A Simple Model for a Superconducting RF cavity with a Vector Phase Modulator

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October 27, 2007

Introduction

This note is not meant to be an original work for understanding RF cavities. It is intended to be a summary of equations that are used in a simple model to describe the behavior of a superconducting RF cavity that is driven with a vector modulator between the cavity and the klystron.

RF Cavity Equivalent Circuit

The response of a cavity can be modeled by a parallel RLC circuit as shown in Figure 1. The inductor represents the magnetic energy stored in the cavity. The voltage across the inductor is given as:

$$v(t) = L\frac{di_L}{dt} \tag{1}$$

The resistor is inversely proportional to the power lost in the cavity walls. The voltage across the resistor is:

$$v(t) = Ri_R \tag{2}$$

The capacitor represents the electric energy stored in the cavity. The current though the capacitor is:

$$i_C = C\frac{dv}{dt} \tag{3}$$

The source current is divided amount the parallel LRC network:

$$i_s = i_L + i_R + i_C \tag{4}$$

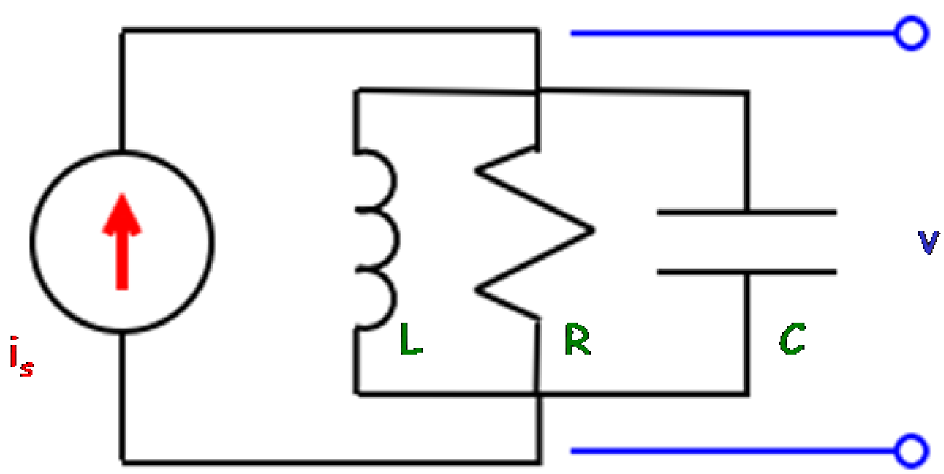

Figure 1. Equivalent Circuit for an RF Cavity

The following quantities are defined:

$$Q = \omega_o RC \tag{5}$$

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{6}$$

Defined this way, the Q is the ratio of the energy stored to the energy lost in the circuit during one cycle. Note that this definition uses the energy lost in circuit and not just the cavity. An external circuit attached to the cavity will load the cavity circuit and dissipate power. The quantity ω_o is the resonant frequency of the cavity. Then:

$$L = \frac{R}{\omega_o Q} \tag{7}$$

$$C = \frac{Q}{\omega_o R} \tag{8}$$

The quantity R/Q is independent of R and is a function of the geometry only. The current flowing though the parallel LRC network becomes:

$$\frac{1}{{\omega_o}^2}\frac{d^2 i_L}{dt^2} + \frac{1}{\omega_o Q}\frac{di_L}{dt} + i_L = i_s \tag{9}$$

<u>Homogenous Solution</u>

The homogenous solution (i_s=0) is of the form:

$$i_L = Ae^{st} \tag{10}$$

where:

$$s = -\frac{\omega_o}{2Q} \pm j\sqrt{{\omega_o}^2 - \left(\frac{\omega_o}{2Q}\right)^2} \tag{11}$$

Define:

$$\alpha = \frac{\omega_o}{2Q} \tag{12}$$

and:

$${\omega_r}^2 = {\omega_o}^2 - \alpha^2 \tag{13}$$

Then the homogenous solution has the form:

$$i_L = Ae^{-\alpha t}cos(\omega_r t) + Be^{-\alpha t}sin(\omega_r t) \tag{14}$$

<u>Impulse response</u>

The impulse response is the solution to:

$$\frac{1}{{\omega_o}^2}\frac{d^2 i_L}{dt^2} + \frac{1}{\omega_o Q}\frac{di_L}{dt} + i_L = q\delta(t - t') \tag{15}$$

where q is the bunch charge. Equation 15 can be re-written in the form:

$$\frac{1}{\omega_o}\frac{d}{dt}\left(\frac{1}{\omega_o}\frac{di_L}{dt}+\frac{1}{\omega_o Q}i_L\right)+i_L=q\delta(t-t') \quad (16)$$

Both sides can be integrated around t'

$$\frac{1}{\omega_o}\left(\frac{1}{\omega_o}\frac{di_L}{dt}+\frac{1}{\omega_o Q}i_L\right)\Bigg|_{t'-\varepsilon}^{t'+\varepsilon}+\int_{t'+\varepsilon}^{t'+\varepsilon}i_L dt=q \quad (17)$$

Since i_L is continuous though t'

$$\frac{di_L}{dt}\Bigg|_{t'-\varepsilon}^{t'+\varepsilon}=q{\omega_o}^2 \quad (18)$$

For t < t'

$$i_L=0 \quad (19)$$

Since i_L must be continuous at t=t', for t> t'

$$i_L=Ae^{-\alpha(t-t\prime)}sin\big(\omega_r(t-t')\big) \quad (20)$$

Differentiating Eqn. 20 at t=t' and using Eqn. 18:

$$i_L=q\omega_o\frac{\omega_o}{\omega_r}e^{-\alpha(t-t\prime)}sin\big(\omega_r(t-t')\big) \quad (21)$$

The cavity voltage can be found by using Eqns. 1 and 7:

$$V=\frac{R}{Q}q\frac{\omega_o}{\omega_r}e^{-\alpha(t-t\prime)}\big[\omega_r cos\big(\omega_r(t-t')\big)-\alpha sin\big(\omega_r(t-t')\big)\big] \quad (22)$$

For a low loss, high frequency cavity, $\omega_r >> \alpha$

$$V=\omega_o q\frac{R}{Q}e^{-\alpha(t-t\prime)}cos\big(\omega_r(t-t')\big) \quad (23)$$

Consider N beam pulses separated by $2\pi/\omega_r$. At the Nth beam pulse, the envelope of the voltage has built up to:

$$V_{env}=\omega_o q\frac{R}{Q}\sum_{n=0}^{N}e^{-\alpha(N-n)\frac{2\pi}{\omega_r}}cos\big(2\pi(N-n)\big) \quad (24)$$

$$V_{env}=\omega_o q\frac{R}{Q}\sum_{n=0}^{N}e^{-\alpha(N-n)\frac{2\pi}{\omega_r}} \quad (25)$$

The sum can be approximated by an integral:

$$\sum_{n=0}^{N}e^{-\alpha(N-n)\frac{2\pi}{\omega_r}}\approx\int_0^N e^{-\alpha(N-n)\frac{2\pi}{\omega_r}}dn \quad (26)$$

$$\sum_{n=0}^{N}e^{-\alpha(N-n)\frac{2\pi}{\omega_r}}\approx\frac{2\pi}{\omega_r}\int_0^t e^{-\alpha(t-t\prime)}dt \quad (27)$$

The envelope voltage becomes:

$$V_{env} = 2\frac{\omega_o}{2\pi} qR\left(1 - e^{-\frac{\omega_o}{2Q}t}\right) \tag{28}$$

But the beam current along the pulse is given as the bunch charge divided by the bunch spacing:

$$I_p = \frac{\omega_o}{2\pi} q \tag{29}$$

The voltage envelope becomes:

$$V_{env} = 2I_p R\left(1 - e^{-\frac{\omega_o}{2Q}t}\right) \tag{30}$$

There are two things to note in Eqn. 30. First the voltage builds up with a time constant of $2Q/\omega_o$ and second, the voltage envelope is proportional to twice the pulse current. This factor of two in front of the pulse current is only true for short bunches. For long bunches, this factor approaches unity.

Envelope Equations

The solution to Eqn. 9 can be broken up into a fast varying and slow varying parts:

$$i_L(t) = Re\{I_L(t)e^{j\omega t}\} \tag{31}$$

$$i_s(t) = Re\{I_s(t)e^{j\omega t}\} \tag{32}$$

where $I_L(t)$ and $I_S(t)$ are slowly varying complex phasors. Substituting Eqns. 31 and 32 into Eqn. 9:

$$\frac{d^2 I_L}{dt^2} + \left(2j\omega + \frac{\omega_o}{Q}\right)\frac{dI_L}{dt} + j\frac{\omega\omega_o}{Q} I_L + ({\omega_o}^2 + \omega^2)I_L = {\omega_o}^2 I_s \tag{33}$$

For large ω, Eqn. 33 becomes:

$$2j\omega\frac{dI_L}{dt} + j\frac{\omega\omega_o}{Q} I_L + ({\omega_o}^2 + \omega^2)I_L \approx {\omega_o}^2 I_s \tag{34}$$

The cavity voltage is given as:

$$v(t) = \frac{R}{\omega_o Q}\frac{di_L}{dt} \tag{35}$$

The cavity voltage can also be separated into fast varying and slow varying parts:

$$v(t) = Re\{V(t)e^{j\omega t}\} \tag{36}$$

Substituting Eqn. 36 into Eq. 35:

$$V \approx j\omega\frac{R}{\omega_o Q} I_L \tag{37}$$

Equation 34 becomes:

$$\frac{2Q}{\omega_o}\frac{dV}{dt} + V - jQ\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)V = RI_s \tag{38}$$

In a steady state situation dV/dt=0:

$$V_\infty = \frac{RI_s}{1 - jQ\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right)} \tag{39}$$

If $\omega \neq \omega_o$, the cavity is not operated on resonance (the cavity is de-tuned). When the cavity is de-tuned, the steady state cavity voltage and the generator are out of phase. The detuning angle is found from Eqn. 39:

$$tan(\varphi_D) = Q\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o}\right) \tag{40}$$

Equation 38 becomes:

$$\frac{2Q}{\omega_o}\frac{dV}{dt} + V - jtan(\varphi_D)V = RI_s \tag{41}$$

The cavity voltage can be separated into real and imaginary parts:

$$V = V_r + jV_i \tag{42}$$

Equation 41 becomes two coupled equations:

$$\frac{2Q}{\omega_o}\frac{dV_r}{dt} + V_r + tan(\varphi_D)V_i = RI_{s_r} \tag{43}$$

and:

$$\frac{2Q}{\omega_o}\frac{dV_i}{dt} + V_i - tan(\varphi_D)V_r = RI_{s_i} \tag{44}$$

Optimum Cavity Coupling for Superconducting Cavities

The cavity coupler can be thought of as a transformer between the power source and the cavity as shown in Figure 2. The generator as seen by the cavity through the coupler is shown in Figure 3.

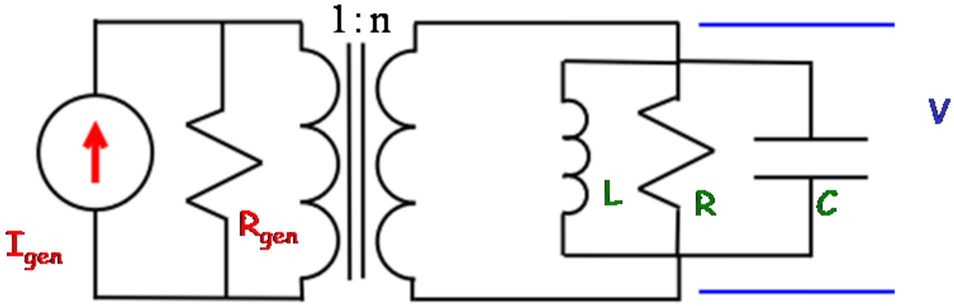

Figure 2. Transformer model of a cavity coupler

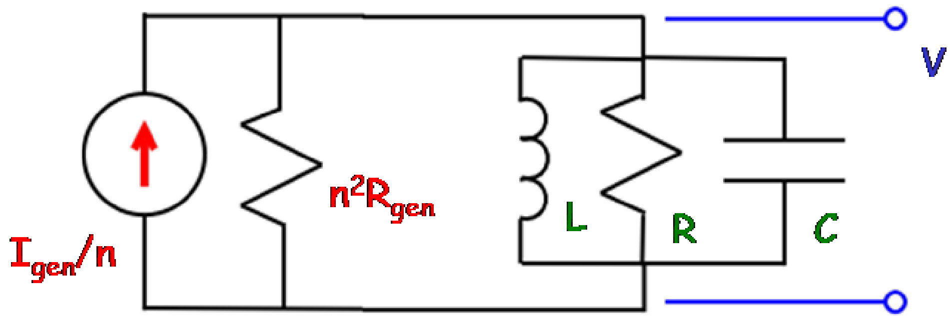

Figure 3. Generator circuit as seen by the cavity though the transformer.

In steady state, this note will use the cavity voltage for the reference phase.

$$V_\infty = V_o + j0 \tag{45}$$

where V_o is the steady state voltage. The current into the cavity as seen by the cavity is:

$$I_{s\infty} = I_g e^{j\varphi_g} - 2I_p e^{j\varphi_b} \tag{46}$$

where the short bunch approximation was assumed so that the RF beam current is twice the pulse beam current. The synchronous phase angle between the beam and the cavity voltage is ϕ_b. Equation 43 becomes:

$$V_o = R_t\left(I_g cos(\varphi_g) - 2I_p cos(\varphi_b)\right) \tag{47}$$

were R_t is the parallel combination of the generator and the cavity resistance. For superconducting cavities, which will be the focus of the rest of this note, the cavity resistance contributes very little to the total resistance.

$$R_t \approx n^2 R_{gen} \tag{48}$$

This resistance determines the loaded Q of the cavity:

$$n^2 R_{gen} = Q_L \frac{R}{Q} \tag{49}$$

where R/Q is purely a geometrical constant of the cavity. Also, I_g is the current seen on the cavity side of the coupler. The current seen on the generator side of the coupler (I_{gen}) is:

$$I_{gen} = nI_g \tag{50}$$

The power delivered to the beam is:

$$P_b = \frac{1}{2} V_o \left(2I_p cos(\varphi_b)\right) \tag{51}$$

Using Eqns. 47-49, the beam power becomes:

$$P_b = RI_{avg} cos(\varphi_b)\left[nI_{gen} cos(\varphi_g) - n^2 2I_p cos(\varphi_b)\right] \tag{52}$$

The beam power is maximized if the coupler is adjusted so that:

$$n = \frac{I_{gen} cos(\varphi_g)}{4I_p cos(\varphi_b)} \tag{53}$$

At optimum power coupling:

$$V_o = Q_L \frac{R}{Q} 2I_p cos(\varphi_b) \tag{54}$$

or at optimum coupling:

$$Q_L = \frac{V_o}{2\frac{R}{Q} I_p cos(\varphi_b)} \tag{55}$$

At optimum coupling, the in-phase generator current as seen by the cavity is:

$$I_g cos(\varphi_g) = 4 I_p cos(\varphi_b) \tag{56}$$

which is twice the in-phase component of the RF beam current. If the cavity voltage is initially zero and the generator is turned on with no beam in the cavity, the final cavity voltage will be $2V_o$. With no beam in the cavity, the cavity voltage will reach V_o at:

$$t_{fill} = ln(2) \frac{2Q_L}{\omega_o} \tag{57}$$

At $t=t_{fill}$, the beam should be injected.

Non-Optimum Cavity Coupling for Superconducting Cavities

Assume that a superconducting cavity has been set for optimum coupling at a given beam current I_{p0}. The loaded Q (Q_L) of the cavity is given as:

$$Q_L = \frac{V_o}{2\frac{R}{Q} I_{p_0} cos(\varphi_b)} \tag{58}$$

Using Equation 47:

$$I_g = 2\left(I_{p_0} + I_p\right) \frac{cos(\varphi_b)}{cos(\varphi_g)} \tag{59}$$

Using Equations 44-46 and Eqn. 58, to compensate beam loading, the generator phase must be set to:

$$tan(\varphi_g) = \frac{I_p tan(\varphi_b) - I_{p_0} tan(\varphi_D)}{I_{p_0} + I_p} \tag{60}$$

Incident and Reflected Power

The incident voltage into the cavity is:

$$V_{inc} = \frac{I_g e^{j\varphi_g}}{2} R_t \tag{61}$$

The total voltage is the sum of the incident and reflected voltages:

$$V = V_{inc} + V_{refl} \tag{62}$$

The reflection coefficient (Γ) is the ratio of the reflected and incident voltage:

$$\Gamma + 1 = \frac{V}{V_{inc}} \tag{63}$$

or:

$$\Gamma + 1 = \frac{4 I_{p_0} cos(\varphi_b)}{I_g} \frac{V}{V_o} e^{-j\varphi_g} \tag{64}$$

Vector Phase Modulators

A vector phase modulator is constructed from 3dB couplers in which a pair of ports is terminated with short circuits. The electrical lengths of the short circuit ports are independently adjustable. The ports that are shorted do not normally couple to each other. The 3 dB hybrid is a four port device with the following scattering matrix:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 & C & 0 & jC \\ C & 0 & jC & 0 \\ 0 & jC & 0 & C \\ jC & 0 & C & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{65}$$

where:

$$C = \frac{1}{\sqrt{2}} \tag{66}$$

The shorts would be located on ports 2 and 4:

$$a_2 = -e^{-j\theta_2} b_2 \tag{67}$$

and:

$$a_4 = -e^{-j\theta_4} b_4 \tag{68}$$

The scattering matrix in Eqn. 65 becomes:

$$\begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = -je^{-j\frac{\theta_2+\theta_4}{2}} \begin{bmatrix} -sin\left(\frac{\theta_2-\theta_4}{2}\right) & cos\left(\frac{\theta_2-\theta_4}{2}\right) \\ cos\left(\frac{\theta_2-\theta_4}{2}\right) & sin\left(\frac{\theta_2-\theta_4}{2}\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \tag{69}$$

The gain of the vector phase modulator is:

$$G = cos\left(\frac{\theta_2-\theta_4}{2}\right) \tag{70}$$

The phase shift of the vector modulator is:

$$\Delta\varphi = -\frac{\theta_2+\theta_4}{2} \tag{71}$$

Or conversely, the phase shifts needed on legs 2 and 4 of the modulator for a given gain and phase shift are:

$$\theta_2 = \cos^{-1}(\mathrm{G}) - \Delta\varphi \tag{72}$$

and:

$$\theta_4 = -\cos^{-1}(\mathrm{G}) - \Delta\varphi \tag{73}$$

Model

A simple model based on the coupled differential equations shown in Eqns. 43-44 was constructed to understand the behavior of a vector modulator inserted between a superconducting cavity and a klystron. In the model, the cavity R/Q of 262 Ohms was used. This value is typical of a superconducting single spoke resonator used in HINS. The cavity voltage and synchronous phase angle were chosen from the design of the first cryomodule in HINS. The cavity voltage of 1472kV with a synchronous angle of 30 degrees was used. It was assumed that the cavity coupling would be optimized for optimum power transfer to the beam at a beam pulse current of 15mA. Using Eqn. 58, the loaded Q of the cavity would then be about $2.2x10^5$ and the fill time of the loaded cavity is about 212uS. Beam injection time should be about 147uS after the generator current turns on. The beam pulse length is 1mS.

Example 1

In this example, the beam current is equal to the optimum beam power current of 15 mA and the cavity is operated on resonance. While filling the cavity, the generator phase is 0 degrees. At beam injection, the generator phase needs to slew to 16 degrees. It was assumed that the vector modulator can slew at a rate of 1 degree/uS. Note that the slew rate is not referenced to the overall phase shift of the vector modulator but to the individual phase shifts on the shorted legs. The parameters are listed in Table 1. Cavity voltages, phase angles, and reflection coefficient are shown in Figures 4-8.

Example 2

In this example, the beam current is equal to the optimum beam power current of 15 mA but the cavity phase is detuned to the synchronous phase angle. At optimum beam current and with detuning the cavity phase to the synchronous phase angle, the generator current is in phase with the cavity voltage during beam pulse. This makes the reflection coefficient zero while the beam is in the cavity. However, this requires the generator phase to be shifted by -10 degrees during the filling of the cavity. The total generator phase swing is less than the phase swing needed while running the cavity on resonance. The parameters are listed in Table 2. Cavity voltages, phase angles, and reflection coefficient are shown in Figures 9-13.

Example 3

In this example, the beam current is equal to the optimum beam power current of 15 mA but the cavity is detuned to less than synchronous phase angle so that the generator phase swing before and after beam arrival is equal but opposite to each other. The total generator phase swing is 12.7 degrees when the beam just arrives in the cavity which is more than the previous example. The reflection coefficient during the beam pulse is about -20dB. The parameters are listed in Table 3. Cavity voltages, phase angles, and reflection coefficient are shown in Figures 14-18.

Example 4

In this example, the beam current is 1mA which is much smaller than the optimum beam power current of 15mA. Since the beam loading is small, the generator phase angle needed during the beam pulse is about 2 degrees. With this small generator phase angle there is little need to detune the cavity so the cavity is run on resonance. However, with small beam loading, the generator current during the beam pulse must be reduced by about a factor of two from what was required during filling the cavity. This requires the phase shifts on the vector modulator shorted legs to each slew about 45 degrees when beam arrives in the cavity. This is about a factor of three larger than the high beam loading cases in the previous examples. For the high beam loading cases, the vector modulator needed to start slewing about 7-8 uS before beam arrived in the cavity. For the low loading beam case in this example, the vector modulator needs to start slewing about 29uS before beam arrives. The reflection coefficient for this example is large during the beam pulse but the power required during the beam pulse is about ¼ of the power required for the high beam loading cases. Of course, the vector modulators must reflect ¾ of the power from the klystron as compared to the large beam loading cases. The parameters are listed in Table 4. Cavity voltages, phase angles, and reflection coefficient are shown in Figures 19-23.

Parameter	Value	Units
Cavity Voltage	1472	kV
Klystron Power	25.57	kW
R/Q	262	Ohms
RF Frequency	325	MHz
Reference Beam Current	15	mA
Actual Beam Current	15	mA
Synchronous Phase Angle	30	degrees
Detuning Angle	0	degrees
Generator phase during filling	0	degrees
Generator phase during beam	16.1	degrees
Vector modulator gain during filling	0.9	
Vector modulator gain during beam	0.9	
Vector modulator slew rate	1	degrees/uS
Beam injection time	0.1384	mS
Vector modulator start time	0.1304	mS
Beam Pulse Length	1	mS

Table 1. Parameters for Example 1.

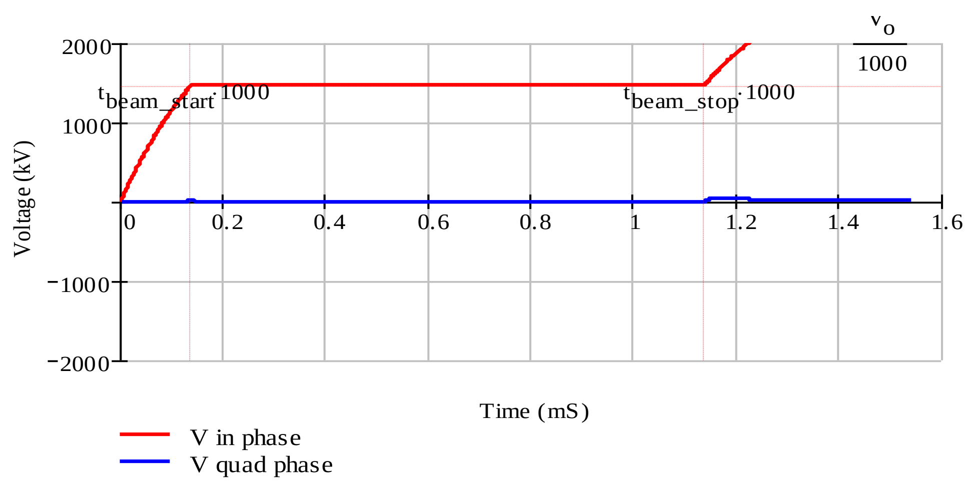

Figure 4. Cavity voltage for Example 1.

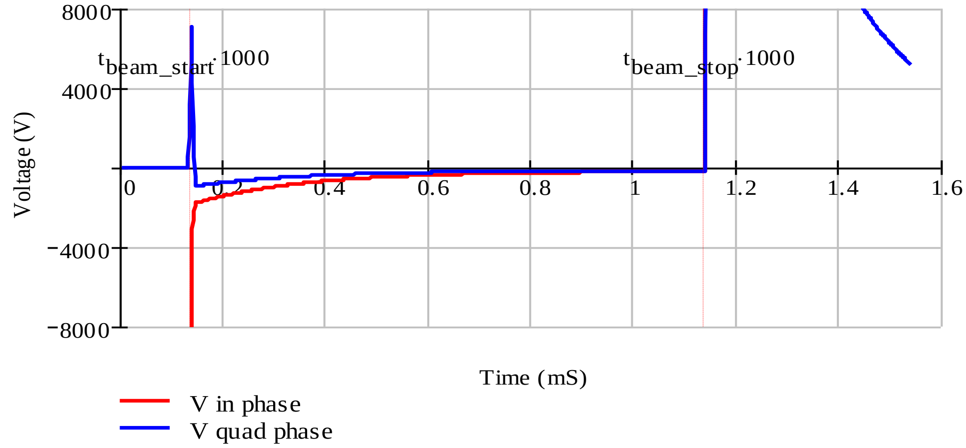

Figure 5. Cavity voltage error for Example 1.

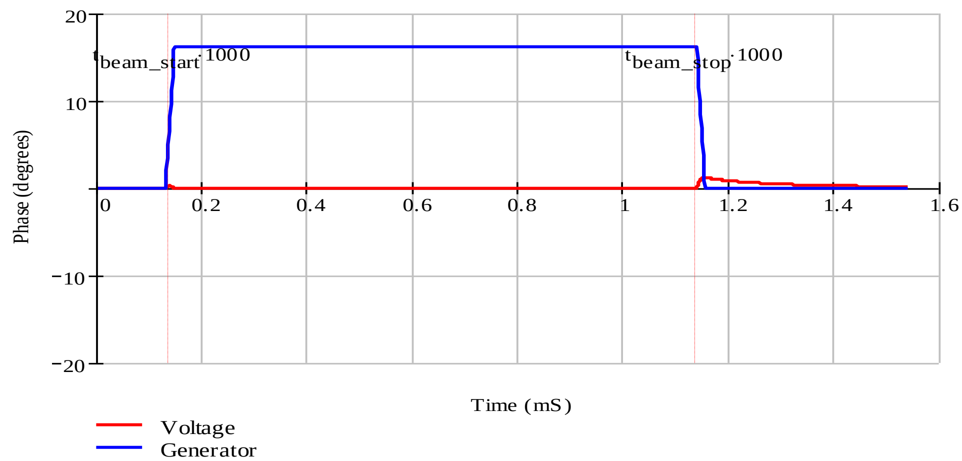

Figure 6. Voltage and Generator phase for Example 1.

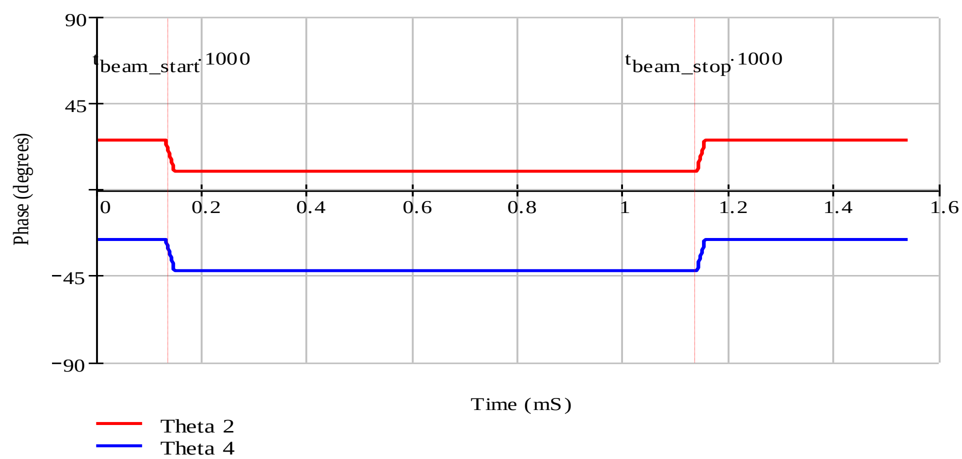

Figure 7. Vector phase modulator shorted leg phase delay angles for Example 1.

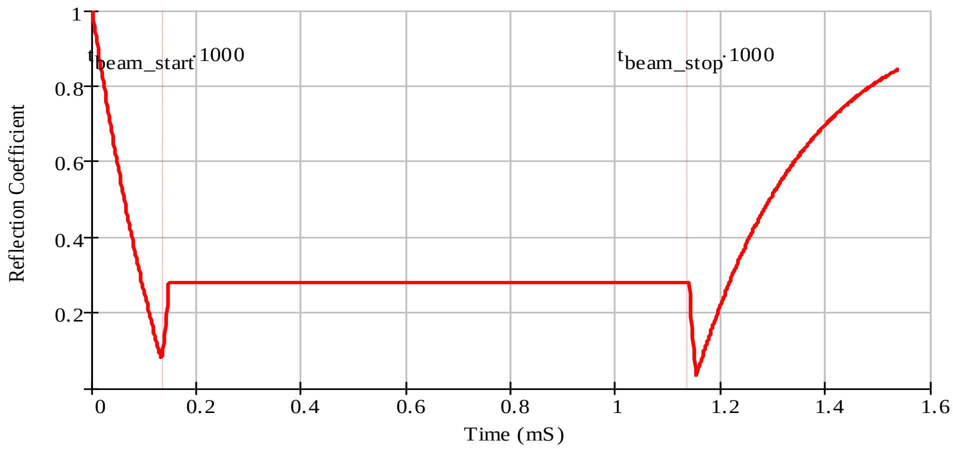

Figure 8. Reflection coefficient for Example 1.

Parameter	Value	Units
Cavity Voltage	1472	kV
Klystron Power	23.61	kW
R/Q	262	Ohms
RF Frequency	325	MHz
Reference Beam Current	15	mA
Actual Beam Current	15	mA
Synchronous Phase Angle	30	degrees
Detuning Angle	30	degrees
Generator phase during filling	-10.5	degrees
Generator phase during beam	0	degrees
Vector modulator gain during filling	0.9	
Vector modulator gain during beam	0.9	
Vector modulator slew rate	1	degrees/uS
Beam injection time	0.148	mS
Vector modulator start time	0.141	mS
Beam Pulse Length	1	mS

Table 2. Parameters for Example 2.

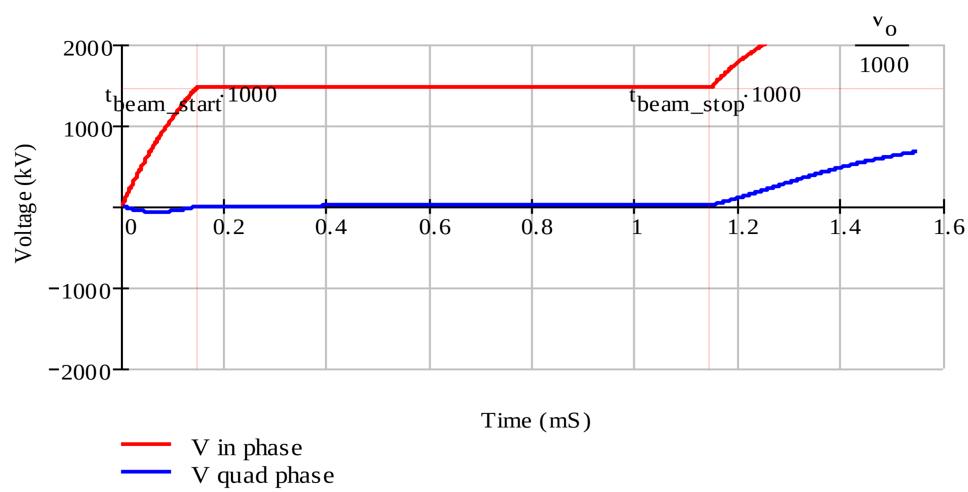

Figure 9. Cavity voltage for Example 2.

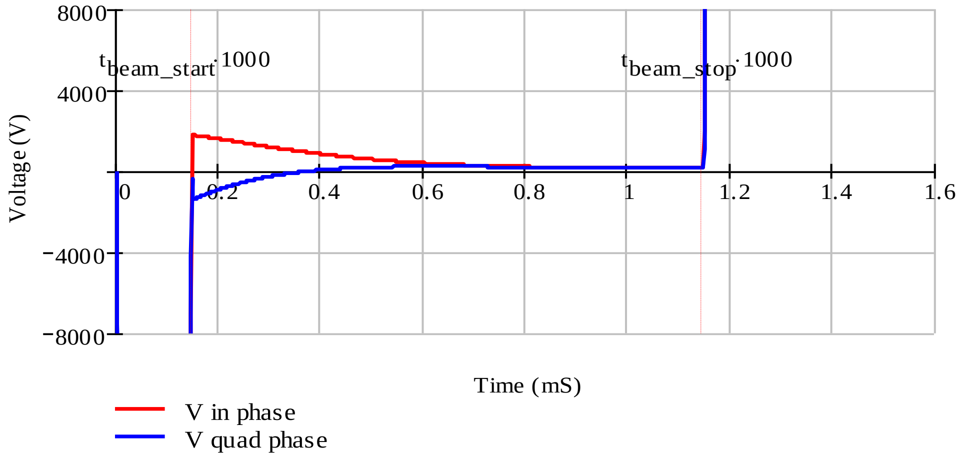

Figure 10. Cavity voltage error for Example 2.

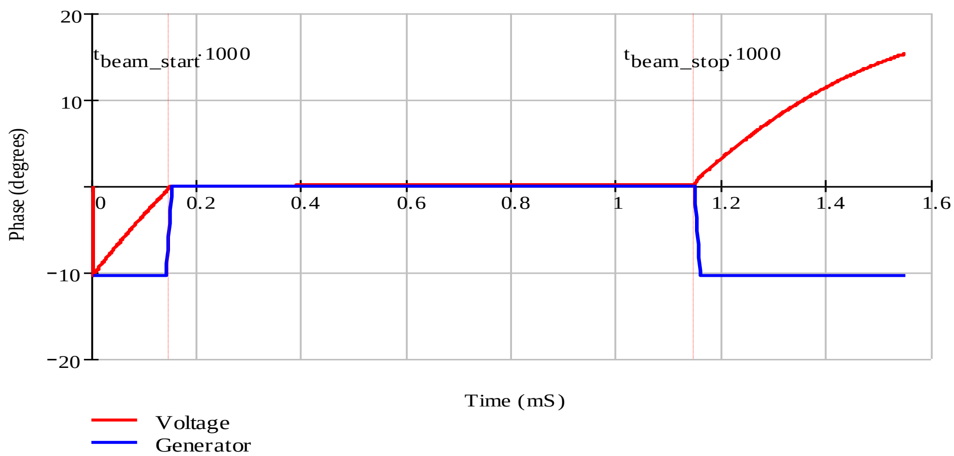

Figure 11. Voltage and Generator phase for Example 2.

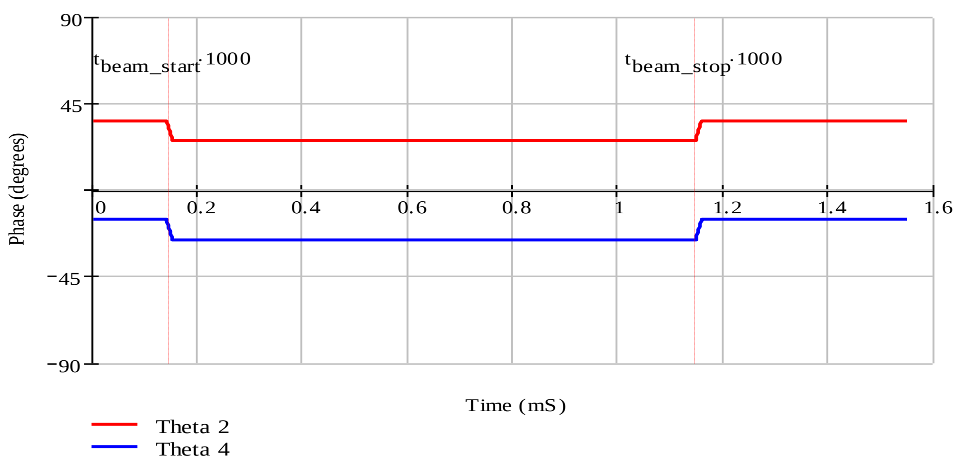

Figure 12. Vector phase modulator shorted leg phase delay angles for Example 2.

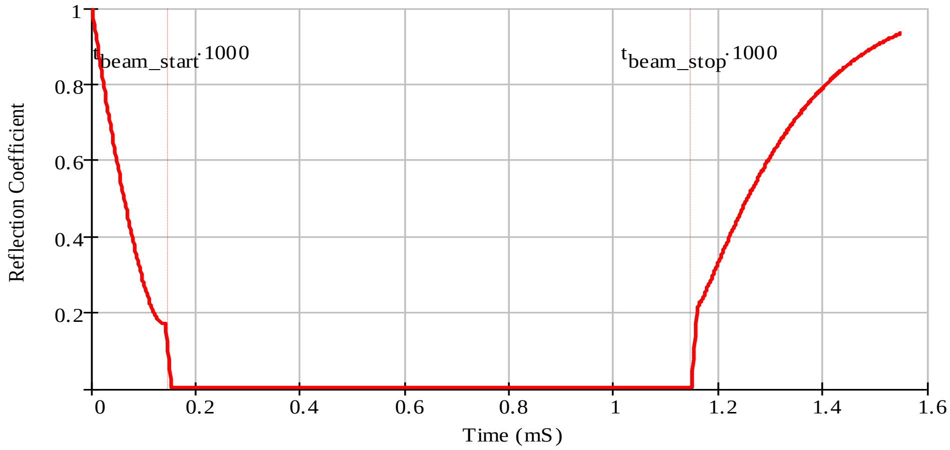

Figure 13. Reflection coefficient for Example 2.

Parameter	Value	Units
Cavity Voltage	1472	kV
Klystron Power	23.88	kW
R/Q	262	Ohms
RF Frequency	325	MHz
Reference Beam Current	15	mA
Actual Beam Current	15	mA
Synchronous Phase Angle	30	degrees
Detuning Angle	20	degrees
Generator phase during filling	-6.5	degrees
Generator phase during beam	6.09	degrees
Vector modulator gain during filling	0.9	
Vector modulator gain during beam	0.9	
Vector modulator slew rate	1	degrees/uS
Beam injection time	0.146	mS
Vector modulator start time	0.138	mS
Beam Pulse Length	1	mS

Table 3. Parameters for Example 3.

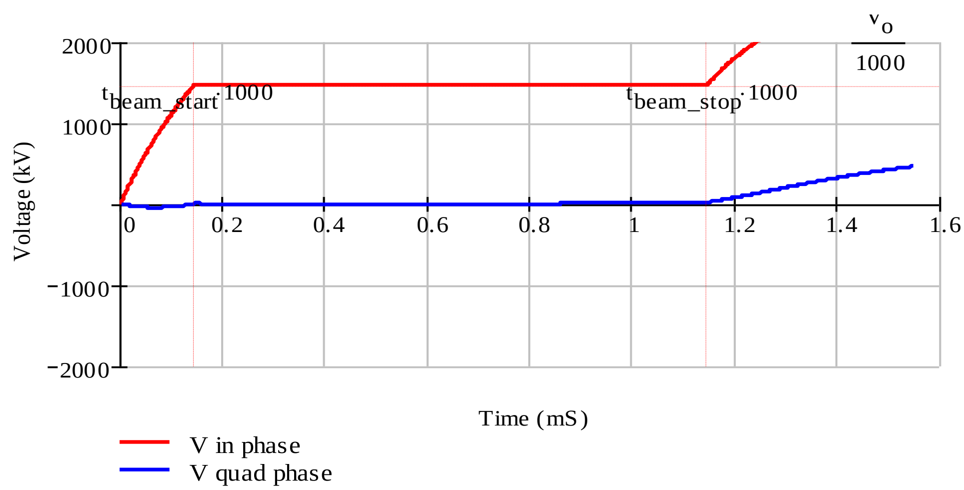

Figure 14. Cavity voltage for Example 3.

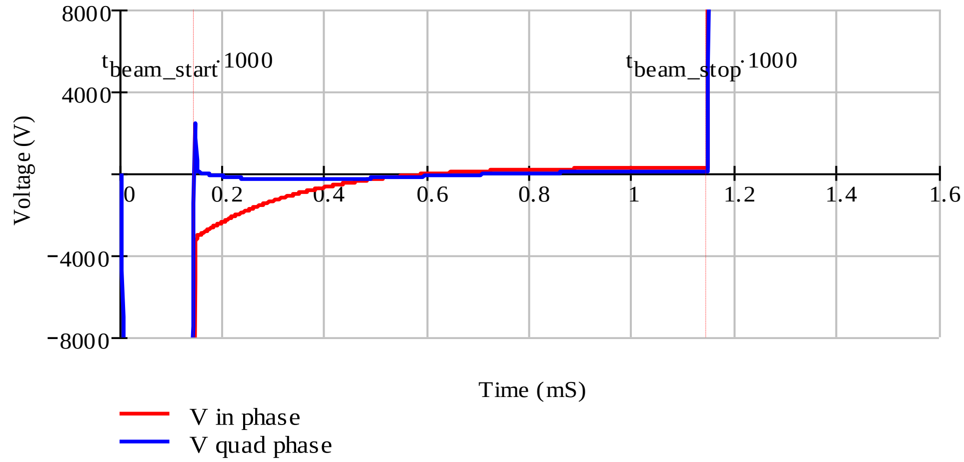

Figure 15. Cavity voltage error for Example 3.

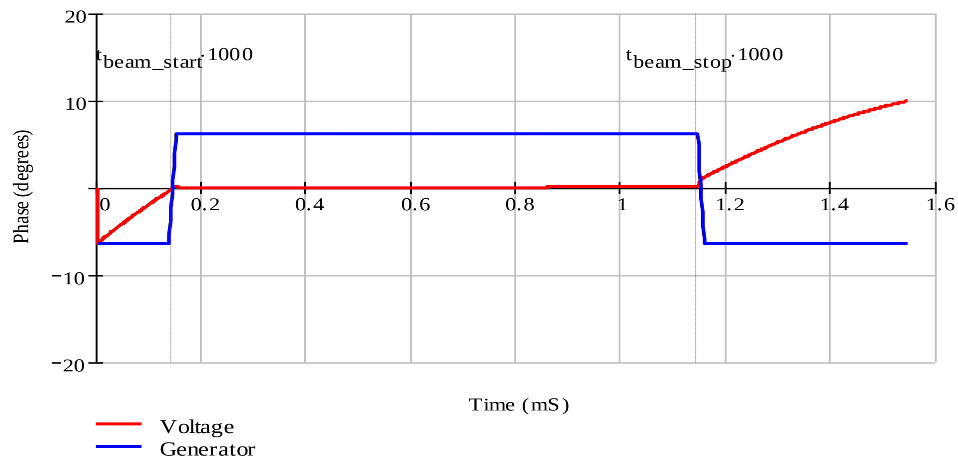

Figure 16. Voltage and Generator phase for Example 3.

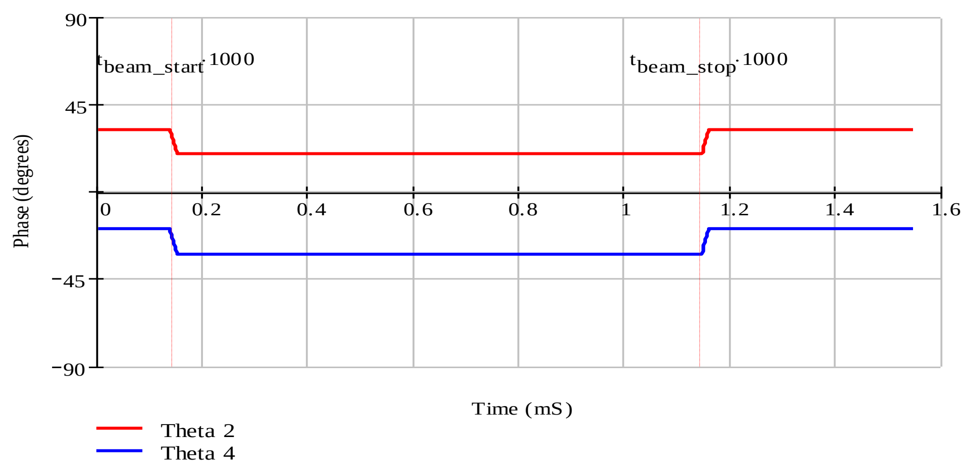

Figure 17. Vector phase modulator shorted leg phase delay angles for Example 3.

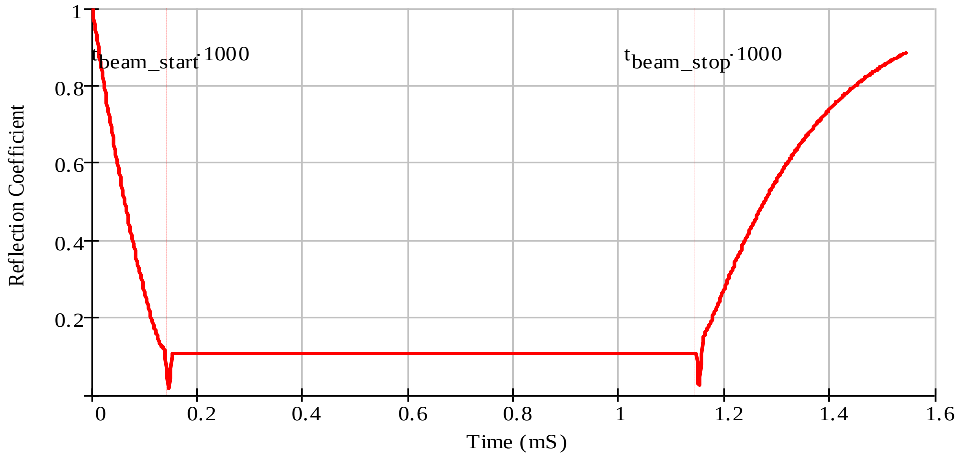

Figure 18. Reflection coefficient for Example 3.

Parameter	Value	Units
Cavity Voltage	1472	kV
Klystron Power	25.57	kW
R/Q	262	Ohms
RF Frequency	325	MHz
Reference Beam Current	15	mA
Actual Beam Current	1	mA
Synchronous Phase Angle	30	degrees
Detuning Angle	0	degrees
Generator phase during filling	0	degrees
Generator phase during beam	2.067	degrees
Vector modulator gain during filling	0.9	
Vector modulator gain during beam	0.4615	
Vector modulator slew rate	1	degrees/uS
Beam injection time	0.146	mS
Vector modulator start time	0.1175	mS
Beam Pulse Length	1	mS

Table 4. Parameters for Example 4.

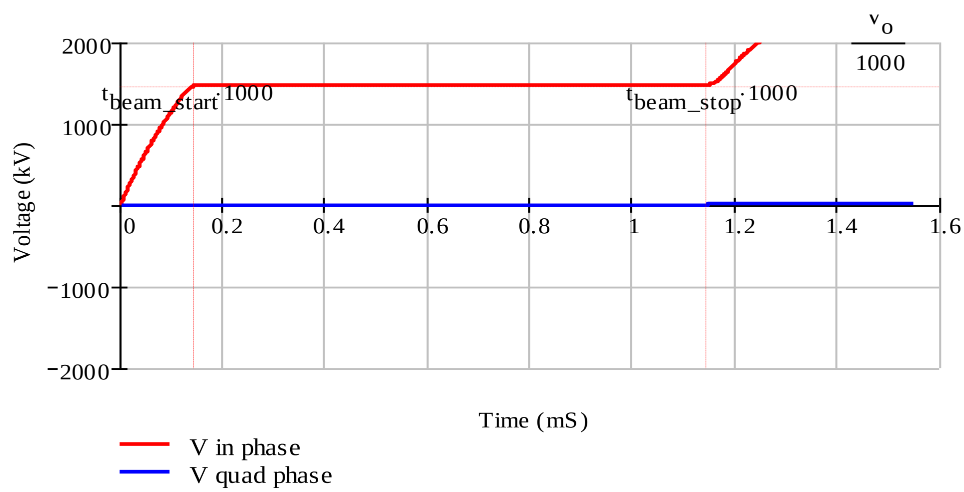

Figure 19. Cavity voltage for Example 4.

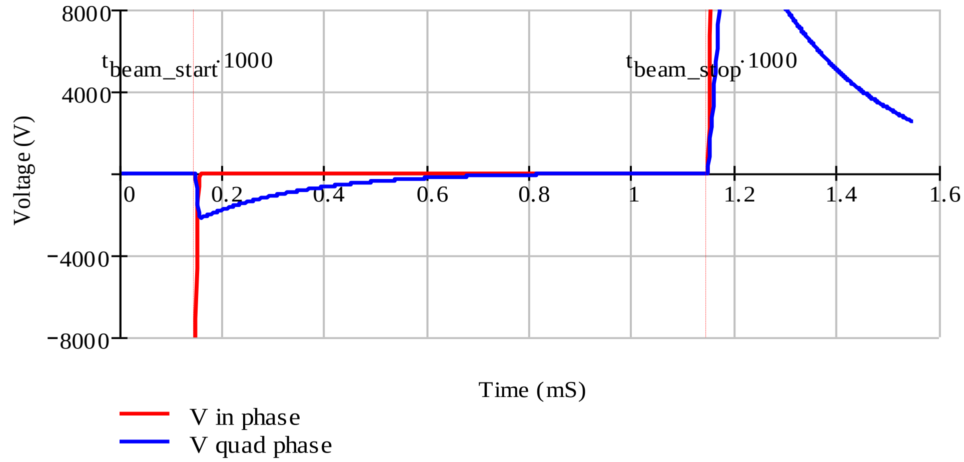

Figure 20. Cavity voltage error for Example 4.

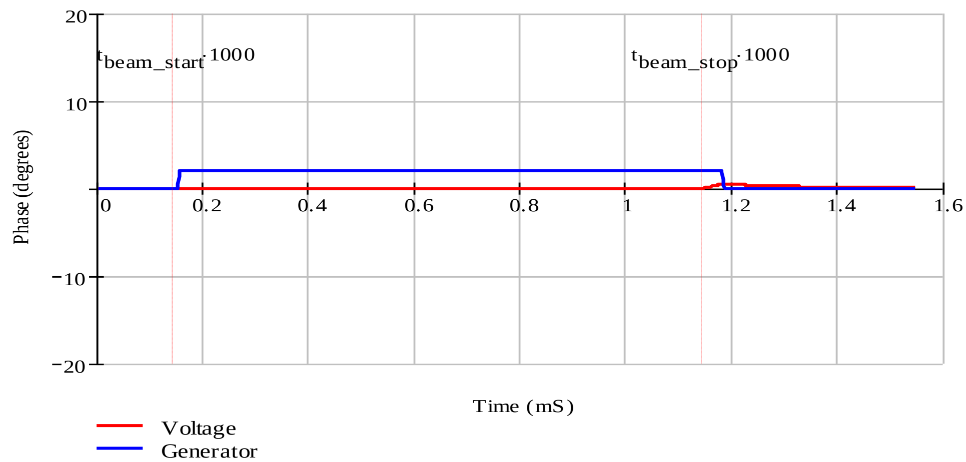

Figure 21. Voltage and Generator phase for Example 4.

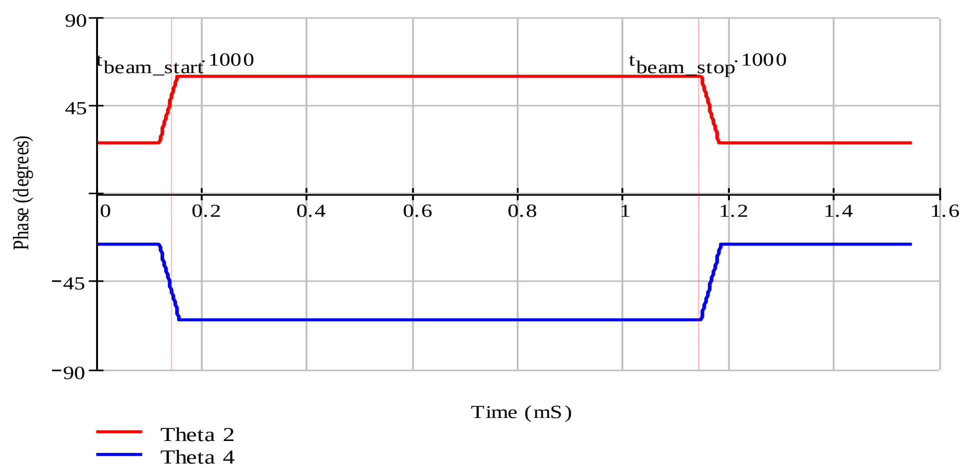

Figure 22. Vector phase modulator shorted leg phase delay angles for Example 4.

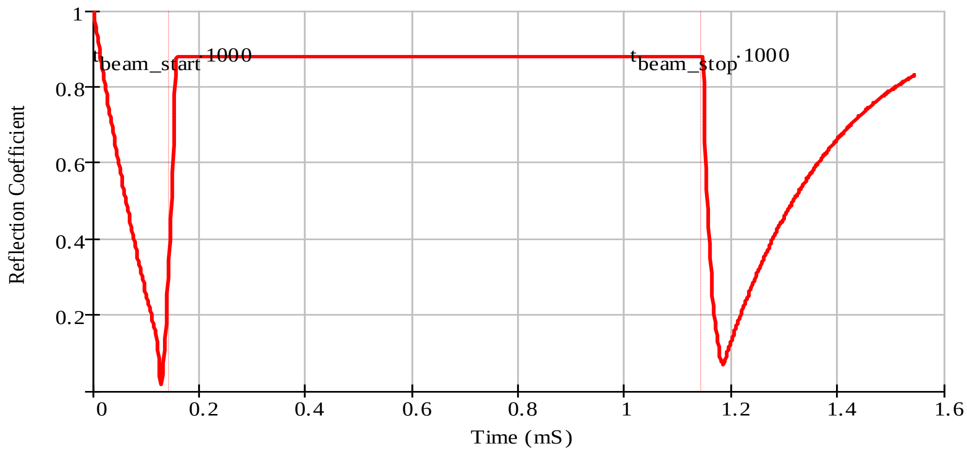

Figure 23. Reflection coefficient for Example 4.