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Effect of Longitudinal Translation of a Quadrupole Magnet on the Downstream Amplitude Function

M. Syphers

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Abstract

The effect on the amplitude function downstream of a quadrupole magnet which is moved in the beam direction is computed. As an example the translation of the first quadrupole in the Fermilab 400 MeV line is examined.

Consider a thin-lens quadrupole magnet with focal length F originally located at $s = 0$ in a beamline. The amplitude function and its slope at the entrance to the magnet are given by β_0 and $\beta_0' \equiv -2\alpha_0$, respectively. Next, imagine the quadrupole moved downstream by a distance Δ, as depicted in Figure 1. At the exit of the moved quadrupole, the new lattice parameters are β_2 and

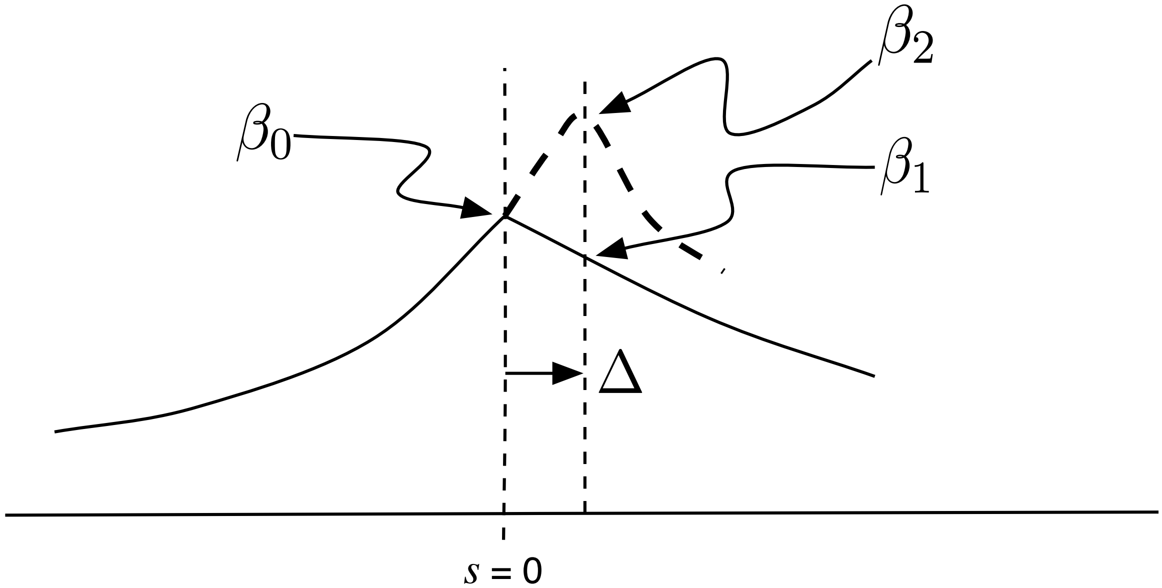

Figure 1: **Sketch of Amplitude Function through a thin quad. The dashed line represents the amplitude function after the quad is moved, the solid line before the move.**

α_2. At this same location, $s = \Delta$, the parameters of the lattice parameters were originally β_1 and α_1 before the move. Downstream of the moved magnet, the new amplitude function will oscillate about the original value – a "beta wave," or "beta mismatch." The amplitude of the mismatch is determined by the determinant of the matrix $\Delta J = J_2 - J_1$, where

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

with $\gamma \equiv (1+\alpha^2)/\beta$. The beta wave will propagate according to[1]:

$$\frac{\Delta\beta}{\beta}(\psi) = \left(\frac{\Delta\beta}{\beta}(0) - \frac{1}{2}|\det\Delta J|\right)\cos 2\psi + \left(\alpha(0)\frac{\Delta\beta}{\beta}(0) - \Delta\alpha(0)\right)\sin 2\psi + \frac{1}{2}|\det\Delta J|$$

where $\psi(s)$ is the original phase advance along the beam line before the perturbation was introduced. The maximum deviation in the downstream amplitude function will be

$$\left|\frac{\Delta\beta}{\beta}\right|_{max} = \frac{|\det\Delta J|}{2} + \sqrt{|\det\Delta J| + \left(\frac{|\det\Delta J|}{2}\right)^2}$$

To lowest order, the magnitude of the beta wave will be just $\sqrt{-\det\Delta J} = \sqrt{\Delta\alpha^2 - \Delta\beta\,\Delta\gamma}$. Downstream of the perturbation, $\det\Delta J$ is an invariant, and so can be evaluated at any convenient location, s.

Determining the Beta Mismatch

We will determine $\det\Delta J$ just downstream of the moved quadrupole, and evaluate it in terms of the quadrupole's focal length, the distance moved, and the values β_0 and α_0 at the original entrance of the magnet.

Consider the matrix M which propagates a particle's trajectory through a drift of length Δ followed by a quadrupole with focal length $F = 1/q$, and the matrix $\tilde{M}$ for going through the quad and *then* the drift. They are

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & \Delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \Delta \\ -q & 1-q\Delta \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ \tilde{M} &= \begin{pmatrix} 1 & \Delta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 1-q\Delta & \Delta \\ -q & 1 \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \Delta J = J_2 - J_1 &= MJ_0M^{-1} - \tilde{M}J_0\tilde{M}^{-1} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha_0 & \beta_0 \\ -\gamma_0 & -\alpha_0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} - \begin{pmatrix} d & b \\ c & a \end{pmatrix}\begin{pmatrix} \alpha_0 & \beta_0 \\ -\gamma_0 & -\alpha_0 \end{pmatrix}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \\ &= \begin{pmatrix} b(a-d)\gamma_0 + c(d-a)\beta_0 & 2b(d-a)\alpha_0 + (a^2-d^2)\beta_0 \\ (a^2-d^2)\gamma_0 + 2c(d-a)\alpha_0 & -b(a-d)\gamma_0 - c(d-a)\beta_0 \end{pmatrix} \end{aligned}$$

from which we can identify

$$\begin{aligned} \Delta\alpha &= b(a-d)\gamma_0 + c(d-a)\beta_0 \\ \Delta\beta &= 2b(d-a)\alpha_0 + (a^2-d^2)\beta_0 \\ \Delta\gamma &= -(a^2-d^2)\gamma_0 - 2c(d-a)\alpha_0 \end{aligned}$$

Substituting the values for a, ... d, we get

$$\begin{aligned}
\Delta\alpha &= q\Delta(\gamma_0\Delta + \beta_0 q) \\
\Delta\beta &= 2q\Delta[(1 - q\Delta/2)\beta_0 - \alpha_0\Delta] \\
\Delta\gamma &= -2q\Delta[(1 - q\Delta/2)\gamma_0 + \alpha_0 q]
\end{aligned}$$

To this point the result is exact (for a thin lens). So, finally, we can evaluate $\det \Delta J$, eventually keeping only lowest-order terms in the displacement:

$$\begin{aligned}
-\det \Delta J &= \Delta\alpha^2 - \Delta\beta\Delta\gamma \\
&= q^2\Delta^2(\gamma_0\Delta + \beta_0 q)^2 + (2q\Delta)^2[(1 - q\Delta/2)\beta_0 - \alpha_0\Delta][(1 - q\Delta/2)\gamma_0 + \alpha_0 q] \\
&\approx q^4\Delta^2\beta_0^2 + 4q^2\Delta^2[\beta_0][\gamma_0 + \alpha_0 q] \\
&= q^4\Delta^2\beta_0^2 + 4q^2\Delta^2[1 + \alpha_0^2 + \beta_0\alpha_0 q] \\
&= q^4\Delta^2\beta_0^2 + 4q^2\Delta^2[1 + (\alpha_0 + \beta_0 q/2)^2 - (\beta_0 q/2)^2] \\
&= (2q\Delta)^2[1 + (\alpha_0 + \beta_0 q/2)^2]
\end{aligned}$$

and so the beta wave will have amplitude

$$\left|\frac{\Delta\beta}{\beta}\right|_{max} = 2q\Delta\sqrt{1 + (\alpha_0 + \beta_0 q/2)^2}.$$

Note that for a periodic FODO cell, just before or after a quadrupole $\alpha = \mp\beta/2F$ (where F can be positive or negative) and so the relative beta wave downstream of a moved quadrupole in an originally matched FODO cell would have amplitude $|\Delta\beta/\beta|_{max} = |2d/F|$.

As a numerical example consider a magnet with focal length $F = 4.5$ m, such as the first magnet in the Fermilab 400 MeV Linac-to-Booster transfer line, in a FODO cell which is moved upstream by an amount $\Delta = -0.254$ m (10 in). The beta wave introduced would have an amplitude of about 11%. The Q1 magnet in the 400 MeV line is actually at a match point, where α is closer to zero, and β_0 is about 10 m. Then, the additional factor is about $\sqrt{2}$. and so we could expect a beta wave of about 15% downstream of the moved magnet if no further re-tuning of the beamline were performed.

References

[1] M. Syphers, T. Sen, and D. Edwards, "Amplitude Function Mismatch," Proc. 1993 Part. Acc. Conf., IEEE 93CH3279-7, 134-136 (1993).