Single Particle Trajectories and Stability Criterion

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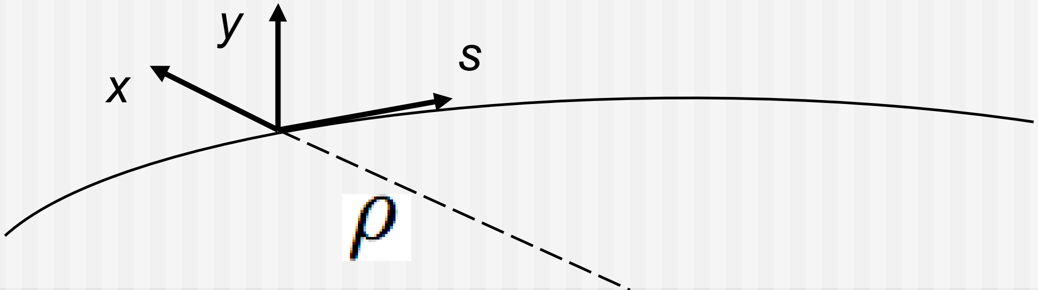
Equations of Motion

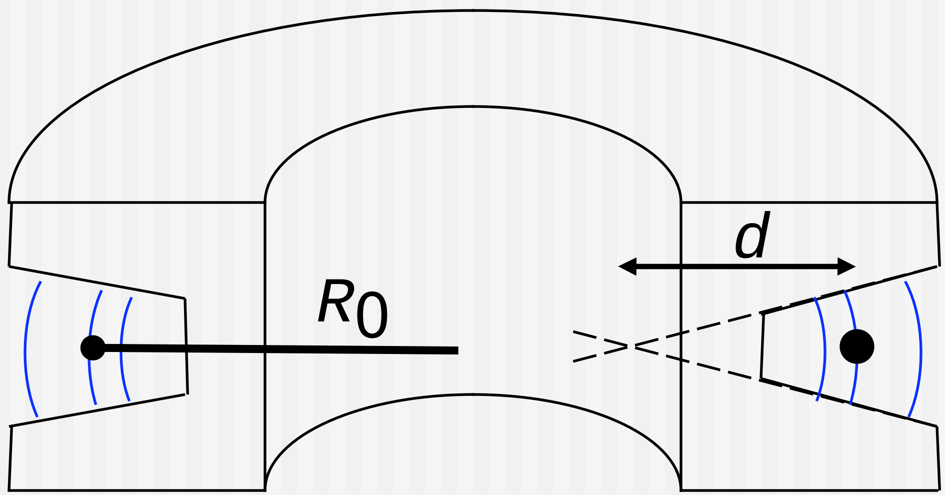
- Lorentz Force $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$
- Magnetic Rigidity $B\rho = \left(\frac{10}{3}\ \frac{\mathrm{T \cdot m}}{\mathrm{GeV/c}}\right) \cdot p$
  - (particle w/ unit charge, *e*)
- Need for Transverse Focusing
  - Uniform bending field stable horizontally, unstable vertically; not all particles (any?) begin “on” the design
- Reference Trajectory and Local Coordinate System



Weak Focusing System

(as it has come to be known…)

- Field varies with radius: $$B = B_0 \left(\frac{R_0}{r} \right)^n$$



n is determined by adjusting the opening angle between the poles

$$n \approx \frac{R_0}{d}$$

$$d = \infty,\ n = 0$$

$$d = R_0,\ n = 1$$

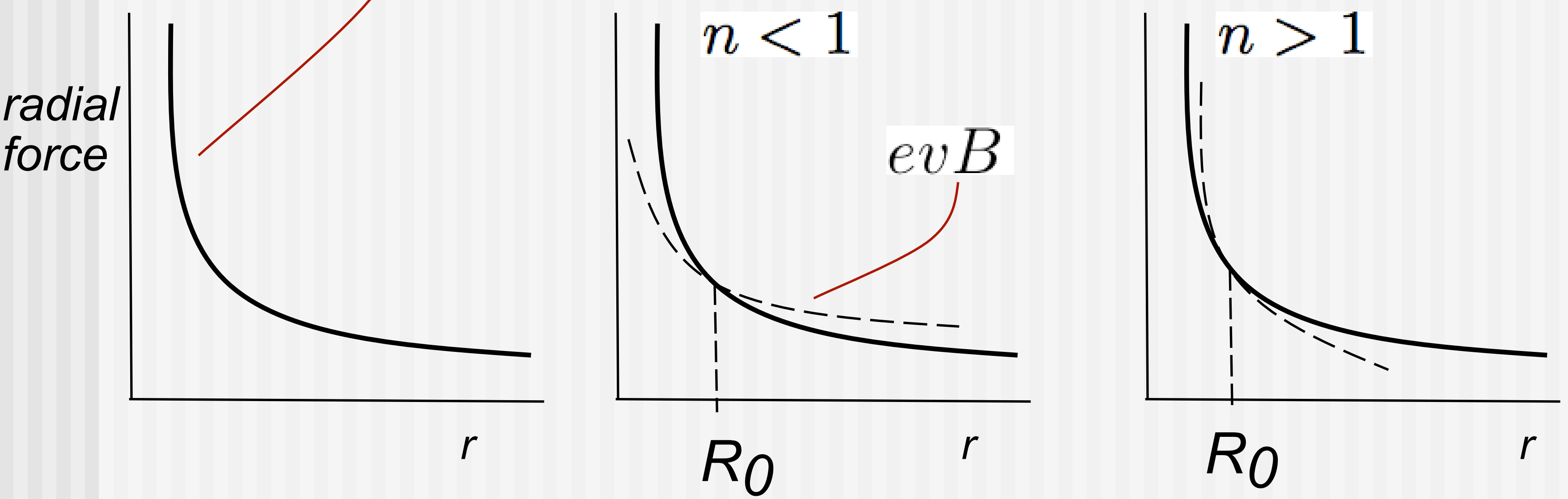
Weak Focusing System

$$B = B_0 \left(\frac{R_0}{r}\right)^n$$

"field index"

Centr. Force: $\frac{mv^2}{r} = evB_0$

$$n \equiv -\frac{R_0}{B_0}\left(\frac{\partial B}{\partial r}\right)_{r=R_0}$$



Weak Focusing:

Differential Equations

Radial:

$$\begin{aligned}
\gamma m(\ddot{r} - r\dot{\theta}^2) &= -evB_y = -evB_0\left(1 - n\cdot\frac{x}{R_0}\right) \\
\gamma m\ddot{r} &= \gamma m\frac{v^2}{r} - evB_0\left(1 - n\cdot\frac{x}{R_0}\right) \\
\ddot{r} &= \frac{v^2}{R_0}\left(1 - \frac{x}{R_0}\right) - \frac{ev^2B_0}{\gamma mv}\left(1 - n\cdot\frac{x}{R_0}\right) \\
\ddot{x} &= \frac{v^2}{R_0}\left(1 - \frac{x}{R_0}\right) - \frac{v^2}{R_0}\left(1 - n\cdot\frac{x}{R_0}\right) \\
\ddot{x} &= -\left(\frac{v}{R_0}\right)^2(1-n)x
\end{aligned}$$

Vertical:

$$\begin{aligned}
\gamma m\ddot{y} &= evB_x \\
&= evB_0\left(-n\cdot\frac{x}{R_0}\right) \\
\ddot{y} &= \frac{ev^2}{\gamma mv}B_0\left(-n\cdot\frac{x}{R_0}\right) \\
&= -n\left(\frac{v}{R_0}\right)^2 x
\end{aligned}$$

- $\ddot{x} + \omega_0^2(1-n)x = 0$

$\ddot{y} + \omega_0^2 nx \;=\; 0$

$\boxed{\text{must have } 0 \le n \le 1 \text{ for stability}}$

- Betatron Tune
  - # osc.'s per turn: $\nu_x = \sqrt{1-n}, \quad \nu_y = \sqrt{n}$

Maximum Excursions

- Solution is Simple harmonic Oscillator:

$$\boxed{\begin{aligned} x &= \text{displacement from} \\ &\quad \text{design trajectory} \\ x' &= dx/ds = \text{slope} \\ &\quad \text{w.r.t. design trajectory} \end{aligned}}$$

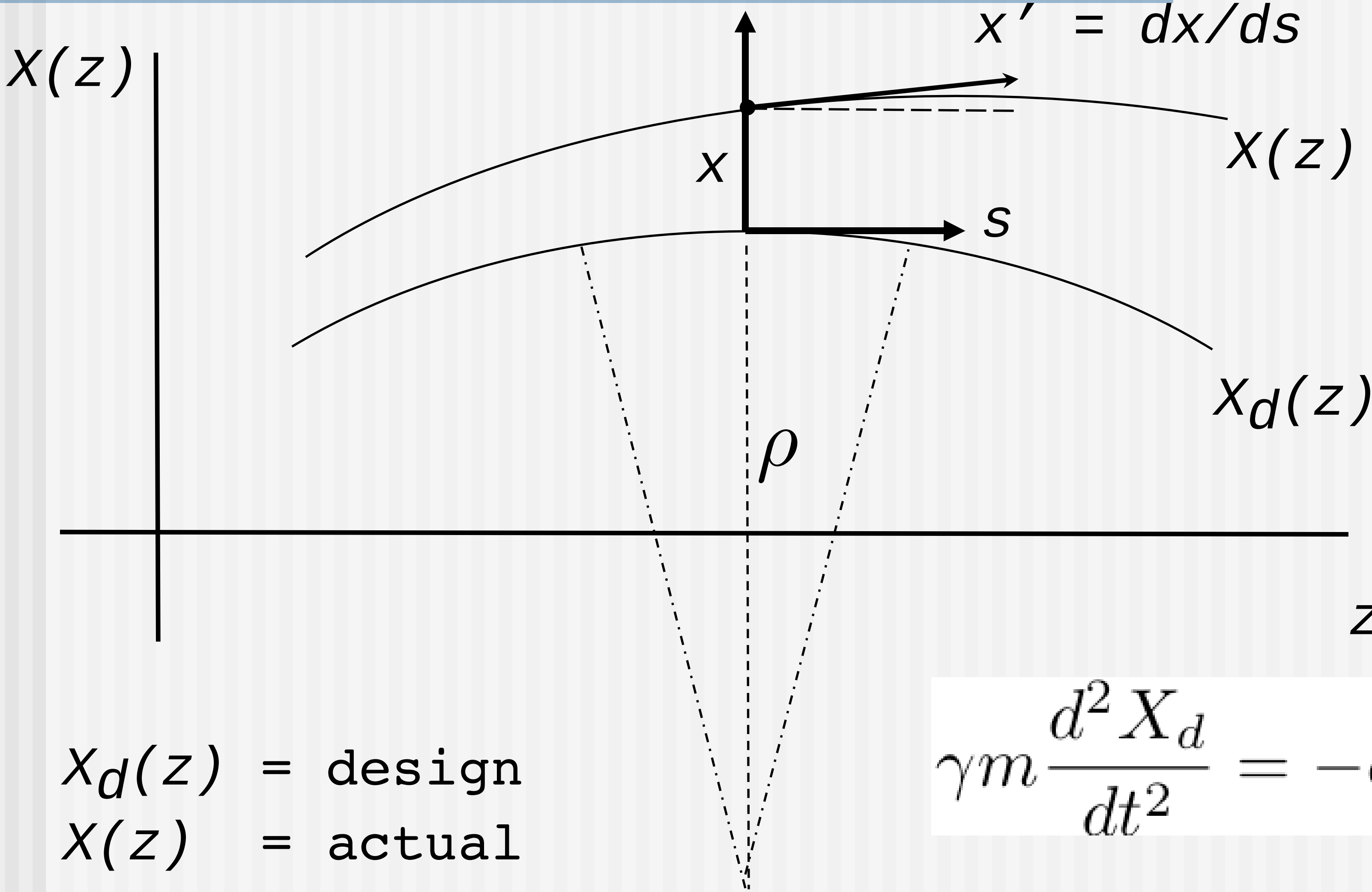
$$\begin{aligned} \ddot{x} + \omega_0^2 \nu^2 x &= 0 \\ v^2 x'' + \omega_0^2 \nu^2 x &= 0 \\ x'' + \left(\frac{\nu}{R_0}\right)^2 x &= 0 \end{aligned}$$

$$\Longrightarrow$$

$$x(s) = x_0 \cos\left(\frac{\nu}{R_0} s\right) + x_0' \frac{R_0}{\nu} \sin\left(\frac{\nu}{R_0} s\right)$$

- For given angular deflection, Maximum Excursion: $\boxed{x_{max} = \frac{R_0}{\nu} x_0'}$
  - Note: 0 < tune < 1,
  - Thus, due to limited range of *n*, then as *R* (*i.e.*, energy) got large, so did the required apertures

Guide Fields and Linear Focusing Fields



$x_d(z)$ = design
$X(z)$ = actual

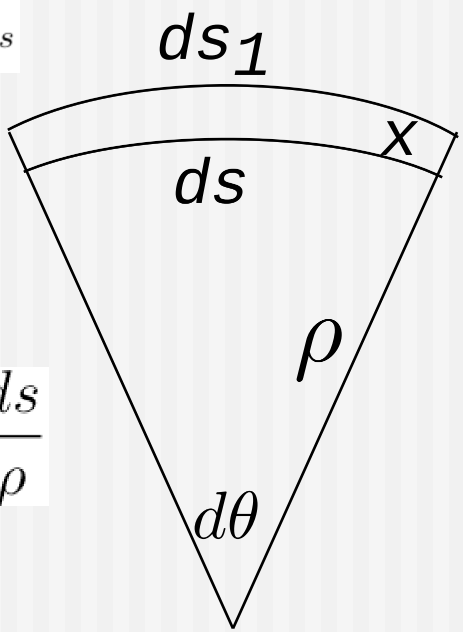
$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Linear Restoring Forces

- Assume linear guide fields: --
-
- Look at radial motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$



$$\frac{ds_1}{\rho + x} = \frac{ds}{\rho}$$

$$\begin{aligned}
\gamma m \frac{d^2(X_d)}{dt^2} &= -e v_s B_0 \\
\gamma m \frac{d^2(X_d + x)}{dt^2} &= -e v_{s1} B_y(X) \\
\gamma m (X_d'' + x'') v_s^2 &= -e v_{s1} B_y(X) \\
\gamma m v_s x'' &= -e \frac{v_{s1}}{v_s} B_y + e B_0 \\
\gamma m v_s x'' &= -e \left[B_y \left(1 + \frac{x}{\rho} \right) - B_0 \right] \\
x'' &= -\frac{e}{p} \left[(B_y - B_0) + B_y \frac{x}{\rho} \right] \\
&\approx -\frac{1}{B\rho} \left[B'x + B_0 \frac{x}{\rho} \right]
\end{aligned}$$

Hill's Equation

- Then, for vertical motion:

$$\begin{aligned}
\gamma m \frac{d^2(Y_d)}{dt^2} &= 0 \\
\gamma m \frac{d^2(Y_d + y)}{dt^2} &= e v_{s1} B_x(Y) \\
\gamma m v_s^2 y'' &= e v_{s1} B_x(Y) \\
\gamma m v_s y'' &= e \frac{v_{s1}}{v_s} B_x \\
\gamma m v_s y'' &= e B_x \left(1 + \frac{x}{\rho}\right) \\
y'' &= \frac{e}{p}\left[B_x(1 + \frac{x}{\rho})\right] \\
&\approx \left(\frac{B'}{B\rho}\right) y
\end{aligned}$$

  - So we have,

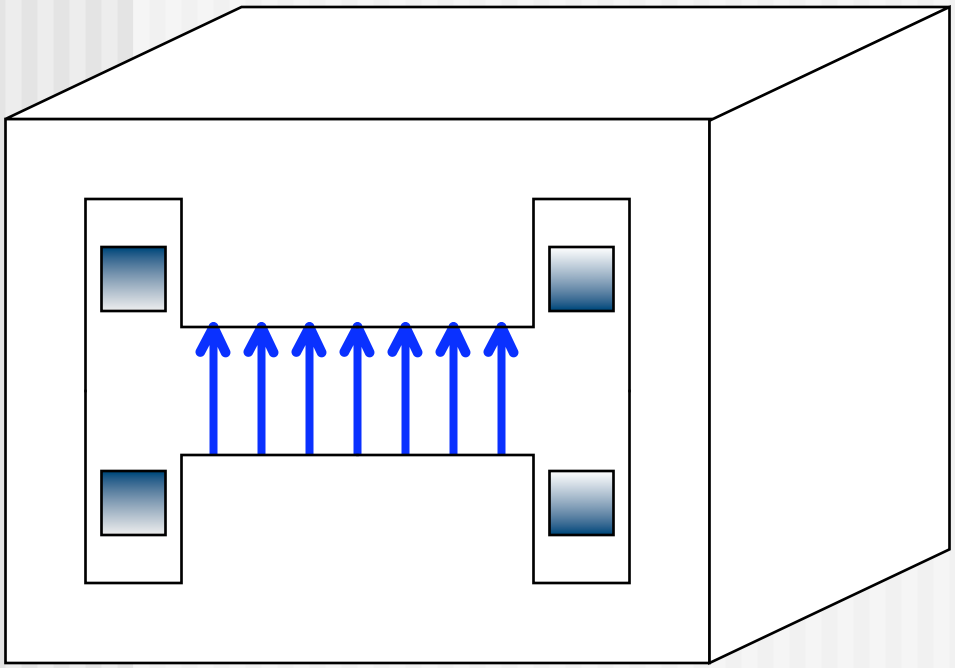
    to lowest order,

$$\begin{aligned}
x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right) x &= 0 \\
y'' - \left(\frac{B'}{B\rho}\right) y &= 0
\end{aligned}$$

General Form: $\longrightarrow$ $\boxed{x'' + K(s)x = 0}$

Hill's Equation

Magnetic Elements



$B_y = B_0$

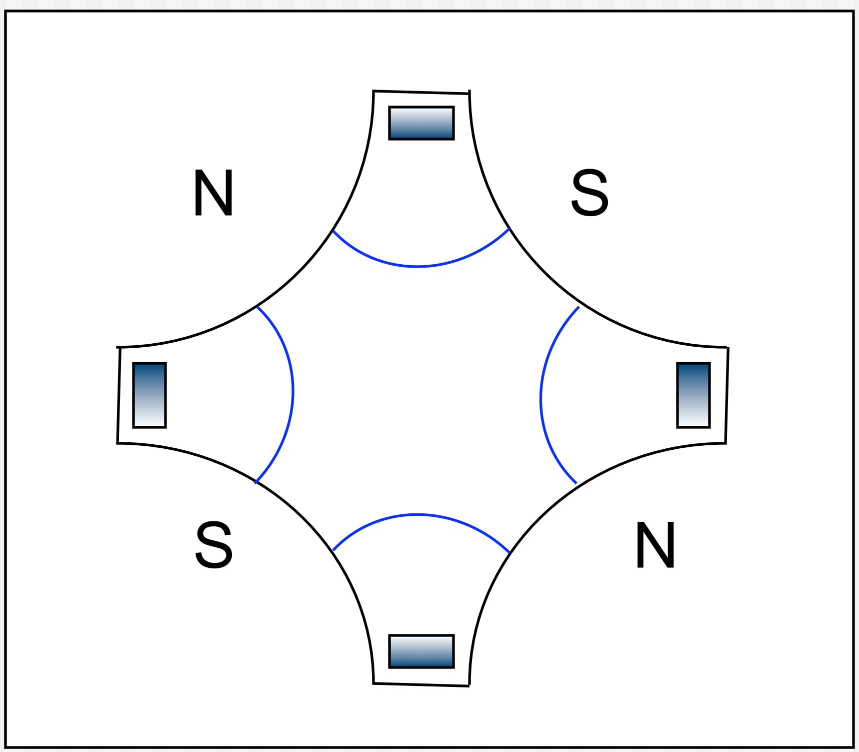
$B_x = 0$

(Dipole Magnet)

$B_y = B'x$

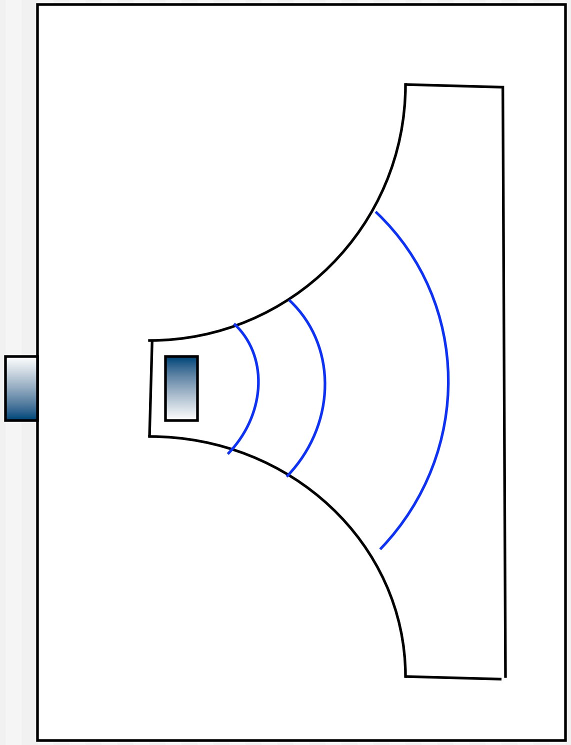
$B_x = B'y$

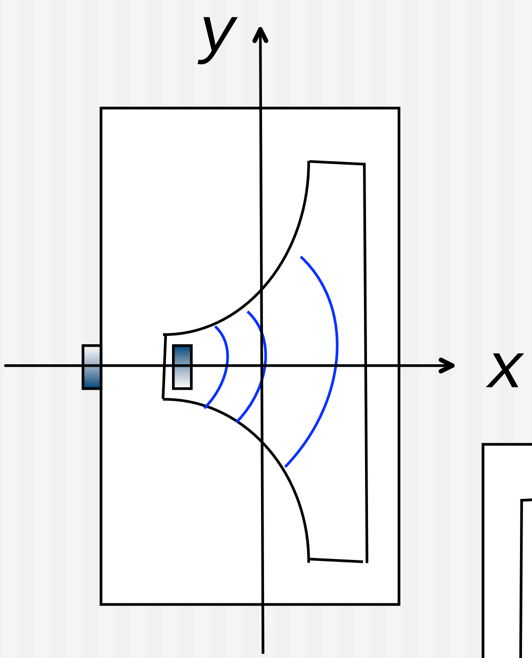
(Quadrupole Magnet)



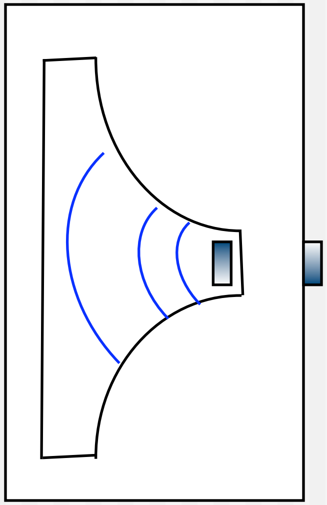
Gradient Magnets

$$B_y = B_0 + B'x$$
$$B_x = B'y$$





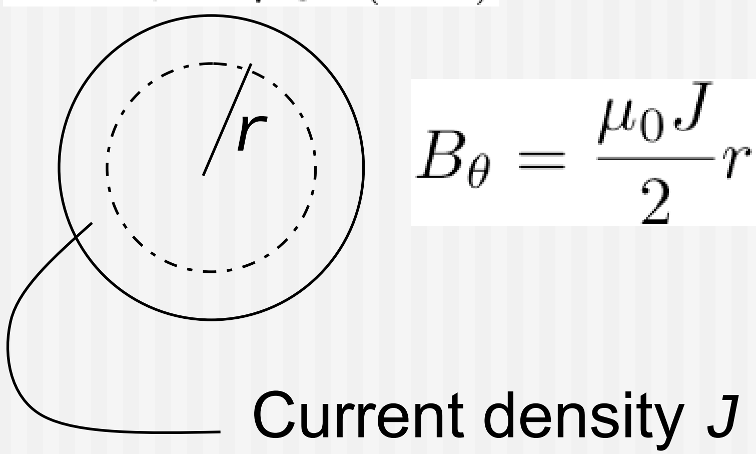
"Alternating
Gradient"



Superconducting Magnets

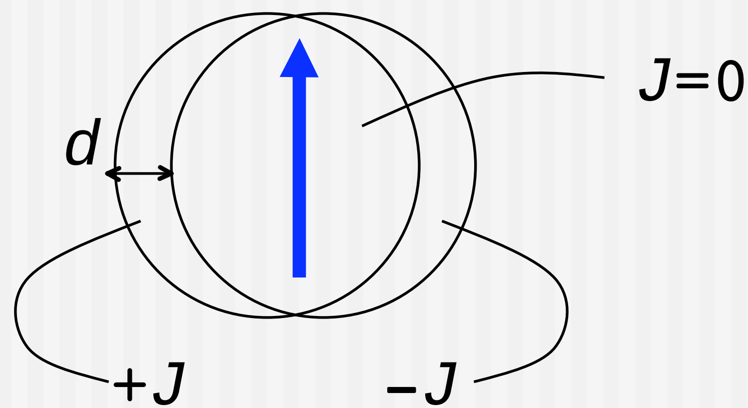
- Here, field is not shaped by iron pole tips, but rather is shaped by placement of the conductor
- Example: dipole magnet…

$$2\pi r B_\theta = \mu_0 J(\pi r^2)$$



$$B_\theta = \frac{\mu_0 J}{2} r$$

$$B_y = \frac{\mu_0 J}{2} d, \quad B_x = 0$$



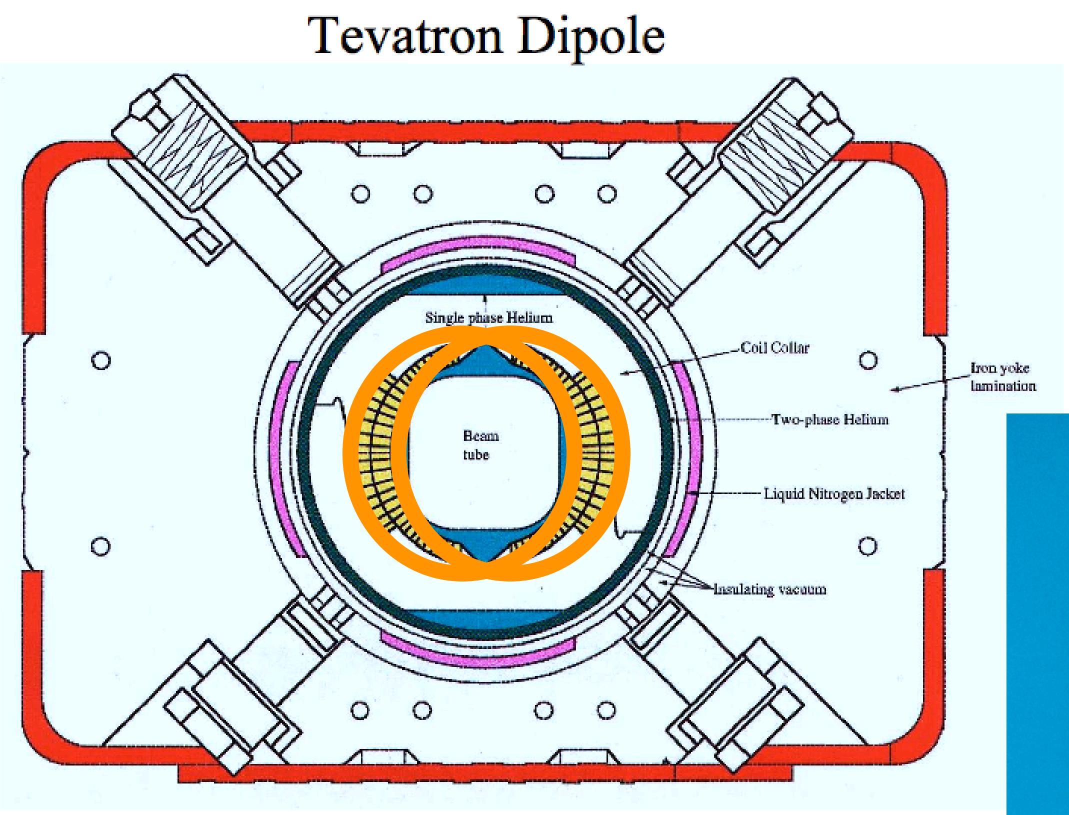
Superconducting Magnets

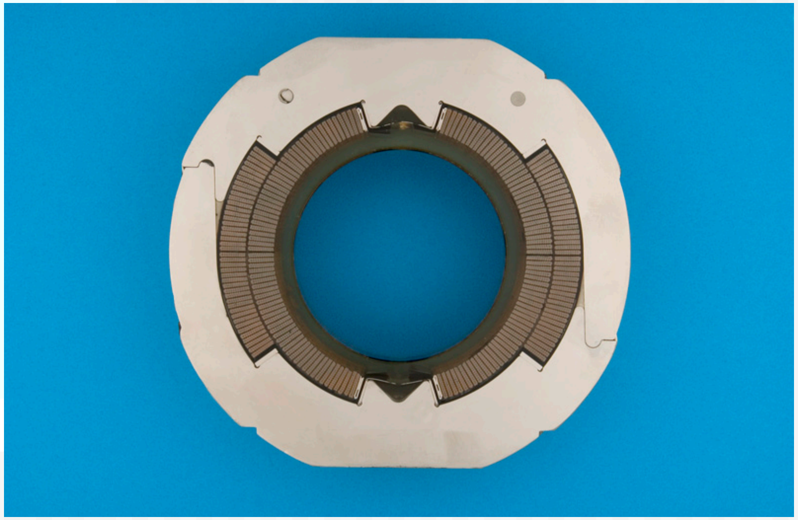
- Example:
  - $J \sim 1000\ \text{A/mm}^2$, $d \sim 1$ cm

    $\Rightarrow B \sim (2\pi\ 10^{-7})(1000*10^6)(10^{-2}) = 2\pi$ Tesla
- Tevatron -- ~4.4 Tesla
- SSC (above parameters) -- 6.6 Tesla
- LHC -- 8 Tesla
- LBNL model magnet -- 16 Tesla

Note: Higher fields a “plus,” but field quality typically easier to control with iron pole tips shaping the field …

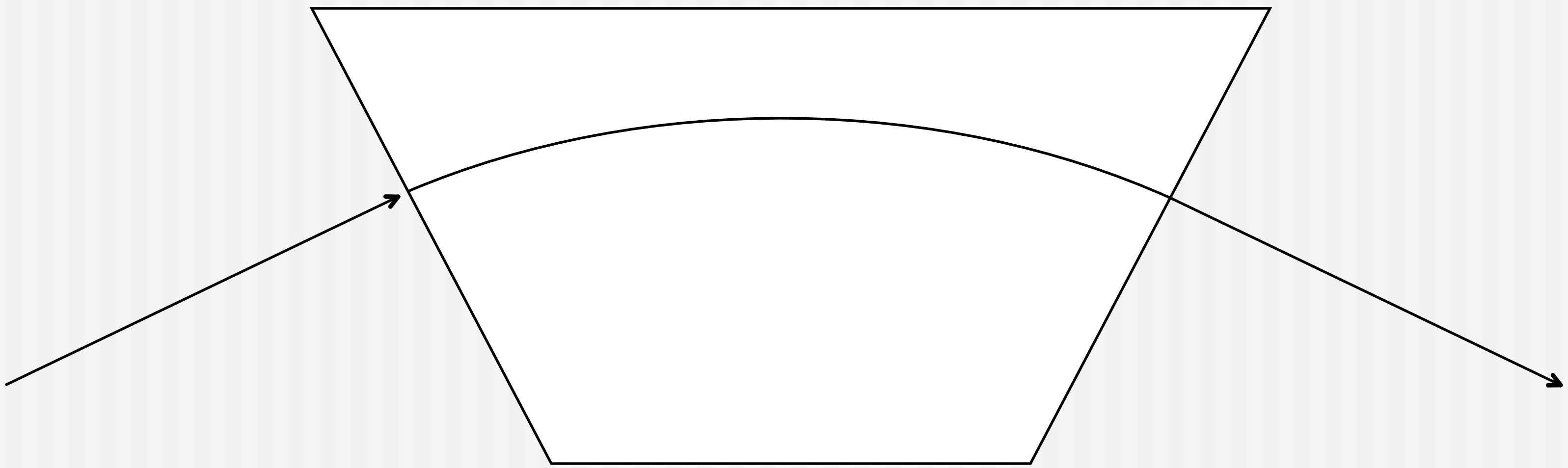
Tevatron Dipole Magnet





Sector Magnets

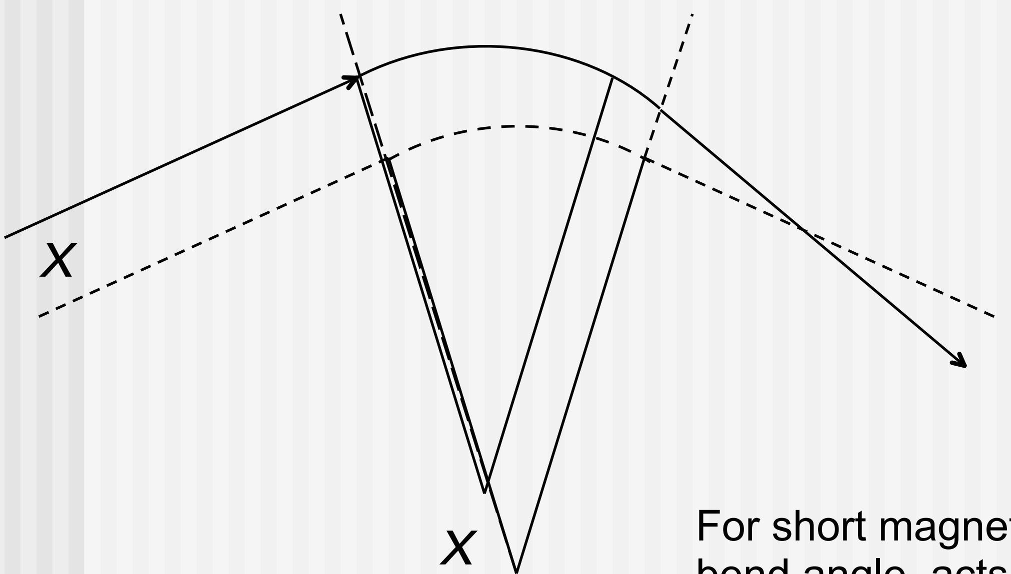
- Sector Dipole Magnet: "edge" of magnetic field is perpendicular to incoming/outgoing design trajectory:



Field points "*out of the page*"

Sector Magnets & Sector Focusing

- Incoming ray displaced from ideal trajectory will experience more/less bending field, thus is "focused" toward axis in the *bend plane*:



$$\text{Extra path length} = ds = d\theta\ x$$
$$\text{so extra bend angle} = dx' = -ds/\rho$$
$$dx' = -(d\theta/\rho)x = -(1/\rho^2)x\ ds$$
$$\text{or, } x'' = -(1/\rho^2)x$$

$$\text{Thus, } K_x = 1/\rho^2,\ K_y = 0.$$

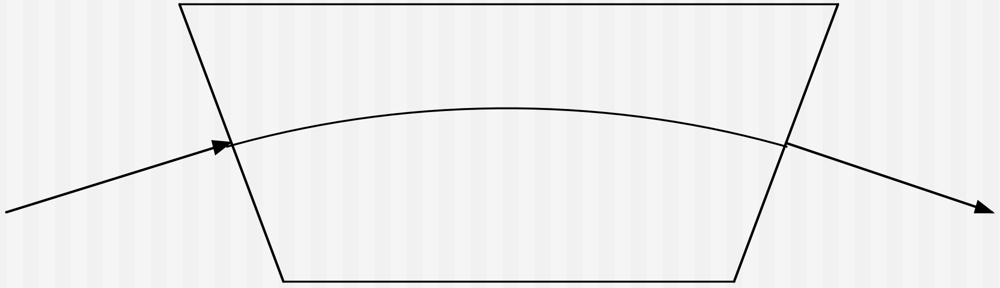
(as seen previously, with B' = 0)

For short magnet with small bend angle, acts like lens in the bend plane with

$$\frac{1}{f_x} = \frac{\theta}{\rho}$$

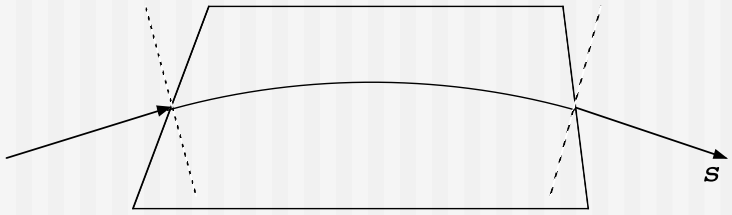
Edge Focusing

- In an ideal *sector magnet*, the magnetic field begins/ends exactly at $s = 0, L$ independent of transverse coordinates x, y relative to the design trajectory.
  - *i.e.*, the face of the magnet is perpendicular to the design trajectory at entrance/exit



Edge Focusing

- However, could (and often do) have the faces at angles *w.r.t.* the design trajectory -- provides “edge focusing”
-
-
-

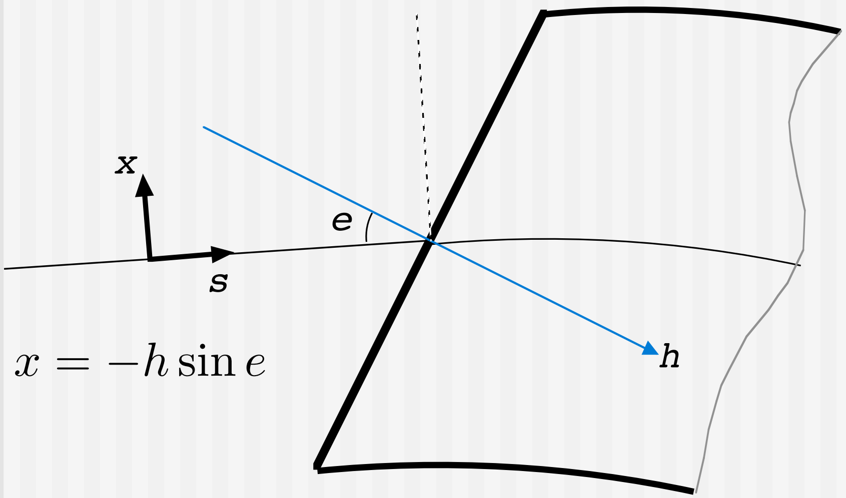


- Since our transverse coordinate *x* is everywhere perpendicular to *s*, then a particle entering with an offset will see more/less bending at the interface...

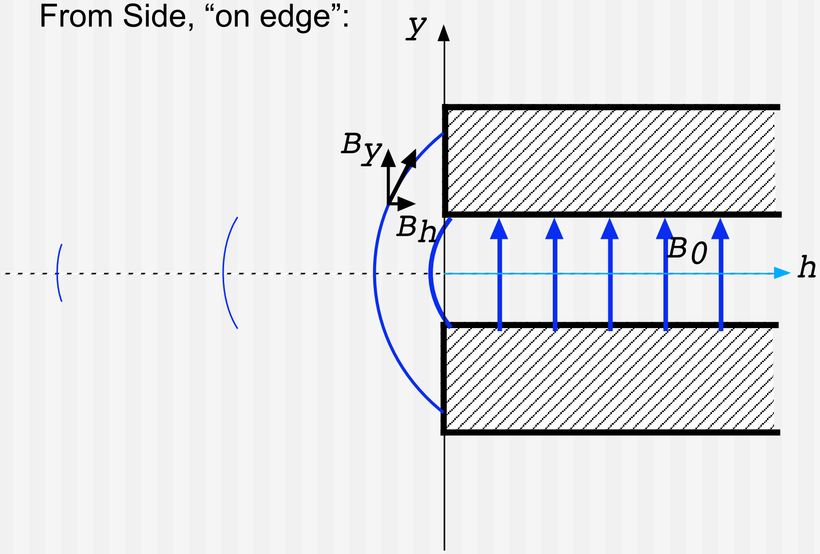
So, How to Model Edges?

- In many cases, can consider edge effects to be perturbations to main motion, and treat as “impulse” kicks -- a “hard edge model” (can do better modeling, if required...)

From Above:

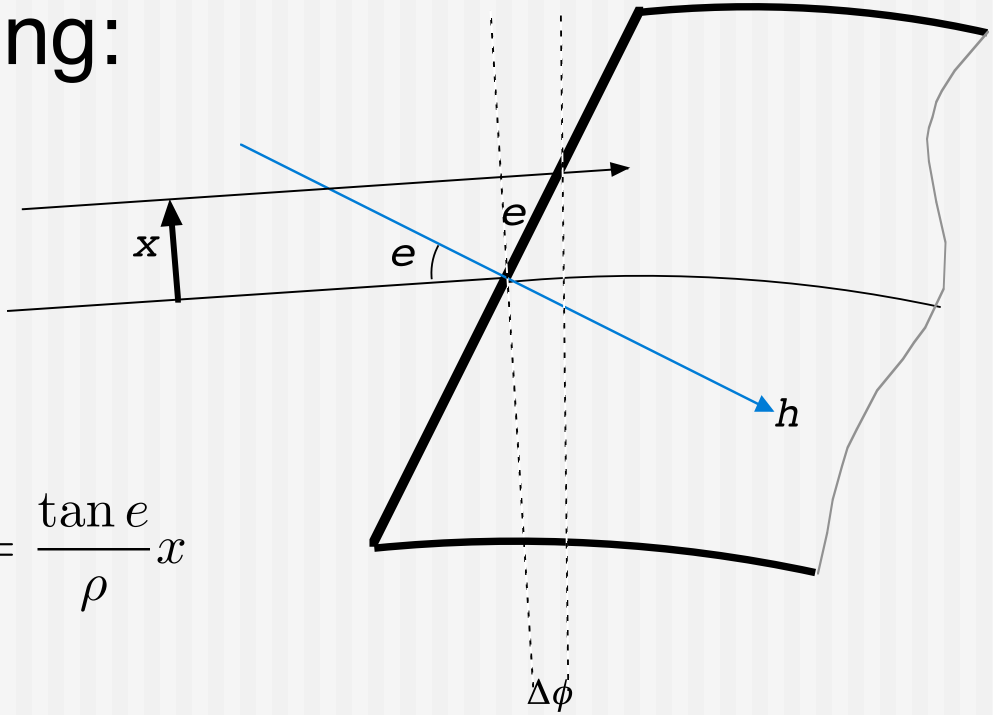


From Side, “on edge”:



Edge Focusing -- radial

- Radial Defocusing:



$$\Delta x' = \Delta\phi = \frac{\Delta s}{\rho} = \frac{x \tan e}{\rho} = \frac{\tan e}{\rho} x$$

- So, for positive *x*, design trajectory "curves away" before particle reaches edge of magnetic field; thus, "defocusing" effect
- Similarly, upon exit

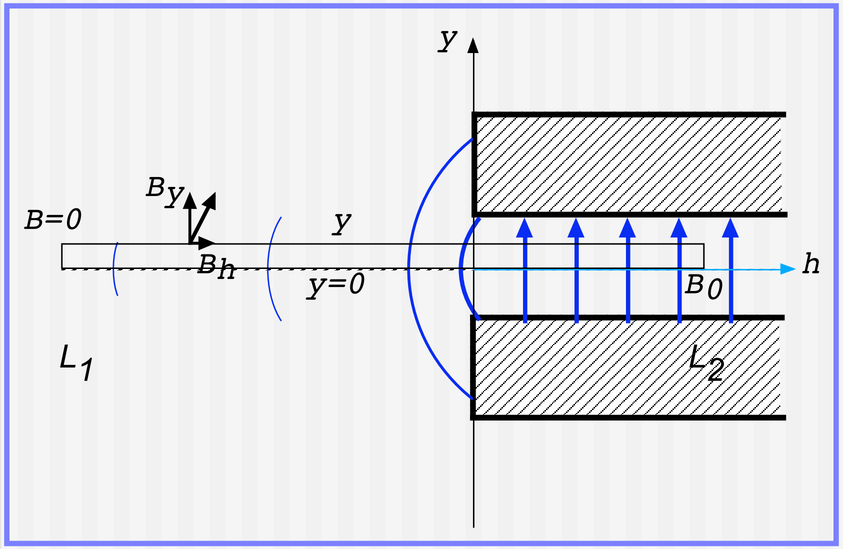
Edge Focusing -- axial

- Vertical (axial) Focusing:

$$\begin{aligned}\Delta y' &= \frac{\Delta p_y}{p} = \frac{ev\int B_x(y)ds}{pv} = \frac{1}{B\rho}\int B_x ds \\ &= -\frac{\sin e}{B\rho}\int B_h ds = -\frac{\tan e}{B\rho}\int (B_h\cos e)ds \\ &= -\frac{\tan e}{B\rho}\int_{L_1}^{L_2}\vec{B}\cdot\vec{ds}\end{aligned}$$

$$B_x = -B_h \sin e$$

$$B_x(y=0) = 0$$



$$\oint \vec{B}\cdot\vec{ds} = 0 + \int_{L_1}^{L_2}\vec{B}\cdot\vec{ds} - B_0\cdot y + 0 = 0$$

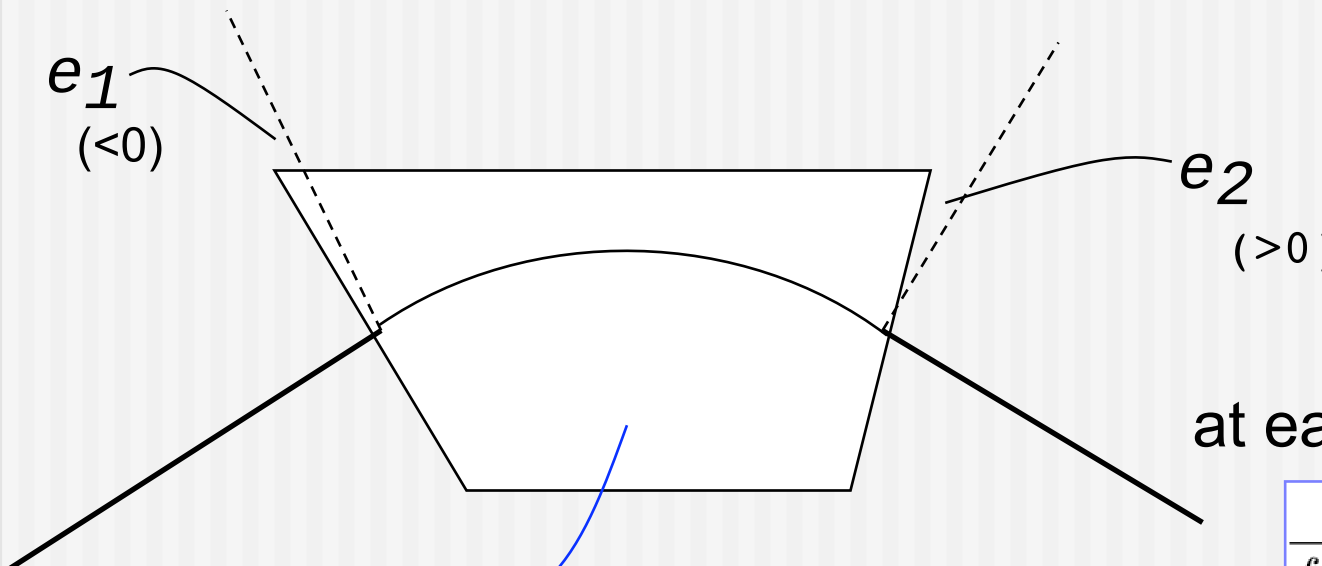
$$\Downarrow$$

$$\int_{L_1}^{L_2}\vec{B}\cdot\vec{ds} = yB_0$$

$$\Longrightarrow \Delta y' = -\left(\frac{\tan e}{B\rho}\right)B_0 y = -\frac{\tan e}{\rho}y$$

Total Bend Magnet: Sector + Edges

- Treat arbitrary edge angles as separate "lenses" at each end of a sector magnet…



$$\boxed{\text{inside, } K_x = 1/\rho^2, \quad K_y = 0}$$

at each edge:

$$\boxed{\frac{1}{f_{x,y}} \approx \pm\frac{\tan e}{\rho}}$$

$$M_{mag} = M_{e_2} M_{sect} M_{e_1}$$

Rectangular Bending Magnet

- "Rectangular" Dipole Magnet:

In bending plane, each edge acts as a lens with focal length:

$$\frac{1}{f} = -\frac{\theta/2}{\rho} = -\frac{\theta}{2\rho}$$

For Sector Magnet, then

hor: $\frac{1}{f_x} \approx \frac{\theta}{\rho}$

ver: $\frac{1}{f_y} \approx 0$

For Rectangular Magnet, then

hor: $\frac{1}{f_x} \approx -\frac{\theta}{2\rho} + \frac{\theta}{\rho} - \frac{\theta}{2\rho} = 0$

ver: $\frac{1}{f_y} \approx \frac{\theta}{2\rho} + \frac{\theta}{2\rho} = \frac{\theta}{\rho}$

Back to Transverse Motion…
Piecewise Method of Solution

- Hill's Equation $$x'' + Kx = 0$$
  - Though $K(s)$ changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, sector mag, quad, edge, ...)

drift
  - K = 0: $$x'' = 0 \longrightarrow x(s) = x_0 + x_0' s$$
  - K > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

Quad, Gradient Magnet, edge, ...
  - K < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L
- K = 0: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$
- $K > 0$: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$
- $K < 0$: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

"Thin Lens" Quadrupole

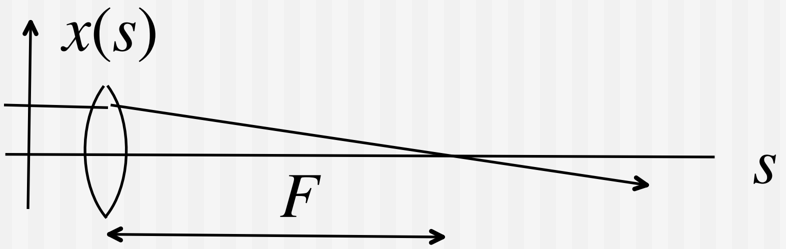
- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics
- Take limit as *L* --> 0, while *KL* remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

  - (similarly, for defocusing quadrupole)

- Valid approx., if *F* >> *L*

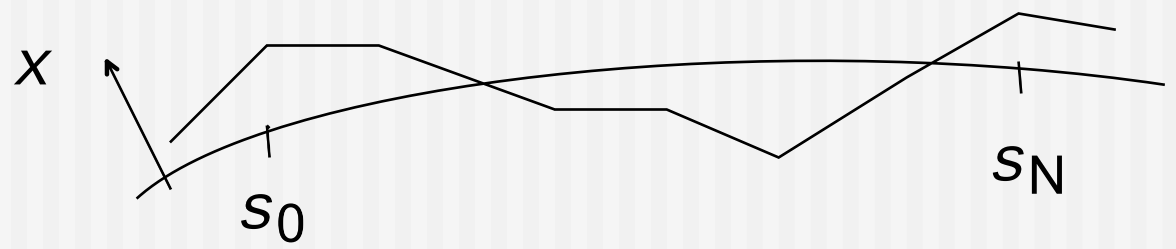
$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$



Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed *via* matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Stability Criterion

- For single pass through a system of elements, above may be enough to describe the system. Suppose the "system" is a synchrotron -- how to show that the motion is stable for many (infinite?) revolutions? (24hrs x 50K rev/sec = …)
- Look at matrix describing motion for one revolution:

$$M = M_N M_{N-1} \cdots M_2 M_1$$

- We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Stability Criterion

V = eigenvector
λ = eigenvalue

$$X_k = M^k X_0 = M^k(AV_1 + BV_2) = A\lambda_1^k V_1 + B\lambda_2^k V_2$$

$$\det M = 1 = \lambda_1\lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ real, gives oscillatory solutions…

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

$$\begin{aligned} &\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\ &\lambda^2 - tr M\lambda + 1 = 0 \\ &\lambda + 1/\lambda = tr M \\ &e^{i\mu} + e^{-i\mu} = 2\cos\mu = tr M \end{aligned}$$

So, μ real (stability)
$\rightarrow |tr M| < 2$

Discovery of Strong Focusing*

- Consider weak focusing system discussed earlier, made up of 2*N* identical gradient magnets. Take every other magnet, turn it around so that the wedge opens inward, and reverse its current.
- Then all magnets have same bend field (in same direction) on the ideal trajectory, but every other magnet has its gradient (*K*) with reversed sign. We now have *N* "cells" of +*K* and -*K.*
- In one degree-of-freedom (vertical, say), each cell has matrix:

$$
\begin{aligned}
M_c &= \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\sqrt{K}L)\cosh(\sqrt{K}L) + \sin(\sqrt{K}L)\sinh(\sqrt{K}L) & \cdots \\ \cdots & \cos(\sqrt{K}L)\cosh(\sqrt{K}L) - \sin(\sqrt{K}L)\sinh(\sqrt{K}L) \end{pmatrix}
\end{aligned}
$$

from which

$$
trM = 2\cos(\sqrt{K}L)\cosh(\sqrt{K}L)
$$

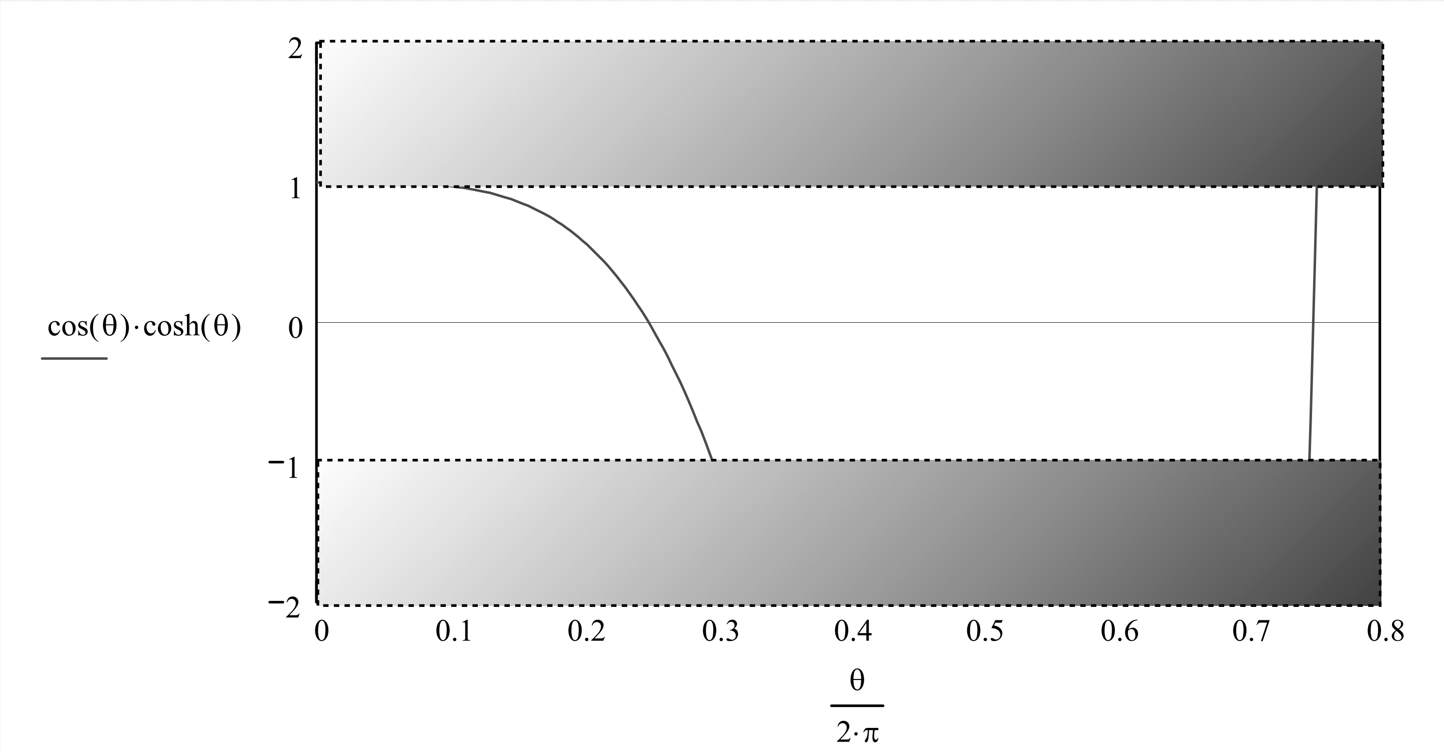
*Courant, Livingston, and Snyder, 1952

So, we need

$$
|\cos(\sqrt{K}L)\cosh(\sqrt{K}L)| < 1
$$

Here, $K = |B'|/B\rho$

The Strong vs. The Weak…



So, could choose $KL^2 \approx [(0.2)2\pi]^2 = 1.58$, say.

$\rightarrow KL^2 = (B'/B\rho)L^2 = (B'\rho/B)(L/\rho)^2 = |n|\ \theta_0^2 = |n|\ (2\pi/2N)^2 = 1.58$

for example, say $N \sim 25$; then $|n| \sim 100 \gg 1$ (*STRONG* focusing!)

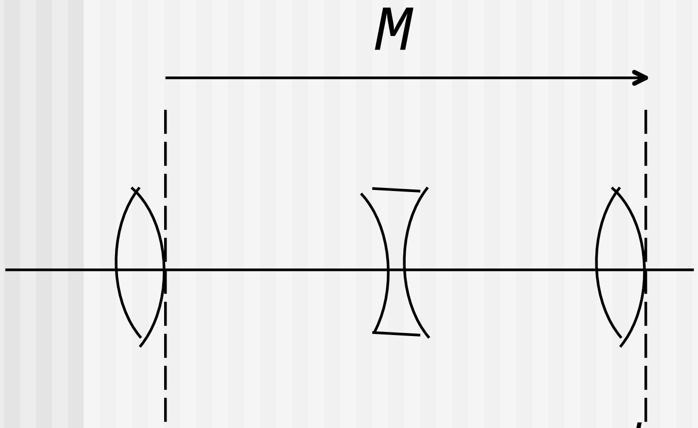
(Note: for Weak Foc. accel., typically $n \sim 2/3 \longrightarrow KL^2 \sim 0.01$!)

Alternating Gradients

- So, now that we can accommodate very strong field gradients, and alternate them over short distances, the extent of radial and vertical excursions becomes decoupled from the orbital radius of the accelerator.
-
-
- The announcement of the AG concept came in 1952, and was immediately applied at Cornell in a 1 GeV electron synchrotron being constructed (Wilson, *et al.*), the world's highest energy at the time. This eventually led to the design and construction of the PS at CERN (1958) and the AGS at Brookhaven National Lab (1960), increasing particle energies to the 30 GeV range. Strong Focusing has been at the heart of every forefront accelerator ever since.

Application to FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $trM = 2 - L^2/F^2$ and thus, for stability,

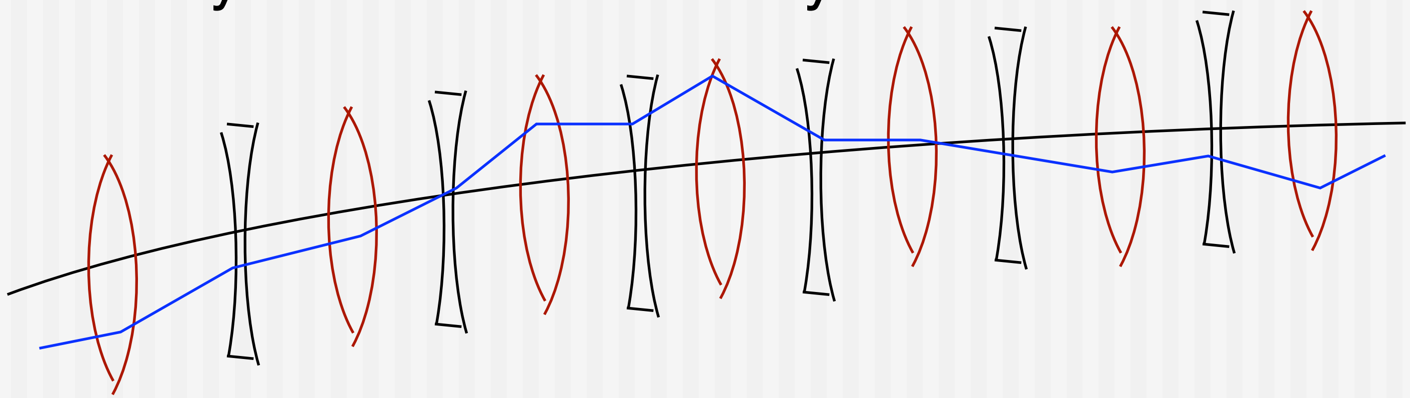
$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

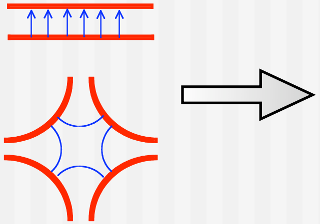
$$F > L/2$$

Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing fields, then in principal can make a synchrotron of arbitrary size

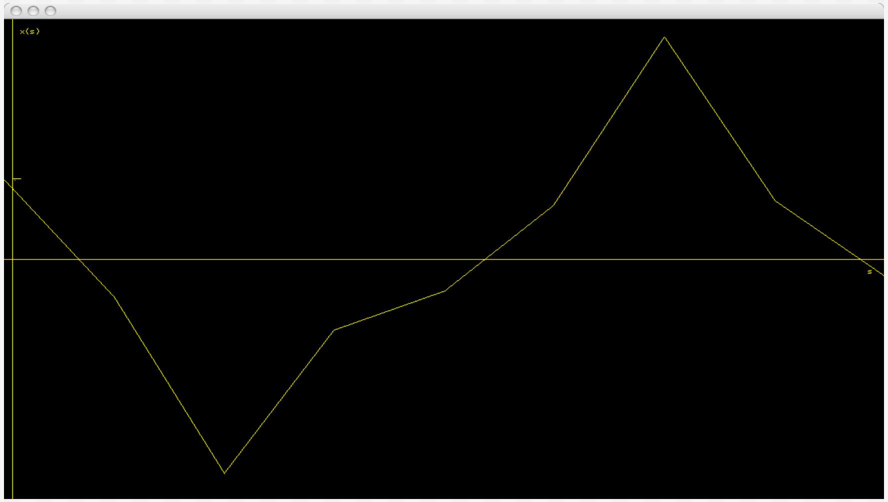


- Can "separate" the bending and focusing "functions"
- First synchrotron to use alternating gradient "thin lenses" + dipole magnets:
  - Fermilab Main Ring

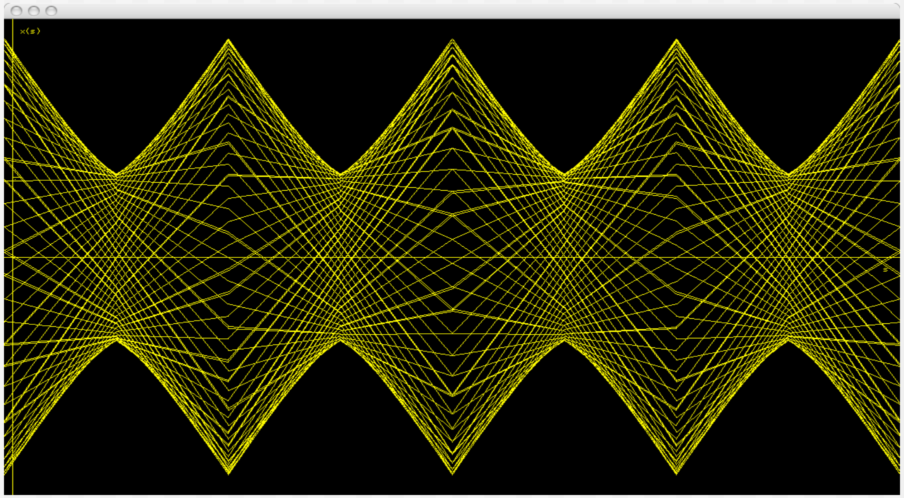


The Notion of an Amplitude Function…

- Track a single particle through a system of FODO cells



- Repeat, representing multiple passages around a synchrotron



Can we describe the maximum amplitude of particle excursions in analytical form?

of course!