Luminosity Evolution in Tevatron

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Run II meeting
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Talk outline

1. Introduction
2. Bunch lengthening due to RF phase noise
3. IBS Growth rates for Gaussian beams
4. IBS in non-linear longitudinal well
5. Comparison with experiment

   Conclusions

1. Introduction

Objective

- Luminosity evolution in Tevatron is driven by
  - Elastic and non-elastic **scattering on the residual gas**
  - Elastic and non-elastic **scattering on counter-rotating beam**
  - **RF noise** & transverse noise (magnetic field fluctuations, quad motion, etc.)
  - **Intrabeam scattering**
  - **Beam-beam effects**
- **The major aim** for the effort is to understand better **the beam-beam effects** and other possible limitations of the luminosity
- To make any practical conclusions **accurate measurements are not less important than good theory**

IBS

- Factors to be taken into account
  - X-Y coupling and actual Tevatron optics
  - Non-linear focusing and Finite size of RF bucket
  - Simultaneous treatment of single and multiple scattering for || degree of freedom

RF noise

- Factors to be taken into account
  - Spectral density of the noise
  - Non-linearity of the potential well
  - Finite size of the well

Measurements

- Longitudinal distribution - Comes from SBD
  - Algorithm for computation of distribution function was suggested by Alvin Tollestrup
    - Additional improvements
      - Restricted fit – $f(I) > 0$
      - Optimal binning
- X and Y emittances - Come from flying wires and Sync Light
  - Data analysis was done by Sasha Valishev
    - Measured optics
    - Light diffraction for sync light
    - Good prediction for initial luminosity

2. Bunch lengthening due to RF phase noise

Theoretical description

$$\ddot{x}+\Omega_s{}^2\sin(x-y(t))=0 \quad\Rightarrow\quad \ddot{x}+\Omega_s{}^2\sin(x)=\Omega_s{}^2\cos(x)y(t)$$

Action - $\quad I=\dfrac{1}{2p}\oint p\,dx$ $\qquad$ Frequency - $\quad w\equiv w(I)=2p\left(\oint\dfrac{dx}{p}\right)^{-1}$

We define the diffusion coefficient using the following form of diff. equation

$$\frac{\partial f}{\partial t}=\frac{1}{2}\frac{\partial}{\partial I}\left(I\frac{D(I)}{w(I)}\frac{\partial f}{\partial I}\right)$$

where diffusion coefficient is

$$D(I)=\frac{w}{I}\frac{d}{dt}\overline{dI^2}=2p\Omega_s{}^2\sum_{n=0}^{\infty}C_n(I)P(nw(I)) \quad ,$$

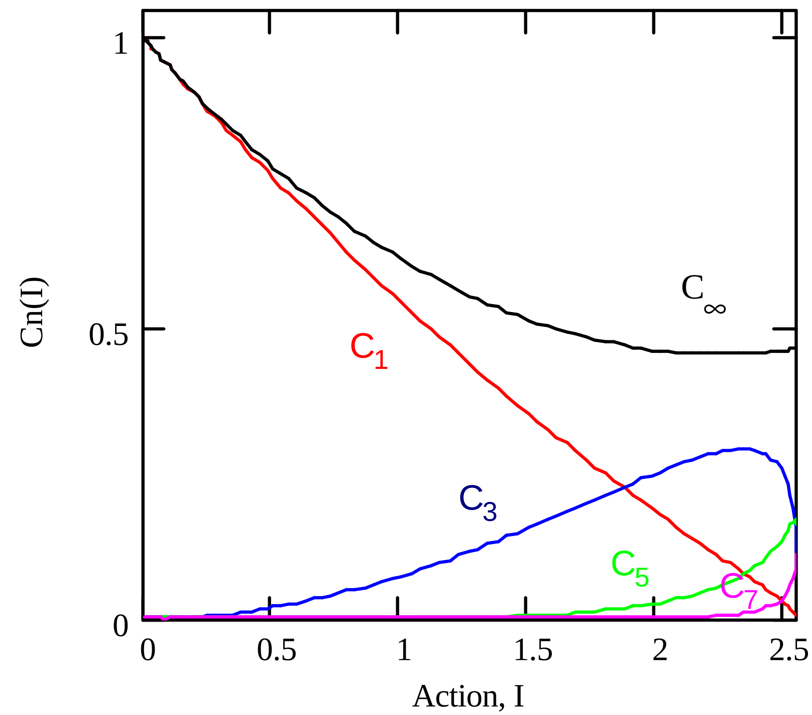
and the spectral density is normalized as

$$\overline{y(t)^2}=\int_{-\infty}^{\infty}P(w)dw \quad .$$

For the white noise, $P(w)=P_0$, it yields

$$D(I)=2p\Omega_s{}^2P_0C_\infty(I)$$

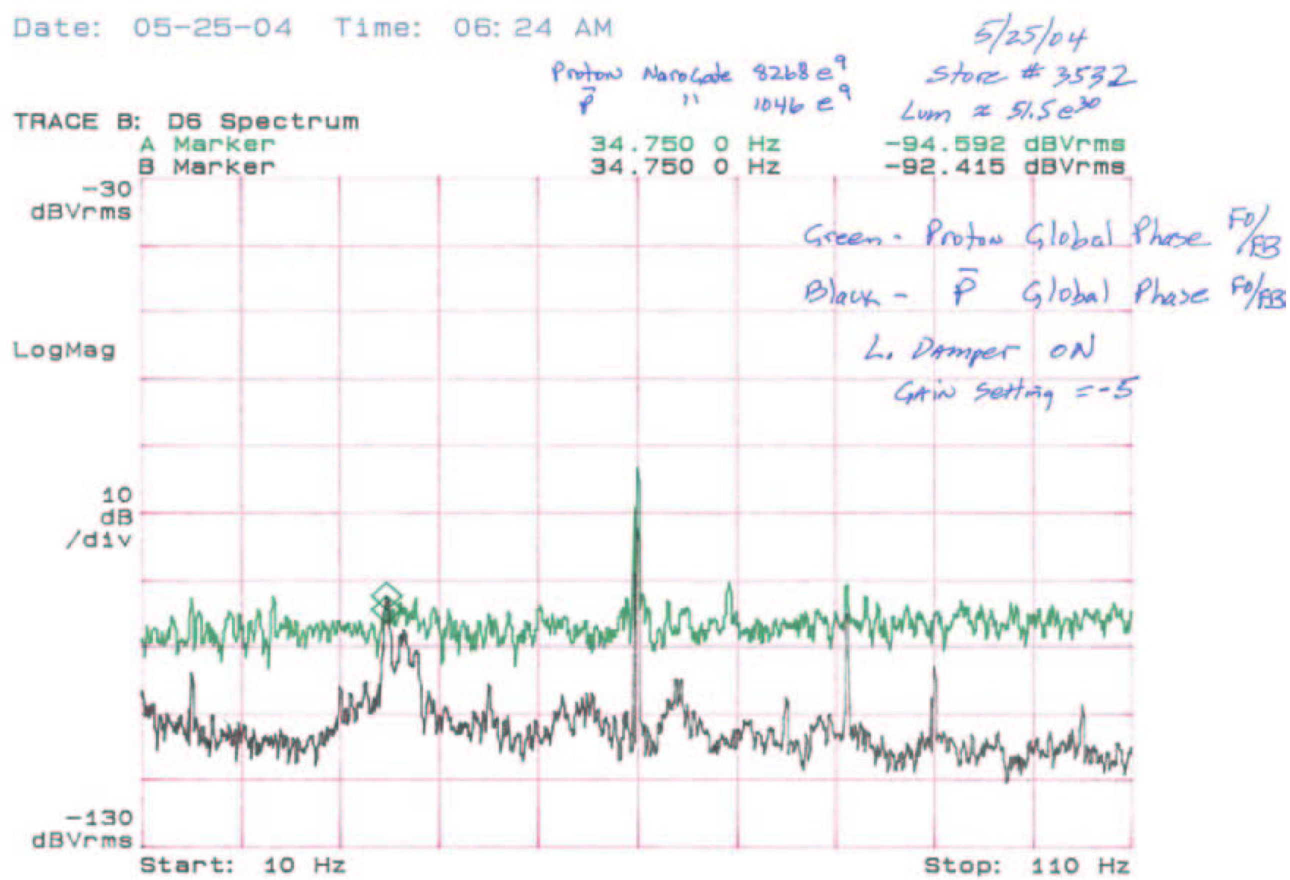
where $\quad C_\infty(I)=\sum_{n=0}^{\infty}C_n(I)$



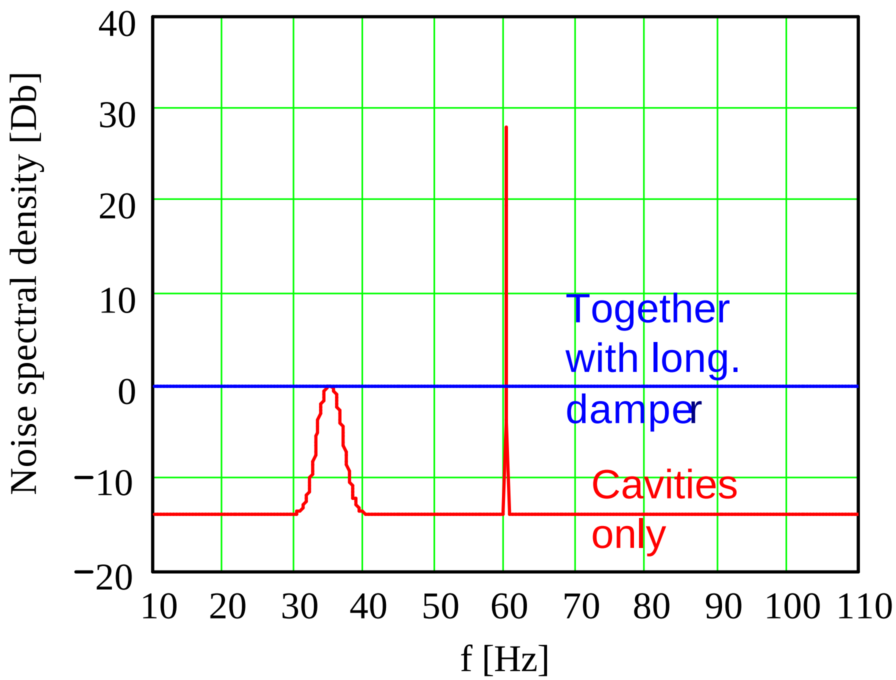
For all even n, $\quad C_n(I)=0$

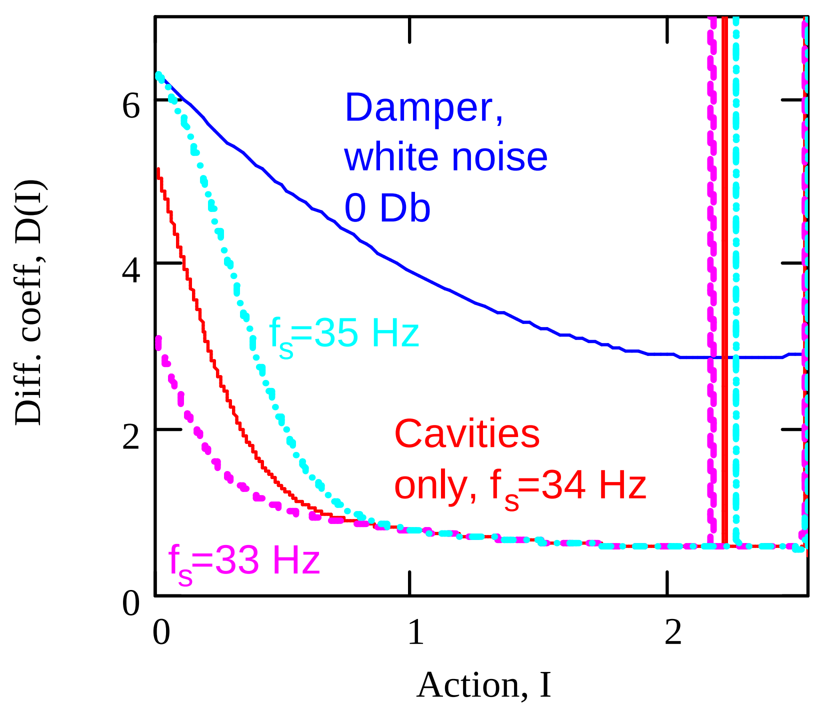
Direct measurement of RF noise performed by John Reid

- Microphonics - cavity mechanical resonances are at synchrotron frequency
  - Phase feedback suppresses microphonics by more than 20 Db
- Longitudinal damper is too noisy
  - Damper “white” noise hides mechanical resonances



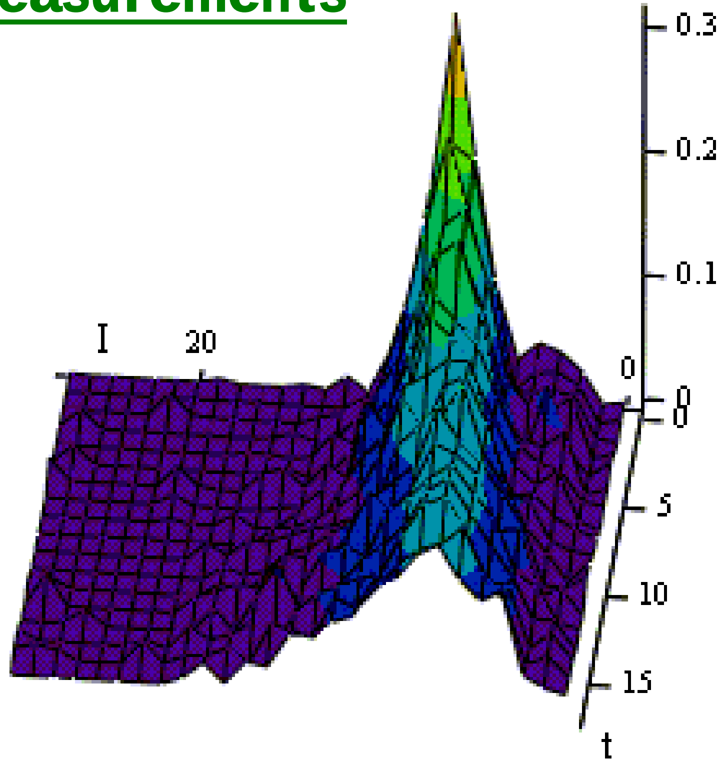
Dependence of Diffusion on the Action

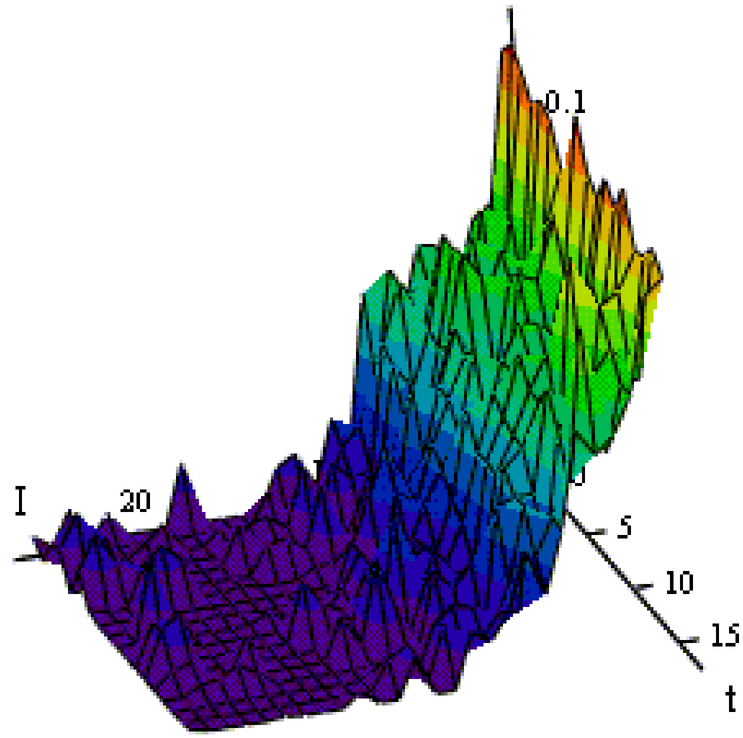




- Small changes of Synchrotron frequency (RF voltage) can significantly change diffusion coefficient if longitudinal damper is off
- The only detailed experimental data we have are for the case when the damper is on

Measurements

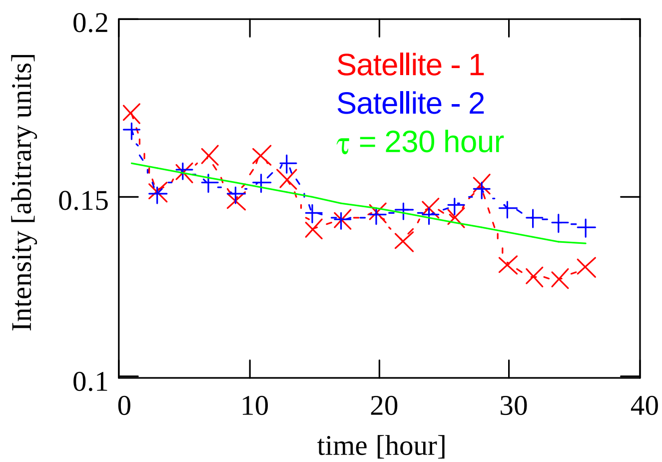


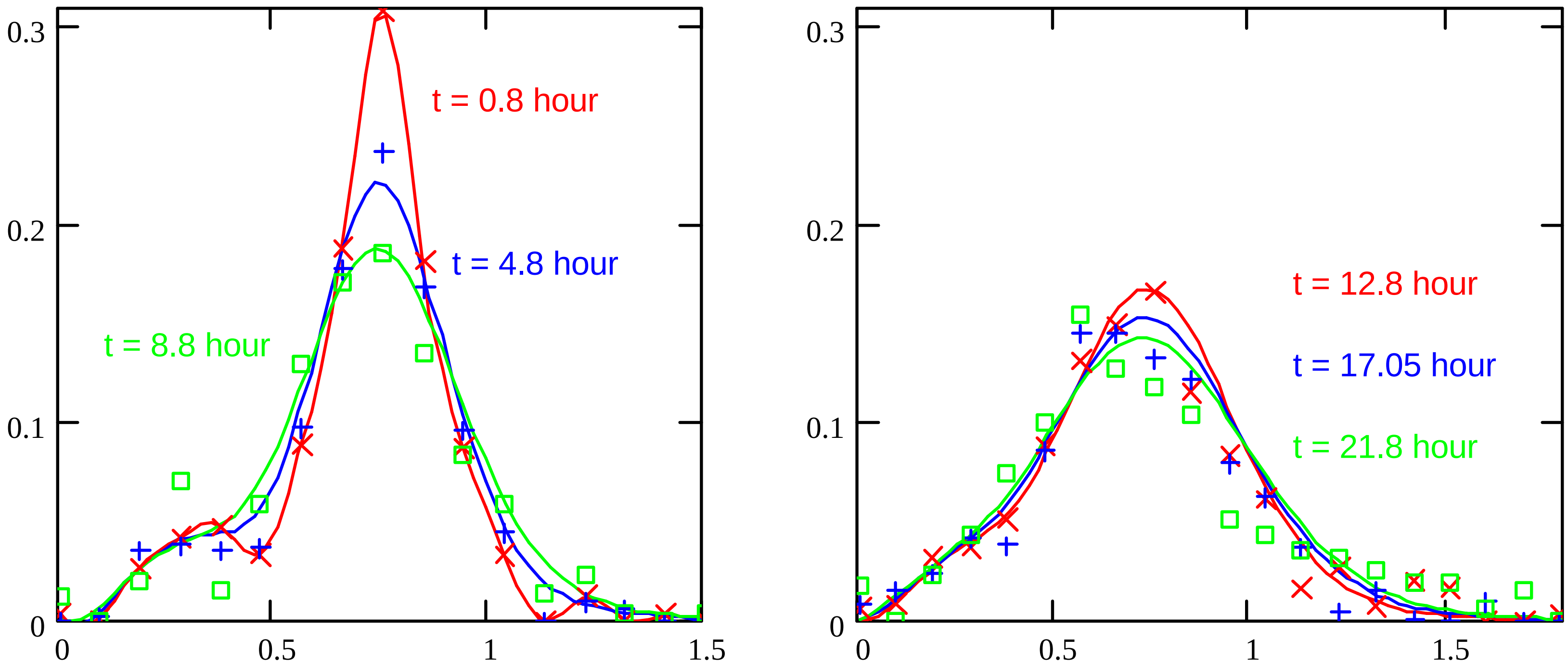


Development of distribution function on time for leading satellites: (-2) – left, and (-1) – right. Time scale [0, 15] corresponds to 37 hours of store time

- Both distributions are corrected for the satellite lifetime of 230 hours
- Longitudinal damper is on –

  Spectral density of RF phase noise is close to the white noise.





Measured and computed distribution functions for satellite (-2) of Store 3678;

RF noise spectral density - 42$\cdot$*10*$^{-12}$ rad^2/Hz, growth rate $d\overline{f^2}/dt = 1.87\cdot10^{-3}$ rad^2/hour

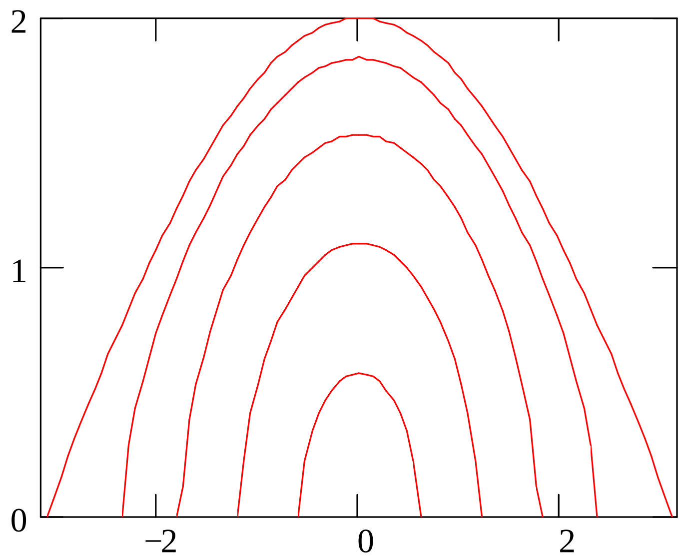
Previous estimate - *50*$\cdot$*10*$^{-12}$ rad^2/Hz *(DoE June 2003 Review)*

Beam lifetime is 230 hour versus >360 of vacuum lifetime

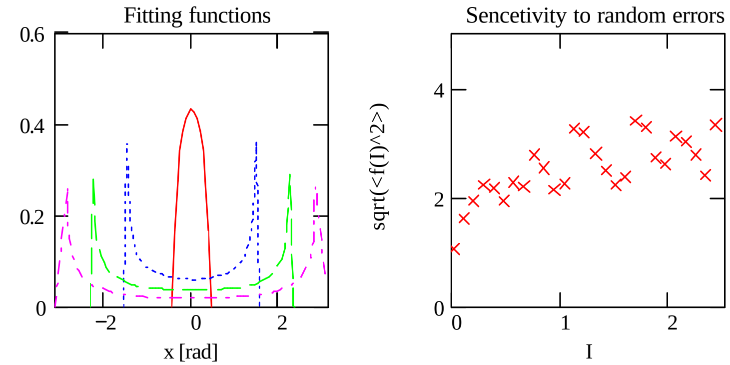
- Diffusion at small amplitudes is described well
  - That allows estimate the noise spectral density with better than 20% accuracy
- Diffusion at large amplitudes is not described well: the peak of distribution function is moved in for the measured distribution but not for computed one
- Beam-beam effects kill particles with large synchrotron amplitudes which, consequently, limits RF bucket size

Restoration of longitudinal distribution from signal of resistive wall monitor

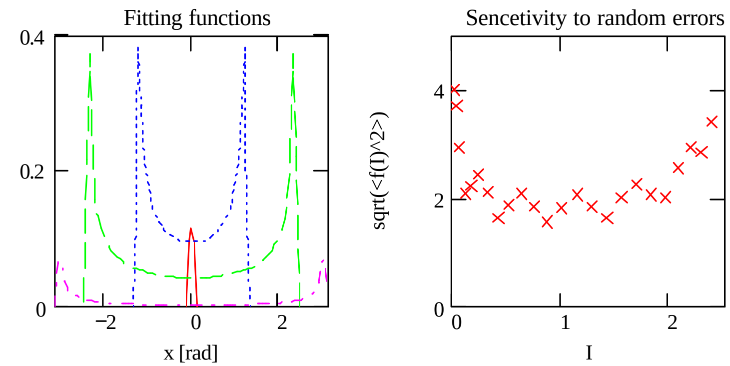
- The method is suggested by Alvin Tollestrup



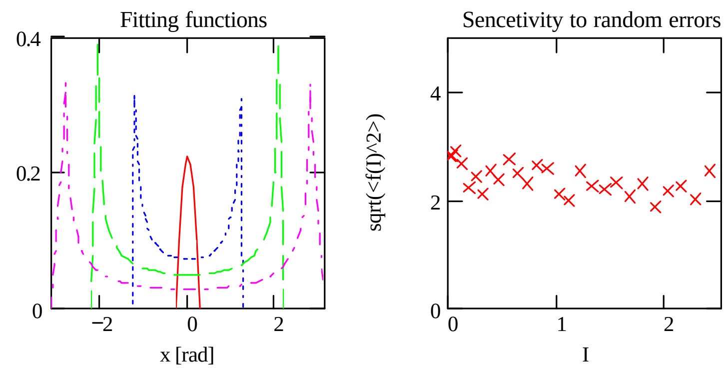
- Further improvements
  - Optimized binning
  - Constrained fit ($f(I)$>0)
  - Fitting for the baseline



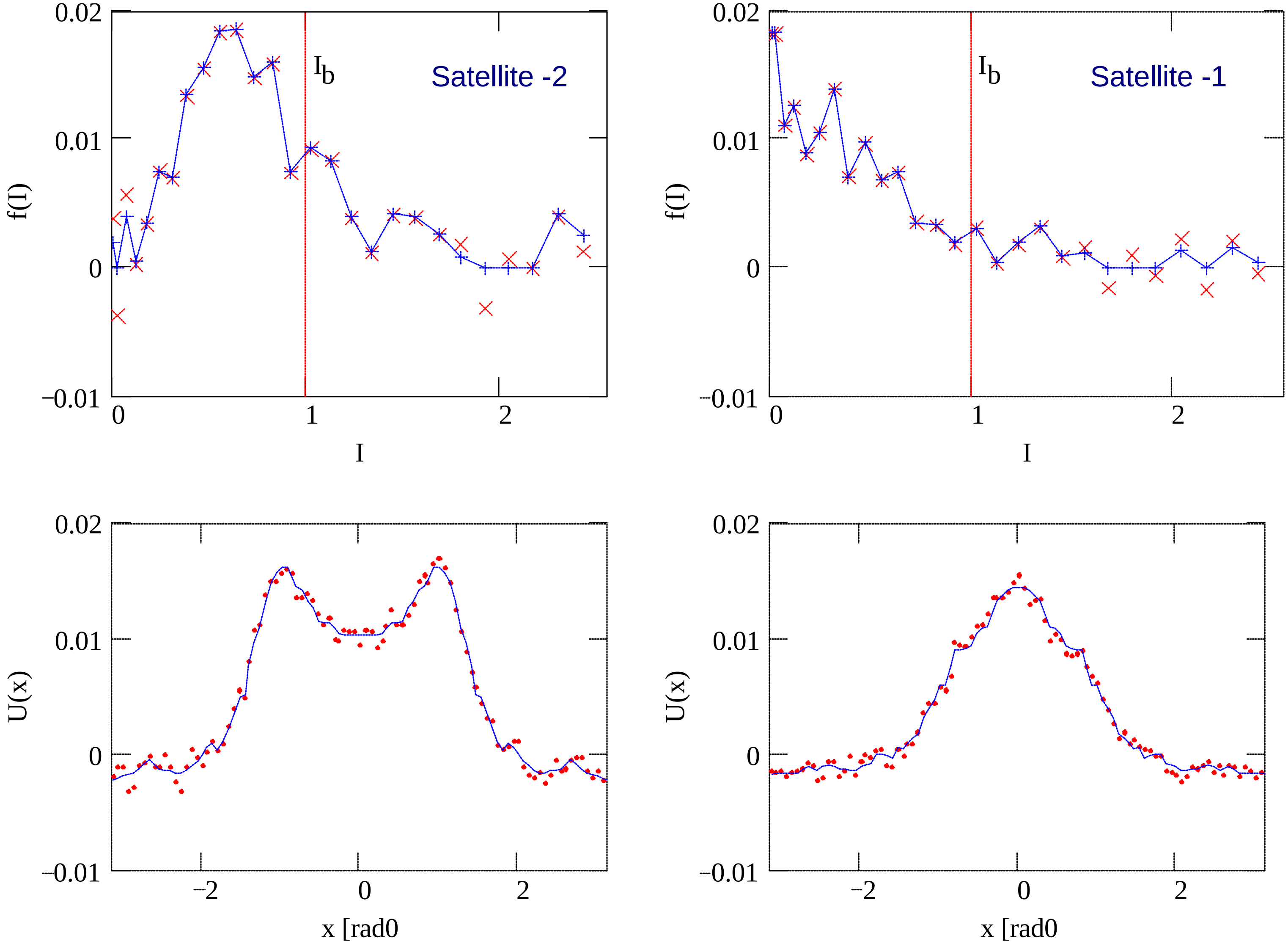
Equal ΔI binning



Equal Δt binning



Optimal binning $I_j = I_{max} (j/N)^{1.4}$



Top – Distr. functions for satellites (-1) and (-2) with constraint and linear fits

Bottom – results of constraint fit for satellites (-1) and (-2)

3. IBS Growth Rates for Gaussian Beams

- Landau collision integral

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial p_i}(F_i f)=\frac{1}{2}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right)$$

$$F_i(p)=-\frac{4pne^4L_c}{m}\int f(p')\frac{u_i}{|\mathbf{u}|^3}d^3p' \quad , \quad \mathbf{u}=\mathbf{v}-\mathbf{v}'$$

$$D_{ij}=4pne^4L_c\int f(p')\frac{u^2d_{ij}-u_iu_j}{|\mathbf{u}|^3}d^3p'$$

- Integration with Gaussian distrib. in all 3 degrees of freedom yields

$$\frac{d}{dt}\overline{\mathrm{v}_x^{\ 2}}=\frac{2p^{3/2}ne^4L}{m\sqrt{\overline{\mathrm{v}_x^{\ 2}}+\overline{\mathrm{v}_y^{\ 2}}+\overline{\mathrm{v}_z^{\ 2}}}}\Psi\left(\sqrt{\overline{\mathrm{v}_x^{\ 2}}},\sqrt{\overline{\mathrm{v}_y^{\ 2}}},\sqrt{\overline{\mathrm{v}_z^{\ 2}}}\right)$$

  - Equations for other degrees of freedom are obtained by cyclic substitution
    Here:

$$\Psi(x,y,z)=\frac{\sqrt{x^2+y^2+z^2}}{p}\int_0^\infty\frac{\sqrt{2}\,t^3dt}{\left(x^2+t^2\right)^{3/2}\left(y^2+t^2\right)^{1/2}\left(z^2+t^2\right)^{1/2}}\left(\frac{y^2-x^2}{y^2+t^2}+\frac{z^2-x^2}{z^2+t^2}\right)$$

- Energy conservation requires: $\Psi(x,y,z)+\Psi(y,z,x)+\Psi(z,x,y)=0$

- The function $\Psi(x,y,z)$ does not depend on $x^2+y^2+z^2$ and therefore for further analysis we choose $x^2+y^2+z^2=1$
- The function $\Psi(x,y,z)$ is determined so that

$$\Psi(0,x,x)=1 \qquad \Psi(x,x,0)=\Psi(x,0,x)=-1/2 \qquad \Psi(x,x,x)=0 \qquad (1)$$

- When two parameters coincide the integral can be computed

$$\Psi\left(x,\sqrt{\frac{1-x^2}{2}},\sqrt{\frac{1-x^2}{2}}\right)=2\hat{\Psi}(x)$$

$$\Psi\left(\sqrt{\frac{1-z^2}{2}},\sqrt{\frac{1-z^2}{2}},z\right)=-\hat{\Psi}(z)$$

$$\hat{\Psi}(x)=\frac{1}{\sqrt{2p}}\frac{1}{3x^2-1}\left(\frac{1}{\sqrt{2}}\frac{3x^2+1}{\sqrt{3x^2-1}}\ln\left(\frac{\sqrt{2}x-\sqrt{3x^2-1}}{\sqrt{2}x+\sqrt{3x^2-1}}\right)+6x\right)$$

- For practical applications the function $\Psi(x,y,z)$ can be approximated as

$$\Psi(x,y,z)\approx\frac{\sqrt{2}}{p}\ln\left(\frac{y^2+z^2}{\sqrt{x^2+y^2}\sqrt{x^2+z^2}}\right)+0.688\frac{(x-y)(x-z)}{\sqrt{x^2+y^2}\sqrt{x^2+z^2}}-0.055\left(y^2-z^2\right)^2-$$

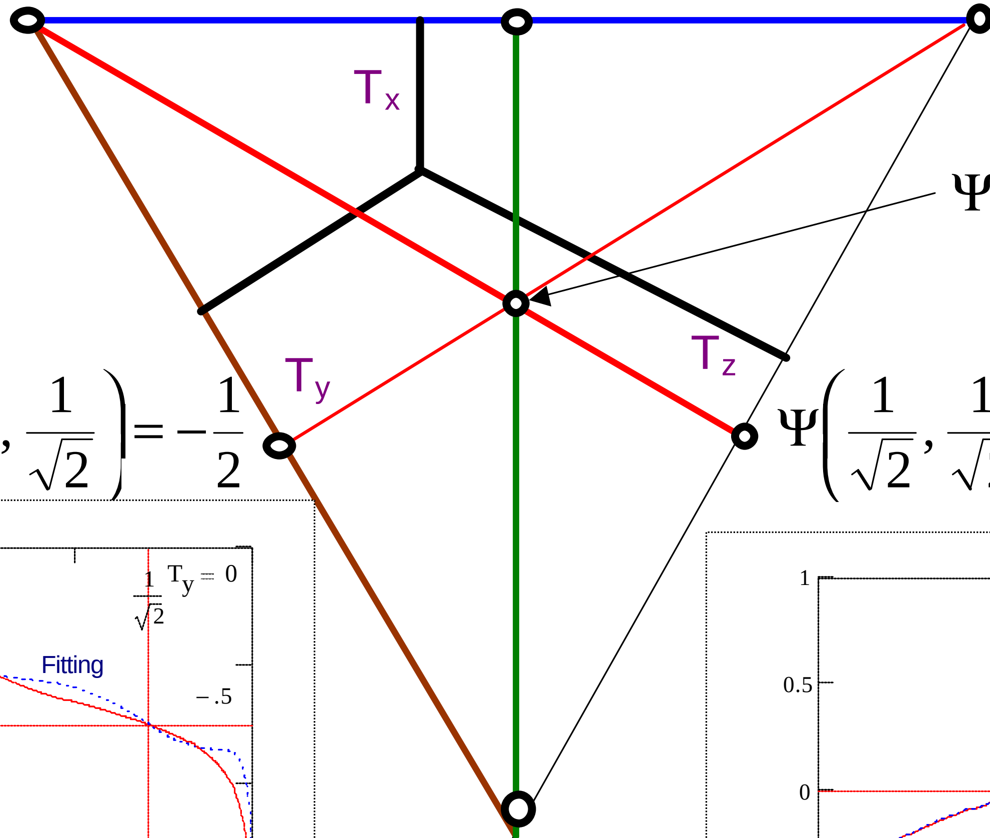
$$7.47x^6|y-z|^3-0.8x\left(1-3x^2\right)^3 J\left(1-3x^2\right) \quad , \qquad J(x)=\begin{cases}1, & x\ge 0\\ 0, & x<0\end{cases}$$

  - This function has correct asymptotics, satisfy conditions (1), and coincide with exact expression within
  - ~1% for x=0, ~10% for entire range of parameters

Dalitz plot for temperatures

$$T_x = x^2,\ T_y = y^2,\ T_z = z^2,$$

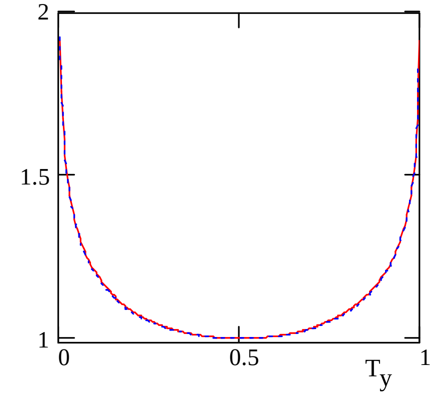
$$\Psi \propto -\ln(x^2+z^2) \qquad \Psi\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 \qquad \Psi \propto -\ln(x^2+z^2)$$



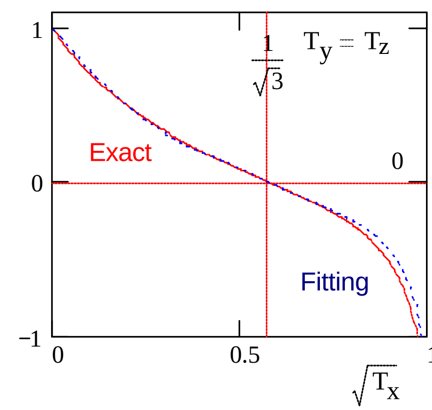
$$\Psi\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 0$$

$$\Psi\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \qquad \Psi\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{2}$$

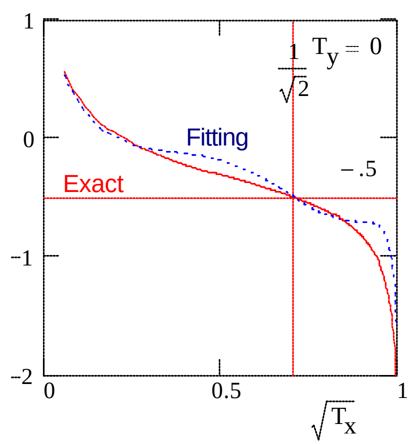
$$\Psi \propto \ln(y^2+z^2)$$



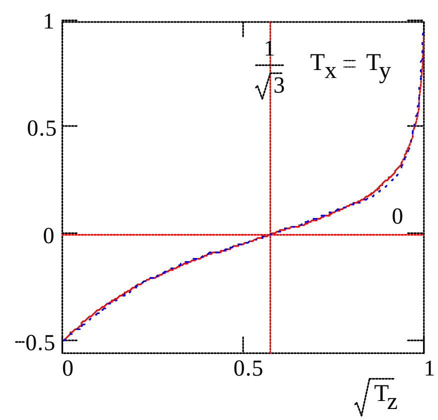
$T_x = 0$ (Along blue line)



$T_y = T_z$ (Along green line)



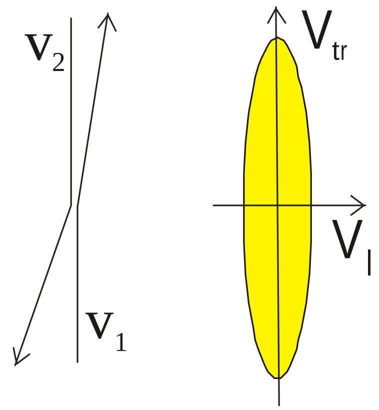
$T_y = 0$ (Along brown line)



(Along red line)

Accelerator specific corrections

- **General Recipe**
  - Find local density and velocity spreads and compute average temperature growth across the beam cross-section. Then average along beam orbit.
    - Take into account that axis of 3D ellipsoid of velocities not necessarily coincide with local coordinate frame axis
    - Take into account additional excitation of transverse degrees of freedom due to non-zero dispersions



- **Details**
  - For Tevatron significant simplifications are due to
    - $v_\parallel << v_x, v_y$ in the beam frame
    - $g >> Q_x, Q_y$
  - After averaging over the bunch length and the cross-section

$$\frac{d}{dt}\left(s_p^{\;2}\right) \equiv \left\langle \frac{d}{dt}\left(\overline{\frac{\Delta p_\parallel^{\;2}}{p}}\right)\right\rangle_s = \frac{1}{4\sqrt{2}}\frac{N r_p^{\;2} c}{g^3 b^3}\left\langle \frac{1}{s_1 s_2 s_s}\frac{\Psi\left(s_p/g, q_1, q_2\right)}{\sqrt{q_1^{\;2}+q_2^{\;2}+\left(s_p/g\right)^2}} L_C \right\rangle_s$$

- Here q_1 and q_2 are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and s_1 and s_2 are ellipse semi-axis in the x-y plane

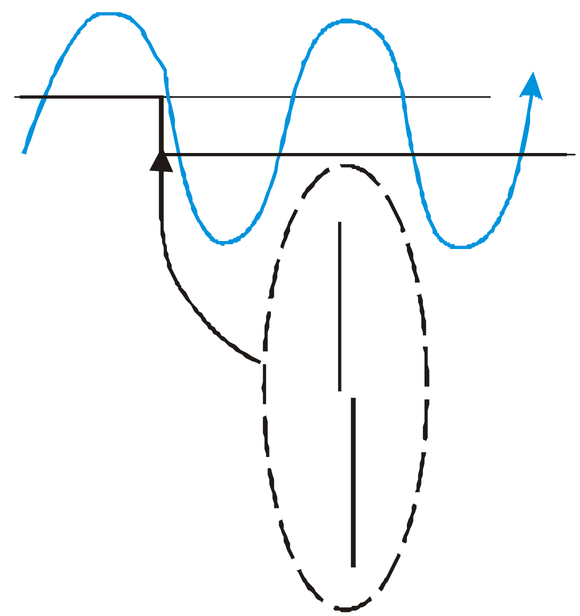
➢ Uncoupled motion

$$s_1 \equiv s_x = \sqrt{e_x b_y + D^2 s_p^{\ 2}}, \qquad s_2 \equiv s_y = \sqrt{e_y b_y},$$

$$q_1 \equiv q_x = \sqrt{\frac{e_x}{b_x}\left(1 + \frac{(D' b_x + a_x D)^2 s_p^{\ 2}}{e_x b_x + D^2 s_p^{\ 2}}\right)}, \qquad q_2 \equiv q_y = \sqrt{e_y / b_y}$$

- Additional transverse emittance growth due to finite dispersion dominates emittance change due to “direct” scattering

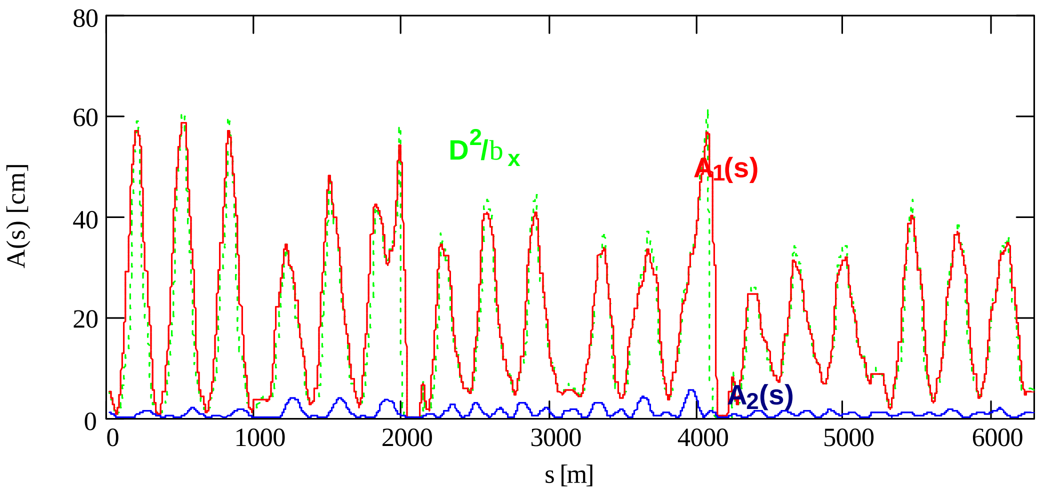
$$\frac{de_x}{dt} = \left\langle \frac{D^2 + (D' b_x + a_x D)^2}{b_x} \frac{d}{dt}\left(\frac{\overline{\Delta p_{\parallel}^{\ 2}}}{p}\right)\right\rangle_s$$

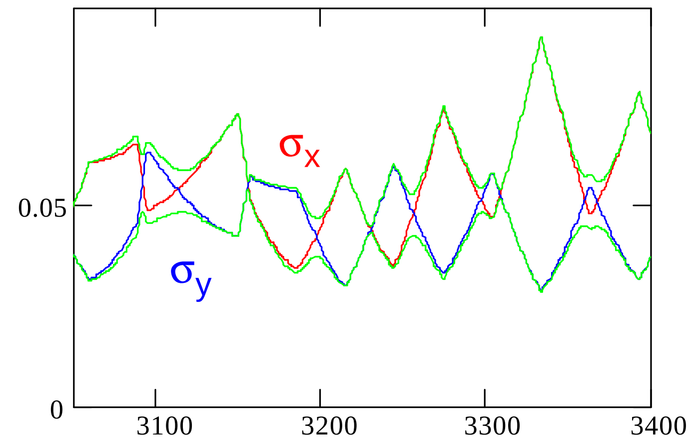


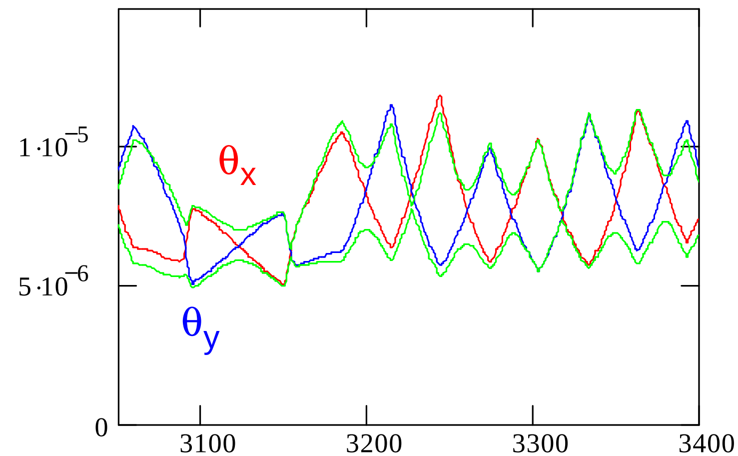
X-Y coupled motion

- Measured Tevatron optics with coupling has been used in calculations!!!
- Coupling effects are sufficiently small
  - little corrections for density and angular spread
- Emittance growth related to mode 2 (y mode) is about 5% of mode 1 (x-mode)

$$\frac{de_1}{dt} = \left\langle A_1 \frac{d}{dt}\left(\frac{\overline{\Delta p_\parallel^2}}{p}\right)\right\rangle_s , \quad \frac{de_2}{dt} = \left\langle A_2 \frac{d}{dt}\left(\frac{\overline{\Delta p_\parallel^2}}{p}\right)\right\rangle_s$$







Top - Beam size projections and ellipsoid semi-axis

Bottom – projections for angular spreads and ellipsoid semi-axis

- In measurements both modes contribute to the beam sizes

$$s_x^{\,2} = b_{1x}e_1 + b_{2x}e_2$$

$$s_y^{\,2} = b_{1y}e_1 + b_{2y}e_2$$

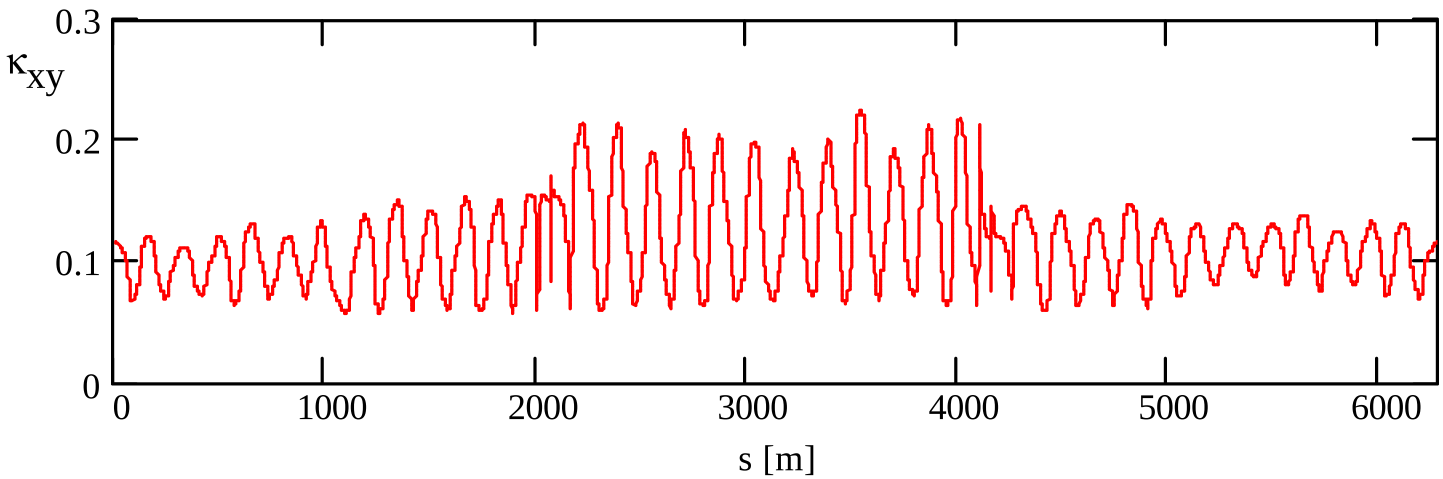
- That yields for observed emittance growth

$$\frac{de_x}{dt} = \frac{de_1}{dt} + \frac{b_{2x}}{b_{1x}}\frac{de_2}{dt}$$

$$\frac{de_y}{dt} = \frac{b_{1y}}{b_{2y}}\frac{de_1}{dt} + \frac{de_2}{dt}$$

- That leads to an increase of observed coupling
  - $k_{xy} \approx 0.1$ at Synchrotron light emittance monitor

$$k_{xy} \equiv \frac{de_y/dt}{de_x/dt + de_y/dt}$$



4. IBS in Non-linear Longitudinal Well

Diffusion equation

- In the case $v_{\parallel} << v_x, v_y$ the friction in Landau collision integral can be neglected
- Diffusion equation

$$\boxed{D(I) = \oint D(p)\,p\,dq \Big/ \oint p\,dq}$$

1D: $\dfrac{\partial f}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial p}\left(D(p)\dfrac{\partial f}{\partial p}\right)$ $\Rightarrow$ 2D: $\boxed{\dfrac{\partial f}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial I}\left(I\,\dfrac{D(I)}{w(I)}\dfrac{\partial f}{\partial I}\right)}$

  - *I* is the action and w is the frequency for dimensionless Hamiltonian of synchrotron motion: $H = \dfrac{p^2}{2} + 2\left(\sin\dfrac{j}{2}\right)^2$
  - Diffusion coefficient depends on distribution, (*I*)

$$\boxed{D(I) = 4L_c\tilde{A}\left(\oint n(j)\,p\,dj \Big/ \oint p\,dj\right)} \qquad \tilde{A} = p^2\sqrt{\frac{p}{2}}\frac{\left(a - 1/g^2\right)e^4 q^2}{eV_0 m_p c b g^2 C}\left\langle \frac{N}{s_1 s_2}\frac{\Psi\left(s_p/g, q_1, q_2\right)}{\sqrt{q_1^2 + q_2^2 + \left(s_p/g\right)^2}}\right\rangle_s$$

Here: $n(j) = \int f\left(I(p,j)\right)dp$, $\displaystyle\int_{-p}^{p} n(j)\,dj = 1$

a	- momentum compaction,		q	- harmonic number
V_0	- RF voltage,		C	- ring circumference

Simultaneous treatment of single and multiple scattering

- Boltzmann type equation
  - In the case $v_{\parallel} << v_{\perp}$ one can write for Coulomb scattering in long. direction

$$\frac{\partial f}{\partial t} = \left\langle \tilde{A}\int n(j)\,\frac{f(p+q) - f(p)}{|q|^3}\,\mathrm{d}q \right\rangle_{period} = \left\langle \tilde{A}\int n(j)\,\frac{f(I') - f(I)}{|p-p'|^3}\,d(j - j')\,\mathrm{d}I'\,dy\,dy' \right\rangle_{period}$$

- After simplification we obtain

$$\boxed{\frac{\partial f(I,t)}{\partial t} = \tilde{A}\int_0^{\infty} W(I,I')\big(f(I',t) - f(I,t)\big)\mathrm{d}I'}$$

$$W(I,I') = \frac{2ww'}{p}\int_0^{\min(a,a')}\frac{dj}{pp'}\,n(j)\left[\frac{1}{|p-p'|^3} + \frac{1}{|p+p'|^3}\right] \xrightarrow{E'\geq E}$$

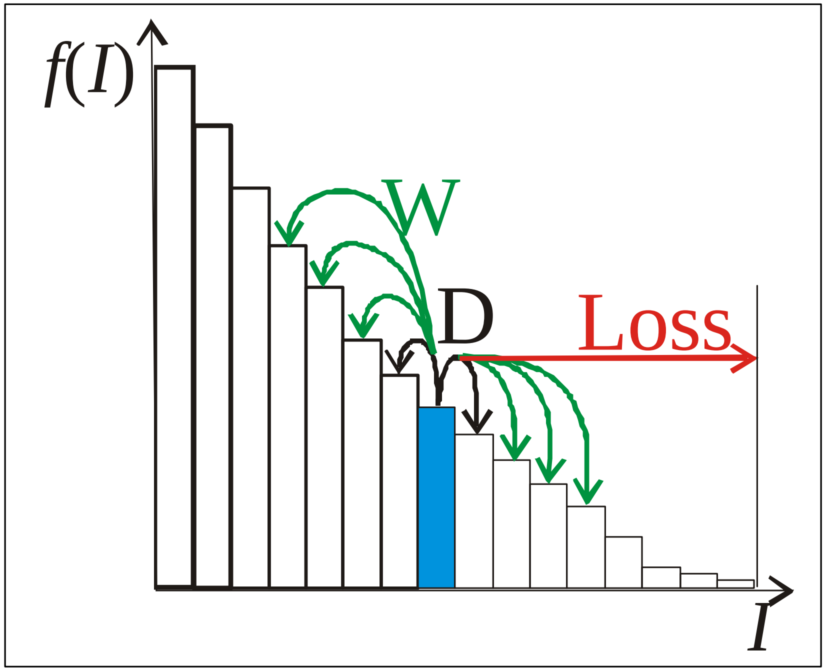
$$\frac{ww'}{p(E-E')^3}\left[(E-E')\int_0^a n(j)\frac{dx}{p} + 2\int_0^a n(x)\,p\,dx\right] \quad .$$

$a \equiv a(I)$ is the motion amplitude

- The kernel is symmetric: $W(I,I') = W(I',I)$,
- The kernel divergence needs to be limited at the minimum action change corresponding to the maximum impact parameter

Numerical model

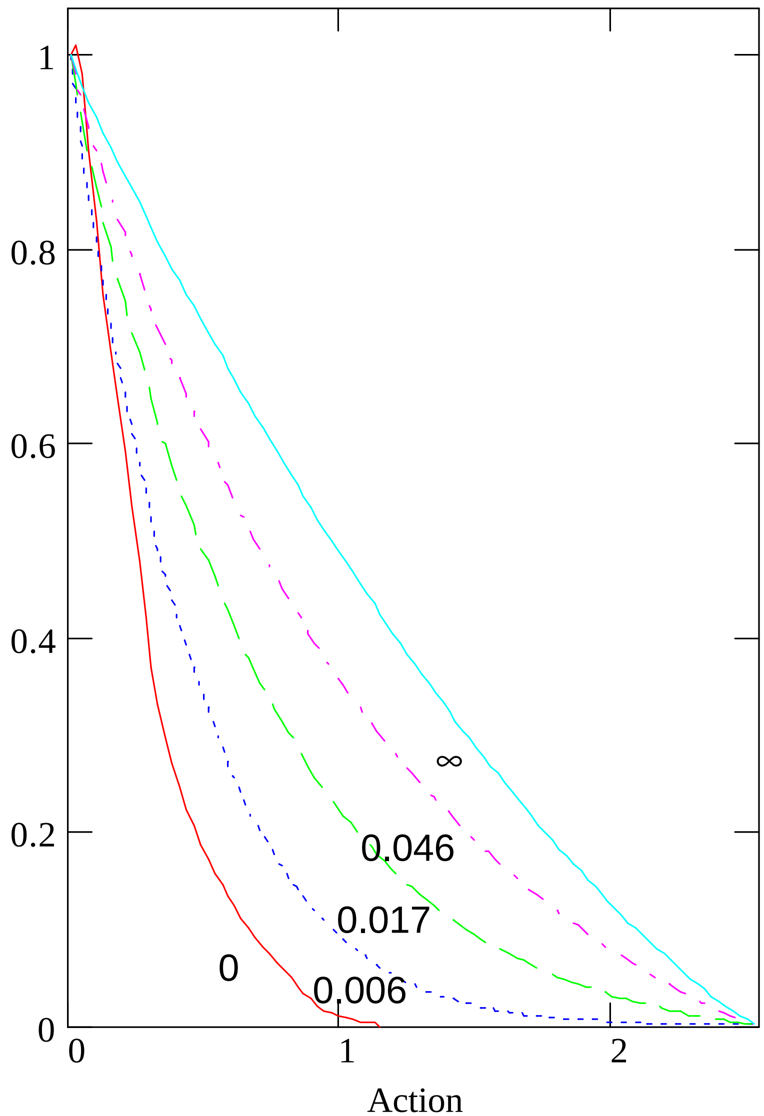
- Set of bins
  - Transition probabilities
    - Nearby bins – diffusion equation to resolve divergence of $W(I, I')$
    - Far away bins – transition probabilities are described by $W(I, I')$
    - Particle loss outside bucket need to be added

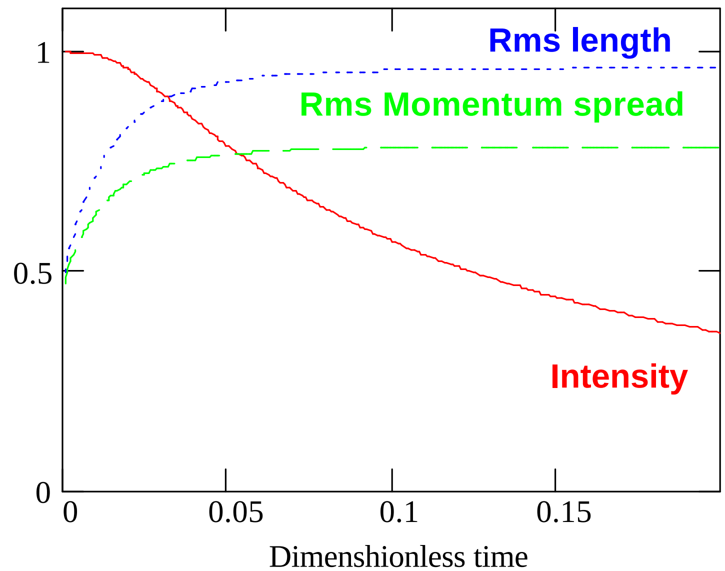


- In matrix form

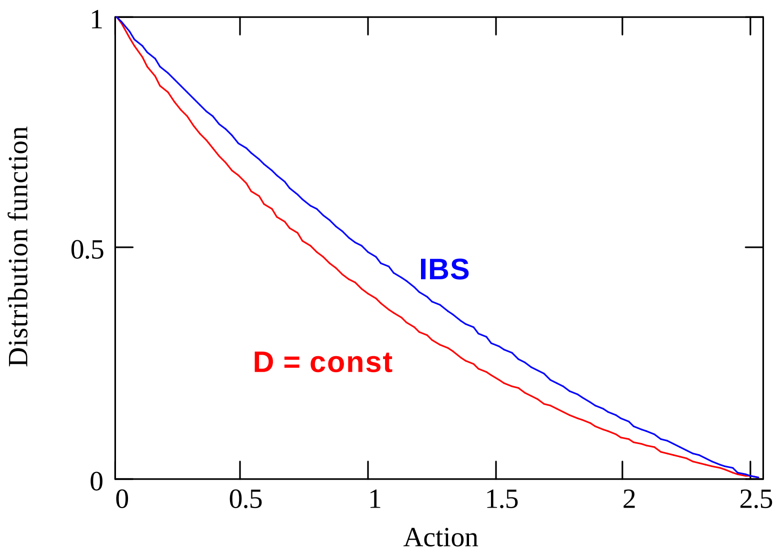
$$\mathbf{f}_{n+1} = \mathbf{f}_n + \mathbf{W}\mathbf{f}_n \Delta t$$

W – is matrix of transition probabilities. It is a symmetric matrix





Dependencies: $N(t)/N_0, \sqrt{\overline{j^2(t)}}, \sqrt{\overline{p^2(t)}}$



Dependence of longitudinal distribution on time for IBS. Measured initial distribution is used

Asymptotic distributions

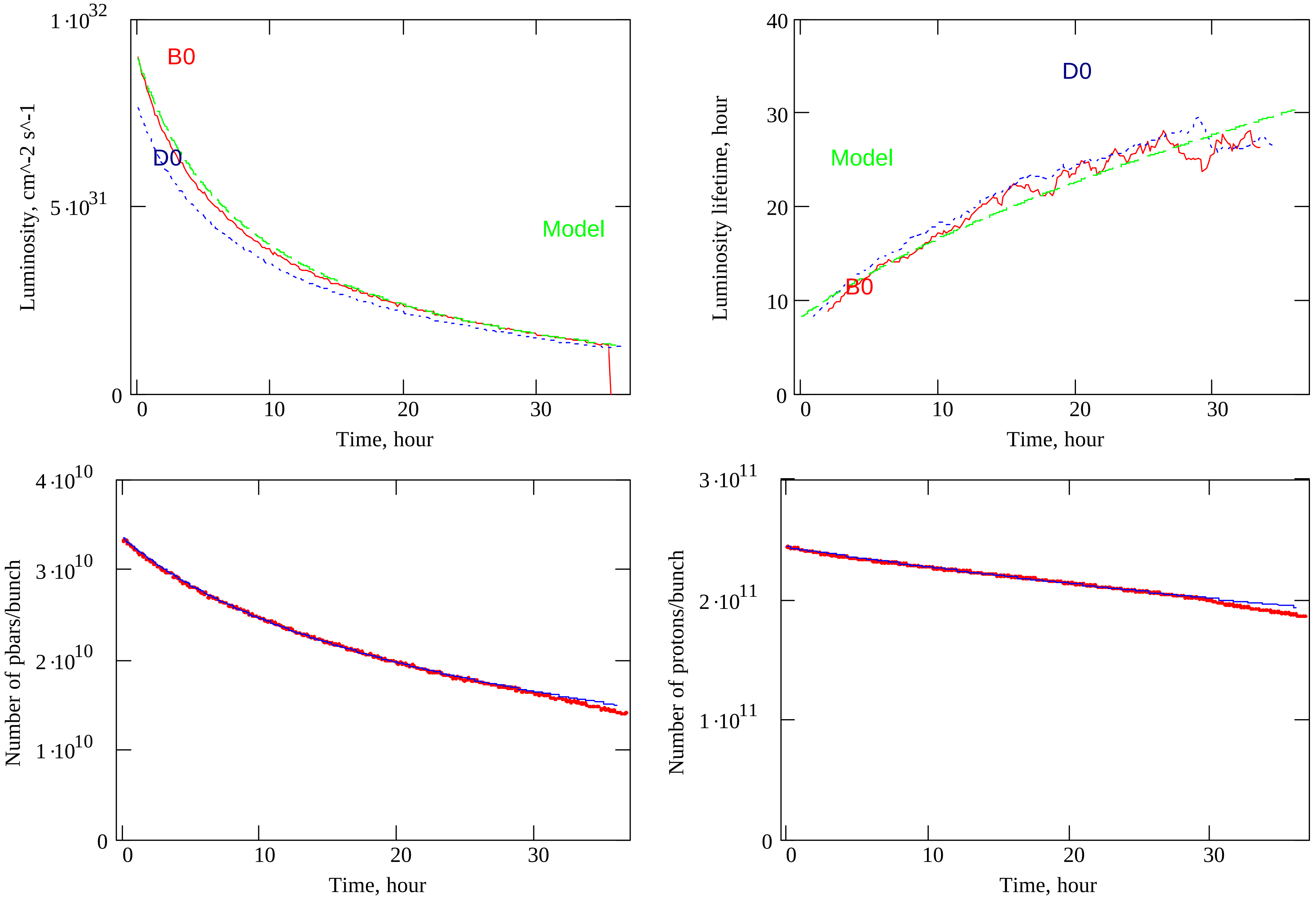
5. Comparison with experiment for Store 3678

Fitting parameters

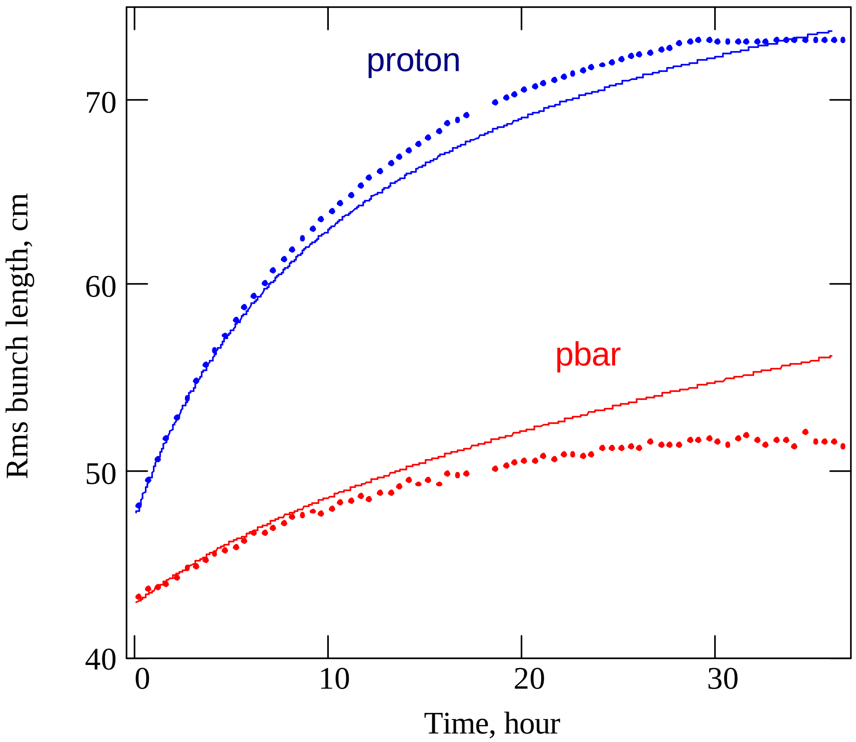
- Effective vacuum – $1.2 \cdot 10^{-9}$ Torr of N_2 equivalent
  - For chosen gas composition it yields
    - Beam lifetime - 360 hour
    - Emittance growth – 0.21 mm mrad/hour
  - Beam lifetime at low intensity more than 600 hour
    - Vacuum worsening with beam intensity is possible reason
    - Neglected beam-beam effects can be another reason
- p – pbar cross-section for particle loss - 69 mbarn
- Coupling parameters: k_p = 0.37, k_{pbar} = 0.32
- Initial measured transverse emittances were multiplied by 1.01 to obtain correct initial luminosity
- **Amplification of IBS diffusion for pbars – 2.4**
  - Beam-beam effects
  - Noise in magnets
    - bunch motion at betatron sideband ~0.1 μm was observed and is consistent with observed increase in diffusion

Additional constraints

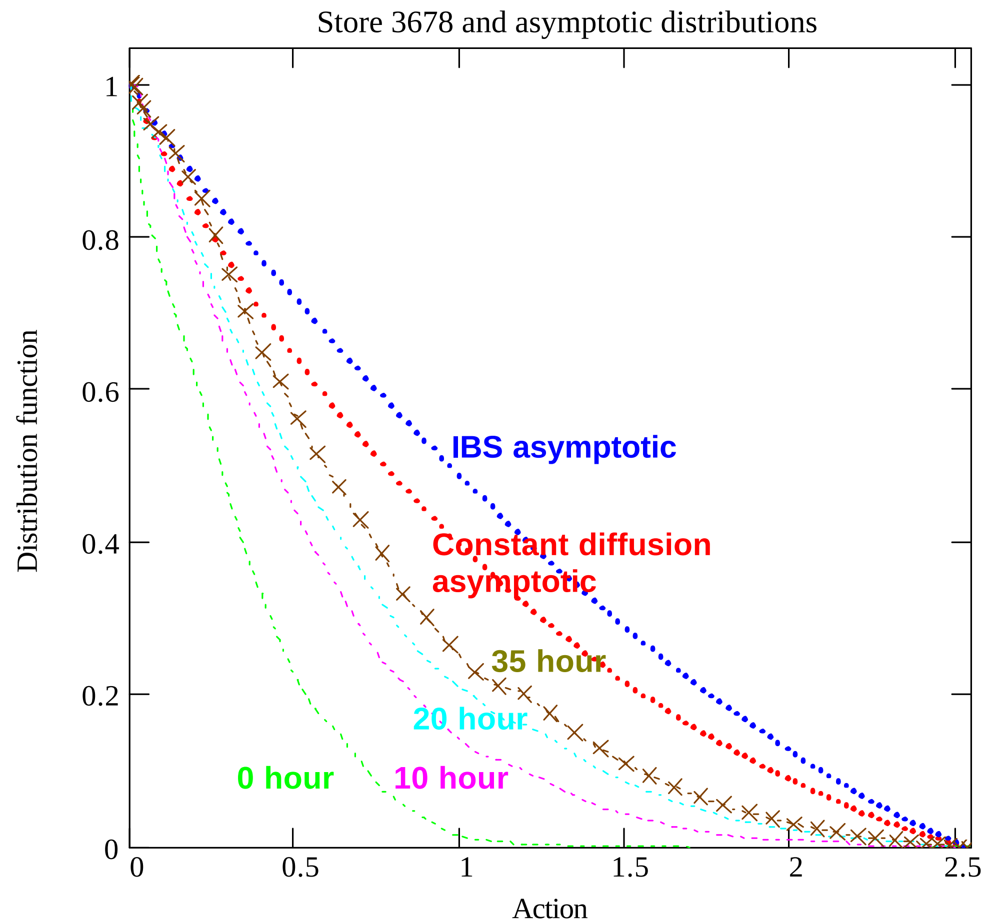
- RF noise was measured for protons by observation of evolution for longitudinal distribution (Damper is on)
  - The same noise was used for pbar beam
  - Suppression of longitudinal emittance growth by longitudinal damper was neglected. For protons it is small contribution anyway.
- Beam-beam effects are completely neglected
  - Increased transverse diffusion
  - Limitation of longitudinal acceptance
    - Well-observed for pbars
    - Less pronounced for protons
- Aperture limitation due to scrapers
- D0 luminosity was taken to be equal to CDF luminosity
  - All measurements we presently have point out that the actual difference should be smaller than the difference between luminosities reported by CDF and D0



Luminosity and beam intensities evolution for Store 3678

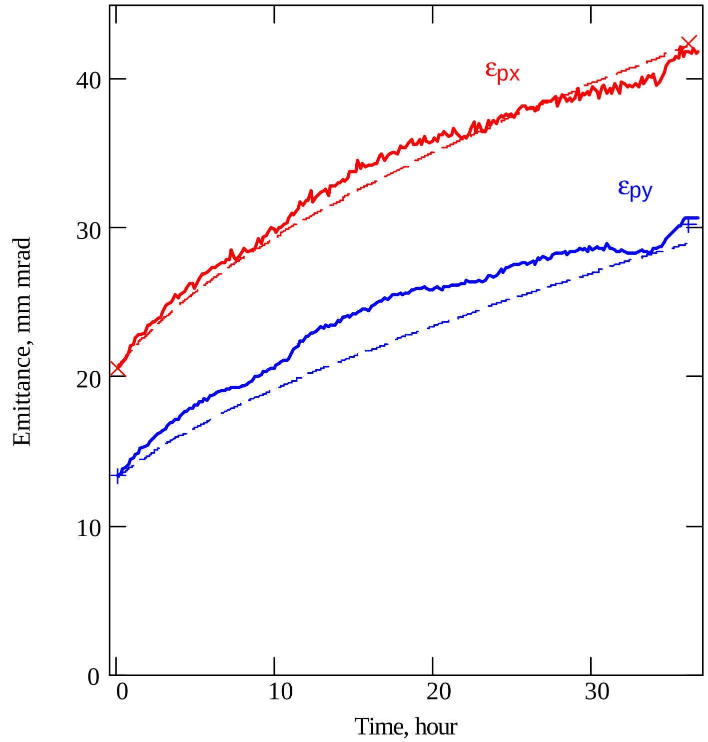


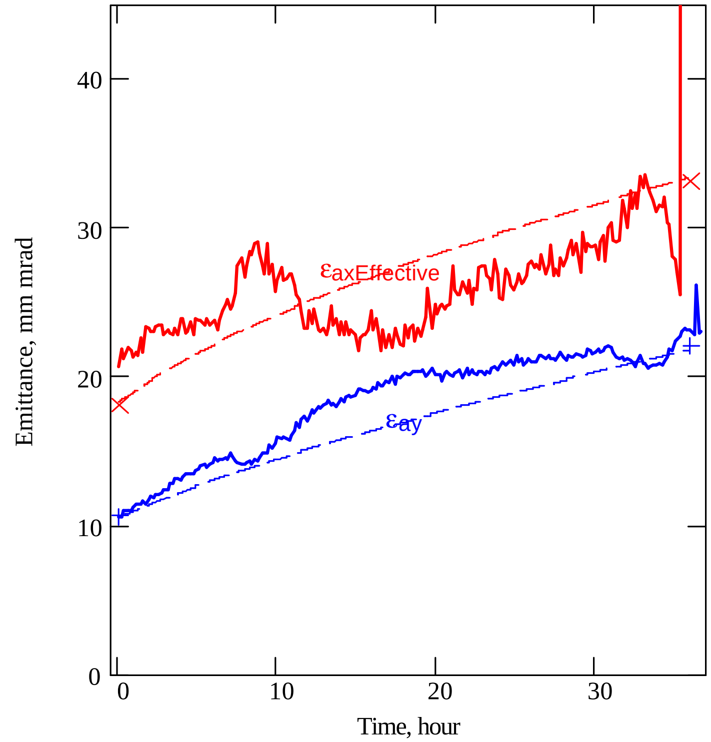
Dependence of computed and measured bunch length on time for Store 3678



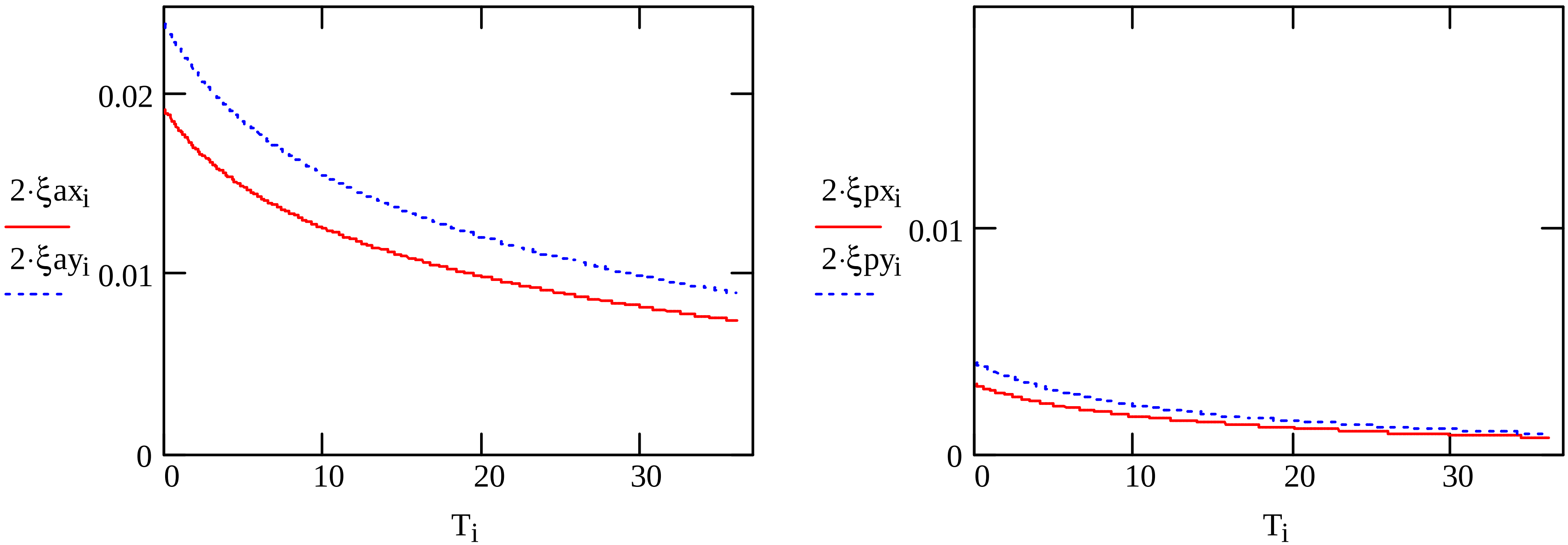
Evolution of proton distribution for Store 3678

- Good coincidence for proton bunch lengthening.. It is not affected by choice of free parameters
- $\bar{p}$ lengthening is affected by RF noise (~20% of IBS): $S_f=42\cdot10^{12}$ rad^{2}s, $d\sigma_\phi{}^2/dt|_{t=0}$= 0.187 rad^2/hour
  - Sextupole power supply was lost at t=28 hour => beam-beam effects



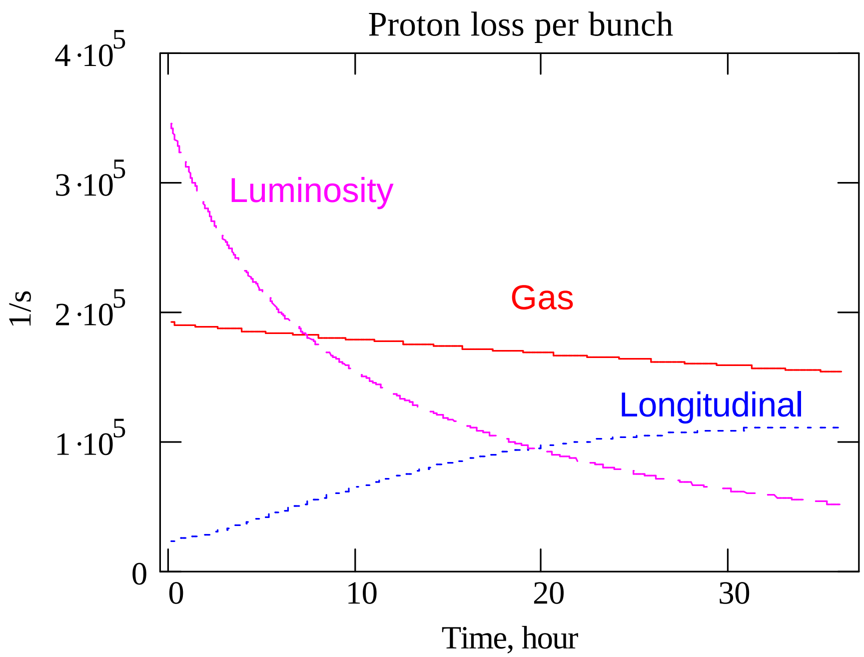


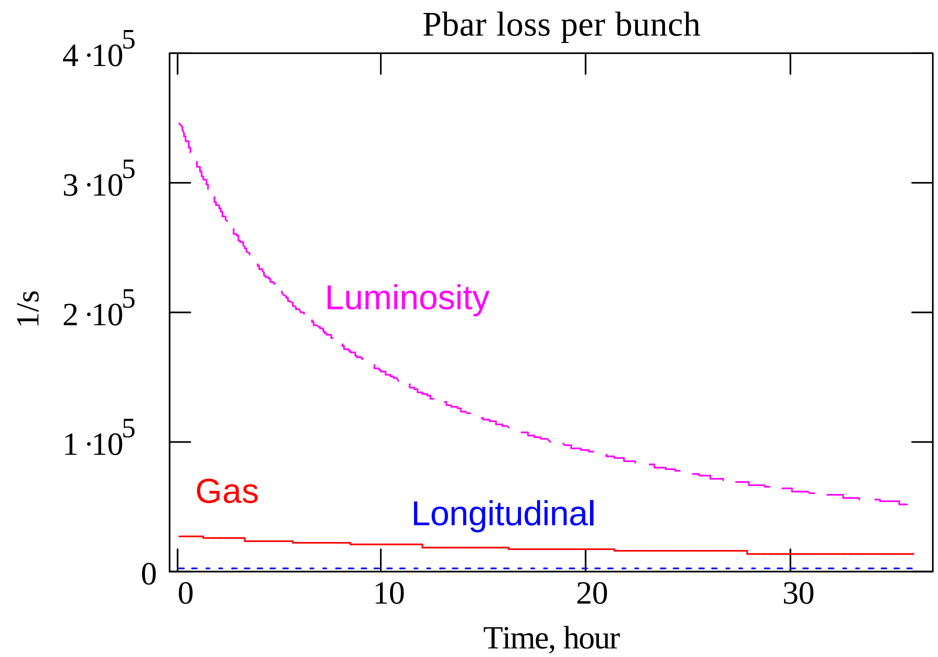
- Both proton and pbar emittance growths have contributions from scattering at the residual gas. At the store beginning it gives:
  - 12% of IBS for protons
  - 60% of IBS for pbars
  - Gas pressure was set by matching particle loss due to nuclear scattering
- For pbars there is unaccounted emittance growth (2.4 times of IBS)
  - Beam-beam effects
  - Noise in magnets
  - Preliminary measurements of bunch motion at ω_b yields σ = 0.1 μm
    - That is consistent with the measurements



Computed linear tune shifts due to beam-beam effects

- Run II design value is achieved for pbars





Particle loss due to different mechanisms

- 33% of lost protons are lost due to luminosity
- 88% of lost pbars are lost due to luminosity
- 6.7% of total number of protons are lost due to luminosity
- 48% of total number of pbars are lost due to luminosity

Conclusions

- Theory describes well observed evolution of parameters for proton beam
- Observed discrepancy for antiprotons is related to other effects which are presently not taken into account
  - Noise in magnets and beam-beam effects are the most probable reasons
- To get such good agreement the improvements in theory as well as in experiment have been required
- The developed IBS theory was applied to the longitudinal emittance growth in Recycler. Good agreement has been obtained

Backup transparencies

1. Emittance Growth Rates for X-Y coupled motion in the case of pancake distribution

- **Growth rates for the momentum spread in the bunched beam**
  - After averaging over the bunch length and the cross-section

$$\frac{d}{dt}\left(s_p^{\;2}\right)\equiv\left\langle\frac{d}{dt}\left(\frac{\overline{\Delta p_\parallel^{\;2}}}{p}\right)\right\rangle_s=\frac{1}{4\sqrt{2}}\frac{r_p^{\;2}c}{g^{3}b^{3}}\left\langle\frac{N}{s_1 s_2 s_s}\frac{\Psi\left(s_p/g,q_1,q_2\right)}{\sqrt{q_1^{\;2}+q_2^{\;2}+\left(s_p/g\right)^2}}L_C\right\rangle_s$$

    - Here q_1 and q_2 are ellipse semi-axis in the plane of local angular spreads (x'-y' plane) and s_1 and s_2 are ellipse semi-axis in the x-y plane
    - This equation is still valid for arbitrary gaussian distribution (pancake distr. is not required)
    - For uncoupled beam

$$s_1\equiv s_x=\sqrt{e_x b_y+D^2 s_p^{\;2}}\,,\qquad s_2\equiv s_y=\sqrt{e_y b_y}\,,$$

$$q_1\equiv q_x=\sqrt{\frac{e_x}{b_x}\left(1+\frac{\left(D'b_x+a_x D\right)^2 s_p^{\;2}}{e_x b_x+D^2 s_p^{\;2}}\right)}\,,\qquad q_2\equiv q_y=\sqrt{e_y/b_y}$$

➢ Optics is described with Mais-Ripken beta-functions

- For coupled motion the eigen-vectors can be parameterized as

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\tilde{e}_1 \mathbf{v}_1(s) e^{-im_1(s)} + \tilde{e}_2 \mathbf{v}_2(s) e^{-im_2(s)}\right),$$

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{b_{1x}} \\ -\dfrac{i(1-u)+a_{1x}}{\sqrt{b_{1x}}} \\ \sqrt{b_{1y}}\, e^{in_1} \\ -\dfrac{iu+a_{1y}}{\sqrt{b_{1y}}} e^{in_1} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \sqrt{b_{2x}}\, e^{in_2} \\ -\dfrac{iu+a_{2x}}{\sqrt{b_{2x}}} e^{in_2} \\ \sqrt{b_{2y}} \\ -\dfrac{i(1-u)+a_{2y}}{\sqrt{b_{2y}}} \end{bmatrix}$$

- To find beam sizes and local angular spreads First introduce bilinear form describing the beam ellipse in 4D space

$$\hat{\mathbf{x}}^T \hat{\Xi} \hat{\mathbf{x}} = 1$$

$$\hat{\Xi}_{11} = \frac{(1-u)^2 + a_{1x}^{\ 2}}{e_1 b_{1x}} + \frac{u^2 + a_{2x}^{\ 2}}{e_2 b_{2x}}, \qquad \hat{\Xi}_{22} = \frac{b_{1x}}{e_1} + \frac{b_{2x}}{e_2},$$

$$\hat{\Xi}_{33} = \frac{u^2 + a_{1y}^{\ 2}}{e_1 b_{1y}} + \frac{(1-u)^2 + a_{2y}^{\ 2}}{e_2 b_{2y}}, \qquad \hat{\Xi}_{44} = \frac{b_{1y}}{e_1} + \frac{b_{2y}}{e_2},$$

$$\hat{\Xi}_{12} = \hat{\Xi}_{21} = \frac{a_{1x}}{e_1} + \frac{a_{2x}}{e_2}, \qquad \hat{\Xi}_{34} = \hat{\Xi}_{43} = \frac{a_{1y}}{e_1} + \frac{a_{2y}}{e_2},$$

$$\hat{\Xi}_{13} = \hat{\Xi}_{31} = \frac{\left[a_{1x}a_{1y} + u(1-u)\right]\cos n_1 + \left[a_{1y}(1-u) - a_{1x}u\right]\sin n_1}{e_1\sqrt{b_{1x}b_{1y}}} + \frac{\left[a_{2x}a_{2y} + u(1-u)\right]\cos n_2 + \left[a_{2x}(1-u) - a_{2y}u\right]\sin n_2}{e_2\sqrt{b_{2x}b_{2y}}} ,$$

$$\hat{\Xi}_{14} = \hat{\Xi}_{41} = \sqrt{\frac{b_{1y}}{b_{1x}}}\frac{a_{1x}\cos n_1 + (1-u)\sin n_1}{e_1} + \sqrt{\frac{b_{2y}}{b_{2x}}}\frac{a_{2x}\cos n_2 - u\sin n_2}{e_2} ,$$

$$\hat{\Xi}_{23} = \hat{\Xi}_{32} = \sqrt{\frac{b_{1x}}{b_{1y}}}\frac{a_{1y}\cos n_1 - u\sin n_1}{e_1} + \sqrt{\frac{b_{2x}}{b_{2y}}}\frac{a_{2y}\cos n_2 + (1-u)\sin n_2}{e_2} ,$$

$$\hat{\Xi}_{24} = \hat{\Xi}_{42} = \frac{\sqrt{b_{1x}b_{1y}}\cos n_1}{e_1} + \frac{\sqrt{b_{2x}b_{2y}}\cos n_2}{e_2} .$$

- **Then we can write the distribution function in the following form**

$$f(\hat{\mathbf{x}}, q_\parallel) = C_1 \exp\left(-\frac{1}{2}\left(\hat{\mathbf{x}} - \mathbf{D}q_\parallel\right)^T \Xi \left(\hat{\mathbf{x}} - \mathbf{D}q_\parallel\right)\right) \exp\left(-\frac{q_\parallel^{\,2}}{2s_p^{\,2}}\right)$$

where $\mathbf{D} = \left[D_x \quad D_x' \quad D_y \quad D_y'\right]^T$

- After integration over momentum spread we obtain

$$f(\hat{\mathbf{x}}) = C_2 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}^T \Xi' \hat{\mathbf{x}}\right), \qquad \Xi' = \Xi - \frac{\Xi \mathbf{D}\mathbf{D}^T \Xi}{s_p^{\,-2} + \mathbf{D}^T \Xi \mathbf{D}}$$

- **Beam sizes**
  - Size projections

$$s_x \equiv \sqrt{\overline{x^2}} = \sqrt{e_1 b_{1x} + e_2 b_{2x} + D_x^{\,2} s_p^{\,2}}$$

$$s_y \equiv \sqrt{\overline{y^2}} = \sqrt{e_1 b_{1y} + e_2 b_{2y} + D_y^{\,2} s_p^{\,2}}$$

$$a_{xy} \equiv \frac{\overline{xy}}{s_x s_y} = \frac{e_1 \sqrt{b_{1x} b_{1y}} \cos n_1 + e_2 \sqrt{b_{2x} b_{2y}} \cos n_2 + D_x D_y s_p^{\,2}}{s_x s_y}$$

  - Ellipse semi-axis

$$s_{1,2} = \sqrt{\frac{2\left(1 - a_{xy}^{\,2}\right)}{s_x^{\,2} + s_y^{\,2} \pm \sqrt{\left(s_x^{\,2} - s_y^{\,2}\right)^2 + 4a_{xy}^{\,2} s_x^{\,2} s_y^{\,2}}}}$$

Local transverse velocity spreads

- Bilinear form for angular spreads

$$\begin{bmatrix} 0 & q_x & 0 & q_x \end{bmatrix} \Xi' \begin{bmatrix} 0 \\ q_x \\ 0 \\ q_y \end{bmatrix} = q_x{}^2 \Xi'_{22} + 2 q_x q_y \Xi'_{24} + q_y{}^2 \Xi'_{44} = 1$$

- Ellipse semi-axis in the plane of local angular spreads (x'-y' plane)

$$q_{1,2} = \sqrt{\frac{2}{\Xi'_{22} + \Xi'_{44} \pm \sqrt{\left(\Xi'_{22} - \Xi'_{44}\right)^2 + 4\Xi'_{24}{}^2}}}$$

Additional transverce emittance growth due to finite dispersion

- For uncoupled motion

$$\frac{d\varepsilon_x}{dt} = \left\langle A_x \frac{d}{dt}\left(\frac{\overline{\Delta p_{\parallel}^2}}{p}\right)\right\rangle_s \quad , \qquad A_x = \frac{D^2 + \left(D'\beta_x + \alpha_x D\right)^2}{\beta_x}$$

- **Coupled motion:** momentum change excites both hor. and vert. motions

$$\begin{bmatrix} D_x \\ D'_x \\ D_y \\ D'_y \end{bmatrix} \frac{\Delta p}{p} \equiv \mathbf{D}\frac{\Delta p}{p} = \mathrm{Re}\left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2\right) = \mathbf{V}\mathbf{a} \quad , \quad V = \left[\mathbf{v}'_1, -\mathbf{v}''_1, \mathbf{v}'_2, -\mathbf{v}''_2\right] \quad , \quad \mathbf{a} = \begin{bmatrix} a'_1 \\ a''_1 \\ a'_2 \\ a''_2 \end{bmatrix}$$

$$\mathbf{a} = \frac{\Delta p}{p}\mathbf{V}^{-1}\mathbf{D}$$

- Then the emittance growth is

$$\frac{d\varepsilon_1}{dt} = \left\langle A_1 \frac{d}{dt}\overline{\left(\frac{\Delta p}{p}\right)^2}\right\rangle_s \; , \qquad \frac{d\varepsilon_2}{dt} = \left\langle A_2 \frac{d}{dt}\overline{\left(\frac{\Delta p}{p}\right)^2}\right\rangle_s$$

Here: $A_{1,2} = \mathbf{D}^T\mathbf{B}_{1,2}\mathbf{D}$, $\mathbf{B}_1 = \left(\mathbf{V}^{-1}\right)^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{V}^{-1}$, $\mathbf{B}_2 = \left(\mathbf{V}^{-1}\right)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{V}^{-1}$

- Expressing matrix V through beta-functions we finally obtain

$$\mathbf{B}_1 = \begin{bmatrix} \dfrac{(1-u)^2 + a_{1x}{}^2}{b_{1x}} & a_{1x} & B_{1_{13}} & B_{1_{14}} \\ a_{1x} & b_{1x} & B_{1_{23}} & \sqrt{b_{1x} b_{1y}}\cos n_1 \\ B_{1_{13}} & B_{1_{23}} & \dfrac{u^2 + a_{1y}{}^2}{b_{1y}} & a_{1y} \\ B_{1_{14}} & \sqrt{b_{1x} b_{1y}}\cos n_1 & a_{1y} & b_{1y} \end{bmatrix}$$

$$B_{1_{13}} = \frac{\left(u(1-u) + a_{1x} a_{1y}\right)\cos n_1 + \left(a_{1y}(1-u) - a_{1x} u\right)\sin n_1}{\sqrt{b_{1x} b_{1y}}}$$

$$B_{1_{14}} = \sqrt{\frac{b_{1y}}{b_{1x}}}\left(a_{1x}\cos n_1 + (1-u)\sin n_1\right)$$

$$B_{1_{23}} = \sqrt{\frac{b_{1x}}{b_{1y}}}\left(a_{1y}\cos n_1 - u\sin n_1\right)$$

$$\mathbf{B}_2 = \begin{bmatrix} \dfrac{u^2 + a_{2x}{}^2}{b_{2x}} & a_{2x} & B_{2_{13}} & B_{2_{14}} \\ a_{2x} & b_{2x} & B_{2_{23}} & \sqrt{b_{2x} b_{2y}} \cos n_2 \\ B_{2_{13}} & B_{2_{23}} & \dfrac{(1-u)^2 + a_{2y}{}^2}{b_{2y}} & a_{2y} \\ B_{2_{14}} & \sqrt{b_{2x} b_{2y}} \cos n_2 & a_{2y} & b_{2y} \end{bmatrix}$$

$$B_{2_{13}} = \frac{\left(u(1-u) + a_{2x} a_{2y}\right)\cos n_2 + \left(a_{2x}(1-u) - a_{2y} u\right)\sin n_2}{\sqrt{b_{2x} b_{2y}}}$$

$$B_{2_{14}} = \sqrt{\frac{b_{2y}}{b_{2x}}}\left(a_{2x} \cos n_2 - u \sin n_2\right)$$

$$B_{2_{23}} = \sqrt{\frac{b_{2x}}{b_{2y}}}\left(a_{2y} \cos n_2 + (1-u) \sin n_2\right)$$

- Finally, for ultra-relativistic beam ($g >> Q_x, Q_y$), we obtain

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_s = \frac{1}{4\sqrt{2}} \frac{r_p{}^2 c}{g^3 b^3} \left\langle \frac{N_i}{s_1 s_2 s_s} \frac{L_C}{\sqrt{q_1{}^2 + q_2{}^2 + (s_p/g)^2}} \begin{bmatrix} \hat{A}_1 \Psi(s_p/g, q_1, q_2) \\ \hat{A}_2 \Psi(s_p/g, q_1, q_2) \end{bmatrix} \right\rangle_s$$