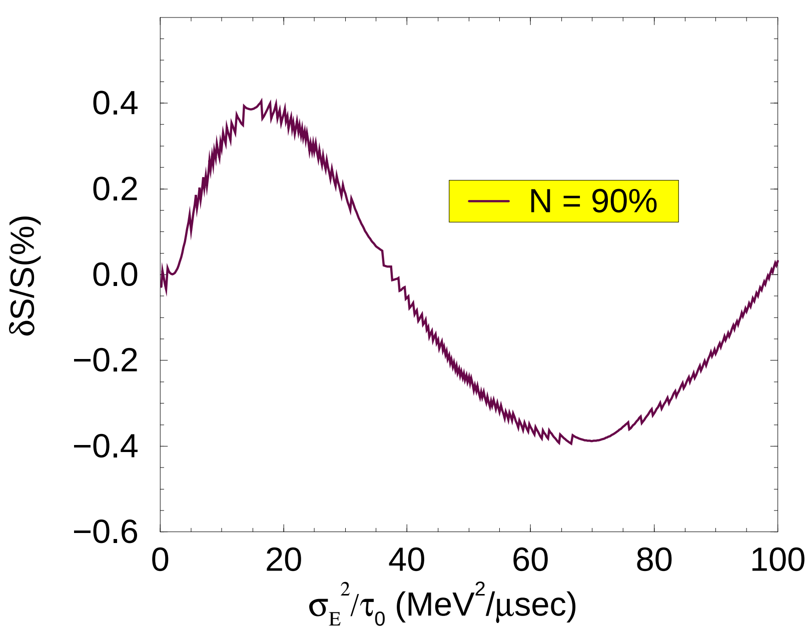

Figure 3: Relative error of the fit Eq.(11-12)