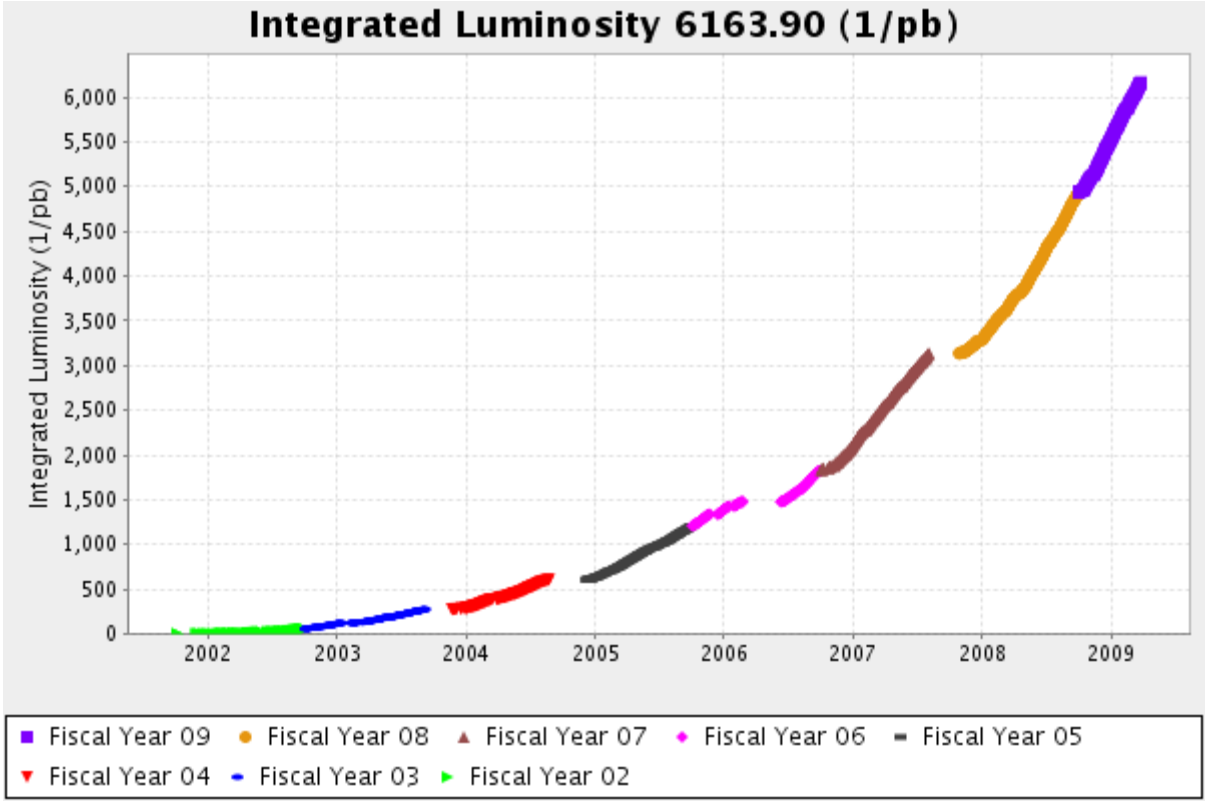
Accelerator Physics Developments for Tevatron Run II



Objective

- The success of Tevatron Run II is based on advances in
  - the accelerator physics,
  - as well as, on the excellence and advances in
    - engineering
    - instrumentation
    - and machine operation.
- Lectures are devoted to review the main advances in Accelerator physics which contributed to the luminosity growth and/or improvement of the Tevatron complex operations.
- The lectures are aimed for the Run II participants who would like to deepen their understanding of the accelerator physics.
- The level of the presented material corresponds to the advanced course of accelerator physics but at the same time we would like to present material so that it could be understandable for less prepared listeners
- It will be presented by real participants

List of the lectures (can be modified on the road)

1. Linear optics fundamentals and linear optics with coupling between degrees of freedom (Lebedev)
2. Linear optics measurements (closed orbit distortion, turn-by turn) (Gianfelice-Wendt)
3. Non-linear dynamics and its measurements in Tevatron (Alexahin)
4. Impedances. (Burov)
5. Longitudinal instabilities
6. Transverse instabilities
7. Single and multiple IBS and gas scattering
8. Stochastic cooling and stacking
9. Stochastic cooling hardware and stochastic cooling measurements
10. Electron cooling and its fundamentals
11. Cooling and beam manipulations in Recycler
12. Antiproton production
13. Luminosity evolution in Tevatron
14. Beam-beam effects and their simulations
15. Instrumentation, feedbacks and their development for Run II

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Lecture 1

Linear Optics Fundamentals and Linear Optics with Coupling Between Degrees of Freedom

Valeri Lebedev

Contents

- Equations of motion, Symplecticity condition, Liouville theorem
- Eigen-vectors and mode emittances of multidimensional motion,
- Parameterization of single dimension motion, Twiss parameters, parameterization of multidimensional motion
- X-Y coupled motion, Edwards-Teng and extended Mais-Ripken parameterizations
- Perturbation theory for symplectic motion

Where coupling effects are important?

- Coupling build-in into design
  - Electron cooler
  - P1 line (required by vertical dispersion match)
- Model without coupling is too rough
  - Tevatron
    - large spread of skew-quadrupole component in dipoles
- "Real machine optics" represents measured coupling
  - Debuncher
  - Accumulator
  - Recycler
- Coupling is not negligible in all other transfer lines and machines

What will be discussed

- X-Y coupling
  - X-S coupling can be described the same way
- X-Y-S coupling is straight forward extension
  - Many results of two dimensional coupling theory can be applied directly

Equations of motion

- Linearized equations of two dimensional motion

$$x'' + \left(K_x{}^2 + k\right)x + \left(N - \frac{1}{2}R'\right)y - Ry' = 0 \quad ,$$

$$y'' + \left(K_y{}^2 - k\right)y + \left(N + \frac{1}{2}R'\right)x + Rx' = 0 \quad .$$

$K_{x,y} = eB_{y,x} / Pc$ - dipole

(linear part of beam space charge force can be added too)

$k = eG / Pc$ - quadrupole,

$N = eG_s / Pc$ - skew-quadrupole,

$R = eB_s / Pc$ - longitudinal magnetic field

From mathematics point of view:

It is a system of ordinary linear differential equations with variable coefficients

The equations for x and y motions are closely coupled

Canonical variables

$\hat{\mathbf{x}} = \mathbf{R}\mathbf{x}$ are the canonical variables

Coincide with usual coordinates (x, θ_x, y, θ_y) if $B_s = 0$

- Canonical coordinates: $x = x$, $y = y$
- Canonical momenta: $p_x = \theta_x - \frac{R}{2} y$, $p_y = \theta_y + \frac{R}{2} x$, $\left(\theta_x \equiv \frac{dx}{ds}, \quad \theta_y \equiv \frac{dy}{ds} \right)$

$$\mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix}, \quad \hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix}, \quad R = eB_s / Pc$$

- Here and below we put a cap above transfer matrices and vectors related to the canonical variables.

- Hamiltonian form of the motion equation
  - In the general case the Hamiltonian of linearized motion can be presented as bilinear form: $H \equiv H(\hat{\mathbf{x}}, s) = \frac{1}{2} h_{ij}(s) \hat{x}_i \hat{x}_j$
  - Then, the motion equations are

$$\frac{d}{ds}\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} = \begin{bmatrix} \partial H / \partial p_x \\ -\partial H / \partial x \\ \partial H / \partial p_y \\ -\partial H / \partial y \end{bmatrix} = \begin{bmatrix} h_{2i}\hat{x}_i \\ -h_{1i}\hat{x}_i \\ h_{4i}\hat{x}_i \\ -h_{3i}\hat{x}_i \end{bmatrix} = \mathbf{UHx}, \quad \mathbf{H} = [h_{ij}], \quad \mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

where $\mathbf{U}$ is called the unit symplectic matrix

- For the above equations of motion the Hamiltonian matrix is

$$\mathbf{H} = \begin{bmatrix} K_x^{\,2} + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^{\,2} - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

  - and their matrix form: $\boxed{\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{UH}\hat{\mathbf{x}}}$

Transfer matrix (linear map)

- Solution of system of four first order linear differential equations can be presented as sum of 4 linearly independent vector-functions

$$\begin{bmatrix} x(s) \\ dx(s)/ds \\ y(s) \\ dy(s)/ds \end{bmatrix} = \sum_{k=1}^{4} c_k \begin{bmatrix} \tilde{x}_k(s) \\ \tilde{\theta}_{xk}(s) \\ \tilde{y}_k(s) \\ \tilde{\theta}_{yk}(s) \end{bmatrix}$$

Or in the vector form : $\mathbf{x}(s) = \tilde{\mathbf{X}}(s)\mathbf{C}$

$$\mathbf{x}(s) = \begin{bmatrix} x(s) \\ dx(s)/ds \\ y(s) \\ dy(s)/ds \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \quad \tilde{\mathbf{X}}(s) = \begin{bmatrix} \tilde{x}_1(s) & \tilde{x}_2(s) & \tilde{x}_3(s) & \tilde{x}_4(s) \\ \tilde{\theta}_{x1}(s) & \tilde{\theta}_{x2}(s) & \tilde{\theta}_{x3}(s) & \tilde{\theta}_{x4}(s) \\ \tilde{y}_1(s) & \tilde{y}_2(s) & \tilde{y}_3(s) & \tilde{y}_4(s) \\ \tilde{\theta}_{y1}(s) & \tilde{\theta}_{y2}(s) & \tilde{\theta}_{y3}(s) & \tilde{\theta}_{y4}(s) \end{bmatrix}$$

- Let's express the solution through it's initial value (s = 0):

$$\boxed{\mathbf{x}(s) = \mathbf{M}(0,s)\mathbf{x}(0)}$$

Taking into account that $\mathbf{x}(0) = \tilde{\mathbf{X}}(0)\mathbf{C}$ we obtain

$$\tilde{\mathbf{X}}(s)\mathbf{C} = \mathbf{x}(s) = \mathbf{M}(0,s)\mathbf{x}(0) = \mathbf{M}(0,s)\tilde{\mathbf{X}}(0)\mathbf{C}$$

$$\Rightarrow \boxed{\mathbf{M}(0,s) = \tilde{\mathbf{X}}(s)\tilde{\mathbf{X}}(0)^{-1}}$$

- **M**(0,s) is transfer matrix from s=0 to s

Unit symplectic matrix

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Antisymmetric

  $\mathbf{U} = -\mathbf{U}^T$

- Useful properties

  $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}$ $\qquad$ $\mathbf{U}\mathbf{U} = -\mathbf{I}$

  $\det \mathbf{U} = 1$

  $\mathbf{a}^T\mathbf{U}\mathbf{a} = 0$ for any vector $\mathbf{a}$

- In the literature it is also denoted as $\mathbf{S}$

Symplecticity condition

- Lagrange invariant
  - Remind that the equation of motion is: $\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{U}\mathbf{H}\hat{\mathbf{x}}$

$$\Rightarrow \quad \frac{d}{ds}\left(\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2\right) = \frac{d\hat{\mathbf{x}}_1{}^T}{ds}\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{U}\frac{d\hat{\mathbf{x}}_2}{ds} = \hat{\mathbf{x}}_1{}^T\mathbf{H}^T\mathbf{U}^T\mathbf{U}\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1{}^T\mathbf{U}\mathbf{U}\mathbf{H}\hat{\mathbf{x}}_2 = 0$$

and, consequently, $\boxed{\hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2 = \text{const}}$

- Symplecticity condition
  - Remind: transfer matrix for canonical variables $\hat{\mathbf{x}} = \hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0$

$$\Rightarrow \quad \hat{\mathbf{x}}_1{}^T\mathbf{U}\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1{}^T\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_2 = \text{const}$$

  - As the above equation is satisfied for any $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ it yields

$$\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s) = \mathbf{U} \quad \Rightarrow \quad \boxed{\hat{\mathbf{M}}(s_1,s_2)^T\mathbf{U}\hat{\mathbf{M}}(s_1,s_2) = \mathbf{U}}$$

    - There 16 equations but only 6 are independent because $\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)$ is antisymmetric
    - i.e. there are six equations which bound transfer matrix elements

$\Rightarrow$ Only 10 of 16 transfer matrix elements are independent

Properties of symplectic matrices

$$\boxed{\hat{\mathbf{M}}^T \mathbf{U} \hat{\mathbf{M}} = \mathbf{U}}$$

$$\mathbf{U}(\ldots = \ldots) \quad \Rightarrow \quad \mathbf{U}\hat{\mathbf{M}}^T \mathbf{U}\hat{\mathbf{M}} = -\mathbf{I}$$

$$(\ldots = \ldots)\hat{\mathbf{M}}^{-1} \quad \Rightarrow \quad \boxed{\mathbf{U}\hat{\mathbf{M}}^T \mathbf{U} = -\hat{\mathbf{M}}^{-1}}$$

$$\hat{\mathbf{M}}(\ldots = \ldots) \quad \Rightarrow \quad \hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T \mathbf{U} = -\mathbf{I}$$

$$\mathbf{U}(\ldots = \ldots) \quad \Rightarrow \quad \boxed{\hat{\mathbf{M}}\mathbf{U}\hat{\mathbf{M}}^T = \mathbf{U}}$$

Liouville's theorem

- Transfer matrix determinant

  Computing determinant of both sides of symplecticity condition

  $\hat{\mathbf{M}}(0,s)^T \mathbf{U}\hat{\mathbf{M}}(0,s) = \mathbf{U}$ we obtain $\left(\det\left(\hat{\mathbf{M}}(0,s)\right)\right)^2 = 1$

  Taking into account that $\hat{\mathbf{M}}(0,0) = \mathbf{I}$ we obtain $\boxed{\left(\det\left(\hat{\mathbf{M}}(0,s)\right)\right) = 1}$

- Liouville's theorem

  Prove that the phase volume, $\Gamma = \int_V dx d\theta_x dy d\theta_y$, is conserved in the course of the motion, $\mathbf{x}(s) = \mathbf{M}(0,s)\mathbf{x}(0)$. Then,

  $$\Gamma' = \int_{V'} dx' d\theta'_x dy' d\theta'_y = \int_V \frac{\partial(x',\theta'_x,y',\theta'_y)}{\partial(x',\theta'_x,y',\theta'_y)} dx d\theta_x dy d\theta_y = \int_V \det\left(\hat{\mathbf{M}}\right) dx d\theta_x dy d\theta_y = \Gamma$$

- Liouville's theorem is justified also in the general case of non-linear motion: $\boxed{\left(\det\left(\hat{\mathbf{M}}(0,s)\right)\right) = 1}$
  - It works even for case of the stochastic cooling
    - Strong filamentation of the phase space
- Next we will see that the symplecticity additionally results in conservation of the mode emittances (analogs of x and y emittances)

Eigen-vectors and eigen-values

- For circular accelerator the turn-by-turn particle positions can be expressed through the eigen-vectors,

$$\boxed{\hat{\mathbf{M}}\hat{\mathbf{v}}_k = \lambda_k \hat{\mathbf{v}}_k}$$, where $\hat{\mathbf{M}} = \hat{\mathbf{M}}(0,L)$ and $k = 1, 2, 3, 4$

$$\hat{\mathbf{x}}_0 = \sum_{k=1}^{4} c_k \hat{\mathbf{v}}_k \quad \Rightarrow \quad \boxed{\hat{\mathbf{x}}_n = \sum_{k=1}^{4} \lambda_k{}^n c_k \hat{\mathbf{v}}_k}$$

$\Rightarrow$ Stability condition $\lambda_k \le 1$

- Complex conjugate of the eigen-value equation results in that the complex conjugate of eigen-vector is also eigen-vector with complex conjugate eigen-value:

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_k^* = \lambda_k^* \hat{\mathbf{v}}_k^*$$

- Product of eigen-values

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^4 + \ldots + \det(\mathbf{M}) = \lambda^4 + \ldots + \det(\mathbf{M}) = \lambda^4 + \ldots + 1$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) = \lambda^4 + \ldots + \lambda_1\lambda_2\lambda_3\lambda_4$$

Comparing we obtain that $\boxed{\lambda_1\lambda_2\lambda_3\lambda_4 = 1}$

$\Rightarrow$ Stability condition $\boxed{|\lambda_k| = 1}$

- Eigen-values of stable motion
  - For stable betatron motion

    $|\lambda_i| = 1$ and

    $\lambda_i \neq \pm 1$ (integer and half-integer resonance condition)
  - Four eigen-values split into two complex conjugate pairs
- Symplectic orthogonality of eigen-vectors

  For any two eigen-vectors the symplecticity condition yields

$$0 = \lambda_j \hat{\mathbf{v}}_j^{\,T} \mathbf{U}\left(\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_i \hat{\mathbf{v}}_i\right) = \left(\hat{\mathbf{M}}\hat{\mathbf{v}}_j\right)^T \mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_j \hat{\mathbf{v}}_j^{\,T} \mathbf{U} \lambda_i \hat{\mathbf{v}}_i = \left(1 - \lambda_j \lambda_i\right) \hat{\mathbf{v}}_j^{\,T} \mathbf{U} \hat{\mathbf{v}}_i$$

i.e. $$\boxed{\left(1 - \lambda_j \lambda_i\right) \hat{\mathbf{v}}_j^{\,T} \mathbf{U} \hat{\mathbf{v}}_i = 0}$$

That yields that for stable motion the only scalar product different from zero is

$$\boxed{\hat{\mathbf{v}}_i^{\,+} \mathbf{U} \hat{\mathbf{v}}_i \neq 0}$$

Note that it is purely imaginary:

$$\left(\hat{\mathbf{v}}^+ \mathbf{U} \hat{\mathbf{v}}\right)^* = \left(\hat{\mathbf{v}}^+ \mathbf{U} \hat{\mathbf{v}}\right)^+ = \hat{\mathbf{v}}^+ \mathbf{U}^+ \hat{\mathbf{v}} = -\hat{\mathbf{v}}^+ \mathbf{U} \hat{\mathbf{v}}$$

Eigen-vectors normalization

- We leave only two of four eigen-vectors in further consideration
  - Two other are complex conjugate and can be omitted

$$\boxed{\begin{array}{ll} \hat{\mathbf{v}}_1^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = -2i \quad , & \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_2 = -2i \quad , \\ \hat{\mathbf{v}}_1^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , & \hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_2 = 0 \quad , \\ \hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , & \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad . \end{array}}$$

- For two top equations
  - Factor of 2 sets correct relation to emittances and phase advances
  - Sign "minus" determines which 2 of 4 eigen-vectors have to be used
- Equations in the second line are actually identities
- 8 independent real variables uniquely determine the eigen-vectors
  - 16 – 2 (arbitrary phases) - 2 (two top eqs.)
    - 4 (two bottom complex eqs.) = 8
- Adding 2 betatron phases we obtain 10 independent real variables describing the motion
  - The same as for the transfer matrix elements considered above

Beam ellipsoid in 4D phase space

- Turn-by-turn particle positions and angles at the lattice beginning

$$\hat{\mathbf{x}} = \mathrm{Re}\left(A_1 e^{-i\psi_1}\hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2}\hat{\mathbf{v}}_2\right)$$

where real parameters, A_1, A_2, ψ_1 and ψ_2 , are the betatron amplitudes and phases.

- Let us introduce the following real matrix: $\boxed{\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right]}$.

- Chosen normalization of eigen-vectors results in that $\hat{\mathbf{V}}$ is a symplectic matrix: $\hat{\mathbf{V}}^T\mathbf{U}\hat{\mathbf{V}} = \mathbf{U}$

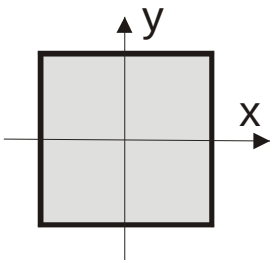
$$\Rightarrow \qquad \hat{\mathbf{V}}^{-1} = -\mathbf{U}\hat{\mathbf{V}}^T\mathbf{U}$$

- Using matrix $\hat{\mathbf{V}}$ allows one to rewrite the top equation in the compact form: $\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\vec{\xi}_A$

where

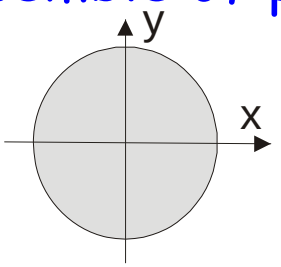
$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \qquad \vec{\xi}_A = \begin{bmatrix} \cos\psi_1 \\ -\sin\psi_1 \\ \cos\psi_2 \\ -\sin\psi_2 \end{bmatrix}.$$

- Single particle phase space:



$$\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\vec{\xi}_A \qquad \vec{\xi}_A = \begin{bmatrix} \cos\psi_1 \\ -\sin\psi_1 \\ \cos\psi_2 \\ -\sin\psi_2 \end{bmatrix}$$

- Ensemble of particles:



$$\vec{\xi} = \begin{bmatrix} \cos\psi_1 \cos\psi_3 \\ -\sin\psi_1 \cos\psi_3 \\ \cos\psi_2 \sin\psi_3 \\ -\sin\psi_2 \sin\psi_3 \end{bmatrix} \qquad \left(\vec{\xi}, \vec{\xi}\right) = 1$$

Vector $\vec{\xi}$ defines sphere with unit radius in 4-D space

$\Rightarrow$ $\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\vec{\xi}$ defines 3D surface of 4D-ellipsoid contained all particles

- Standard form of surface parameterization is a bilinear form

$$\boxed{\hat{\mathbf{x}}^T \boldsymbol{\Xi}\, \hat{\mathbf{x}} = 1}$$

$$\left(\vec{\xi}, \vec{\xi}\right) = 1 \;, \quad \Rightarrow \quad \hat{\mathbf{x}}^T \left(\left(\hat{\mathbf{V}}\mathbf{A} \right)^{-1} \right)^T \left(\hat{\mathbf{V}}\mathbf{A} \right)^{-1} \hat{\mathbf{x}} = 1$$

$$\hat{\mathbf{V}}^{-1} = -\mathbf{U}\hat{\mathbf{V}}^T\mathbf{U} \;, \qquad \mathbf{U}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{U} \quad \Rightarrow \quad \boxed{\begin{aligned} \hat{\boldsymbol{\Xi}} &= \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T \\ \hat{\boldsymbol{\Xi}}' &= diag(A_1^{-2}, A_1^{-2}, A_2^{-2}, A_2^{-2}) \end{aligned}}$$

Mode emittances and total beam emittance

- Symmetric matrix Ξ uniquely determines the beam phase space
  - It has 10 independent parameters
  - It can be expressed through matrix $\hat{\mathbf{V}}$ and 2 amplitudes (also 10 param.)
- 4D beam emittance and mode emittances
  - Inverting equation for matrix $\hat{\Xi}$ we obtain $\hat{\Xi}' \equiv \mathbf{A}^{-1}\mathbf{A}^{-1} = \hat{\mathbf{V}}^T \hat{\Xi} \hat{\mathbf{V}}$
  - i.e. symplectic transform $\hat{\mathbf{V}}$ reduces matrix $\hat{\Xi}$ to its diagonal form but does not change 4D volume of the ellipsoid because $\det \hat{\mathbf{V}} = 1$
  - In new coordinates

$$\hat{\Xi}'_{11} x'^2 + \hat{\Xi}'_{22} p_x'^2 + \hat{\Xi}'_{33} y'^2 + \hat{\Xi}'_{44} p_y'^2 = 1$$

  - Then 4D beam emittance (omit correction $\pi^2/2$ for real 4D volume)

$$\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\Xi}'_{11}\hat{\Xi}'_{22}\hat{\Xi}'_{33}\hat{\Xi}'_{44}}} = \frac{1}{\sqrt{\det(\hat{\Xi}')}} = \frac{1}{\sqrt{\det(\hat{\Xi})}} = (A_1 A_2)^2 \Rightarrow$$

$$\boxed{\varepsilon_{4D} = \varepsilon_1 \varepsilon_2 = \frac{1}{\sqrt{\det(\hat{\Xi})}} \quad , \quad \varepsilon_1 = A_1^2 \quad , \quad \varepsilon_2 = A_2^2}$$

- Beam envelope bilinear form
  - Summarizing above we conclude, if we know
    - beam emittances
    - eigen-vectors and, consequently, matrix $\hat{\mathbf{V}}$
  - then the particle ellipsoid can be described as following:

$$\hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} = 1,$$

$$\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}} \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T \mathbf{U}^T$$

- Symplectic transform/motion does not change mode emittances

$$\hat{\mathbf{x}}_1{}^T \hat{\boldsymbol{\Xi}}_1 \hat{\mathbf{x}}_1 = 1 \xrightarrow{\hat{\mathbf{x}}_2 = \hat{\mathbf{M}}_{12}\hat{\mathbf{x}}_1} \hat{\mathbf{x}}_2{}^T \left(\hat{\mathbf{M}}_{12}{}^{-1}\right)^T \hat{\boldsymbol{\Xi}}_1 \hat{\mathbf{M}}_{12}{}^{-1} \hat{\mathbf{x}}_2 = 1$$

$$\Rightarrow \quad \hat{\boldsymbol{\Xi}}_2 = \left(\hat{\mathbf{M}}_{12}{}^{-1}\right)^T \hat{\boldsymbol{\Xi}}_1 \hat{\mathbf{M}}_{12}{}^{-1} = \left(\hat{\mathbf{M}}_{12}{}^{-1}\right)^T \mathbf{U}\hat{\mathbf{V}}_1 \hat{\boldsymbol{\Xi}}' \hat{\mathbf{V}}_1{}^T \mathbf{U}^T \hat{\mathbf{M}}_{12}{}^{-1} = \mathbf{U}\hat{\mathbf{V}}_2 \hat{\boldsymbol{\Xi}}' \hat{\mathbf{V}}_2{}^T \mathbf{U}^T$$

Where $\hat{\mathbf{V}}_2 = -\mathbf{U}\left(\hat{\mathbf{M}}_{12}{}^{-1}\right)^T \mathbf{U}\hat{\mathbf{V}}_1 = \hat{\mathbf{M}}_{12}\hat{\mathbf{V}}_1$

Second order moments of the Gaussian distribution

- Gaussian distribution function for coupled beam motion

$$f(\hat{\mathbf{x}}) = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \exp\left(-\frac{1}{2} \hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} \right)$$

- The second order moments of the distribution

$$\hat{X}_{ij} \equiv \overline{\hat{x}_i \hat{x}_j} = \int \hat{x}_i \hat{x}_j f(\hat{\mathbf{x}}) d\hat{x}^4 = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \int \hat{x}_i \hat{x}_j \exp\left(-\frac{1}{2} \hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} \right) d\hat{x}^4$$

Transform, $\hat{\mathbf{y}} = \hat{\mathbf{V}}^{-1} \hat{\mathbf{x}}$, reduces matrix $\hat{\boldsymbol{\Xi}}$ to its diagonal form and makes an integration trivial

Finally one obtains

$$\boxed{\hat{\mathbf{X}} = \hat{\mathbf{V}} \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T}$$

- Relationship between bilinear form and second order moments

$$\hat{\mathbf{X}} \hat{\boldsymbol{\Xi}} = \left(\hat{\mathbf{V}} \hat{\boldsymbol{\Xi}}'^{-1} \hat{\mathbf{V}}^T \right) \left(\mathbf{U} \hat{\mathbf{V}} \hat{\boldsymbol{\Xi}}' \hat{\mathbf{V}}^T \mathbf{U}^T \right) = \mathbf{I} \quad \Rightarrow \quad \boxed{\hat{\mathbf{X}} = \hat{\boldsymbol{\Xi}}^{-1}}$$

  - And, consequently, $\boxed{\varepsilon_{4D} = \varepsilon_1 \varepsilon_2 = \sqrt{\det\left(\hat{\mathbf{X}}\right)}}$.

Finding Emittances and Eigen-vectors from $\hat{\mathbf{X}}$ and $\hat{\boldsymbol{\Xi}}$

- Solution of characteristic equation results in the mode emittances
  - Consider equation $\boxed{\det\left(\hat{\boldsymbol{\Xi}} - i\lambda\,\mathbf{U}\right) = 0}$

    It has 4 roots: $\lambda_1 = -\lambda_2 = 1/\varepsilon_1$ and $\lambda_3 = -\lambda_4 = 1/\varepsilon_2$
  - Proof:

$$\det\left(\hat{\boldsymbol{\Xi}} - i\lambda\,\mathbf{U}\right) = \det\left(\mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T - i\lambda\,\mathbf{U}\right) = \det\left(\hat{\boldsymbol{\Xi}}' - i\lambda\,\mathbf{U}^T\hat{\mathbf{V}}^T\mathbf{U}\hat{\mathbf{V}}\mathbf{U}\right) =$$

$$\det\left(\hat{\boldsymbol{\Xi}}' - i\lambda\,\mathbf{U}\right) = \left(\frac{1}{\varepsilon_1^{\,2}} - \lambda^2\right)\left(\frac{1}{\varepsilon_2^{\,2}} - \lambda^2\right) = 0 \quad .$$

- Eigen-vectors of charact. eq. are equal to the motion eigen-vectors:

$$\boxed{\left(\hat{\boldsymbol{\Xi}} - \frac{i}{\varepsilon_l}\mathbf{U}\right)\hat{\mathbf{v}}_l = 0}$$

  - Proof: Rewrite $\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T$ as $\hat{\boldsymbol{\Xi}}\hat{\mathbf{V}}\mathbf{U} = \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'$ and multiply it by $\mathbf{u}_l$ as result one obtains the above equation.

    Here $l = 1, 2;$ $\mathbf{u}_1 = \begin{bmatrix}1 & -i & 0 & 0\end{bmatrix}^T$, $\mathbf{u}_2 = \begin{bmatrix}0 & 0 & 1 & -i\end{bmatrix}^T$

    and we also took into account that: $\hat{\mathbf{V}}\mathbf{u}_l = \hat{\mathbf{v}}_l$, $\mathbf{U}\mathbf{u}_l = -i\mathbf{u}_l$, $\boldsymbol{\Xi}'\mathbf{u}_l = \mathbf{u}_l/\varepsilon_l$

- Similarly for the second order moments: $\boxed{\left(\hat{\mathbf{X}}\mathbf{U} + i\varepsilon_l\mathbf{I}\right)\hat{\mathbf{v}}_l = 0}$.

Eigen-vectors and Beta-functions of Uncoupled Motion

- Eigen-vectors:

$$\hat{\mathbf{v}} = \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+\alpha}{\sqrt{\beta}} \end{bmatrix}, \quad \hat{\mathbf{V}} = \begin{bmatrix} \sqrt{\beta} & 0 \\ -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{\sqrt{\beta}} \end{bmatrix}$$

- Eigen-vectors symplectic orthogonality and normalization:

$$\hat{\mathbf{v}}^{+}\mathbf{U}_2\hat{\mathbf{v}} = \begin{bmatrix} \sqrt{\beta} & \dfrac{i-\alpha}{\sqrt{\beta}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+\alpha}{\sqrt{\beta}} \end{bmatrix} = -2i,$$

$$\hat{\mathbf{v}}^{T}\mathbf{U}_2\hat{\mathbf{v}} = 0$$

- Bilinear form

$$\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}} \begin{bmatrix} 1/\varepsilon & 0 \\ 0 & 1/\varepsilon \end{bmatrix} \hat{\mathbf{V}}^{T}\mathbf{U}^{T} = \begin{bmatrix} \dfrac{1+\alpha^2}{\varepsilon\beta} & \dfrac{\alpha}{\varepsilon} \\ \dfrac{\alpha}{\varepsilon} & \dfrac{\beta}{\varepsilon} \end{bmatrix}$$

  - Courant-Snyder invariant

$$\hat{\mathbf{x}}^{T}\hat{\boldsymbol{\Xi}}\hat{\mathbf{x}} = x^2\frac{1+\alpha^2}{\varepsilon\beta} + 2x\theta\frac{\alpha}{\varepsilon} + \theta^2\frac{\beta}{\varepsilon} = 1 \quad \Rightarrow \quad \varepsilon = \frac{1+\alpha^2}{\beta}x^2 + 2\alpha x\theta + \beta\theta^2$$

Eigen-vectors and Beta-functions of Coupled Motion

- Single-particle phase-space trajectory along the beam orbit

$$\boxed{\begin{aligned}\hat{\mathbf{x}}(s) &= \hat{\mathbf{M}}(0,s)\,\mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1 e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2 e^{-i\psi_2}\right)\\ &= \mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s) e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s) e^{-i(\psi_2+\mu_2(s))}\right)\end{aligned}}$$

  vectors $\hat{\mathbf{v}}_1(s)$ and $\hat{\mathbf{v}}_2(s)$ are the eigen-vectors at coordinate s

  ψ_1 and ψ_2 are the initial phases of betatron motion

- Mais-Ripken parameterization
  - The terms $e^{-i\mu_1(s)}$ and $e^{-i\mu_2(s)}$ are introduced to bring the eigen-vectors to the following standard form:

$$\hat{\mathbf{v}}_1(s) = \begin{bmatrix} \sqrt{\beta_{1x}(s)} \\ -\dfrac{iu_1(s)+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} \\ \sqrt{\beta_{1y}(s)}e^{i\nu_1(s)} \\ -\dfrac{iu_2(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)} \end{bmatrix}, \quad \hat{\mathbf{v}}_2(s) = \begin{bmatrix} \sqrt{\beta_{2x}(s)}e^{i\nu_2(s)} \\ -\dfrac{iu_3(s)+\alpha_{2x}(s)}{\sqrt{\beta_{2x}(s)}}e^{i\nu_2(s)} \\ \sqrt{\beta_{2y}(s)} \\ -\dfrac{iu_4(s)+\alpha_{2y}(s)}{\sqrt{\beta_{2y}(s)}} \end{bmatrix},$$

  - The sign of $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ normalization determines the choice of two of vectors out of two complex conjugate pairs; so that for these vectors u_1, u_4 >0

- Generalized Twiss parameters (10 parameters):

  $\mu_1(s)$ and $\mu_2(s)$ are the phase advances of betatron motion.

  $\beta_{1x}(s)$, $\beta_{1y}(s)$, $\beta_{2x}(s)$ and $\beta_{2y}(s)$ are the beta-functions;

  $\alpha_{1x}(s)$, $\alpha_{1y}(s)$, $\alpha_{2x}(s)$ and $\alpha_{2y}(s)$ are the alpha-functions

  - Six real functions $u_1(s)$, $u_2(s)$, $u_3(s)$, $u_4(s)$, $\nu_1(s)$ and $\nu_2(s)$ are determined by the symplecticity conditions
    - They immediately result in: $u_1 = 1 - u_2$, $u_4 = 1 - u_3$ and $u_2 = u_3$

$\Rightarrow$

$$\boxed{\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}\, e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}\, e^{i\nu_1} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}\, e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x})}}\, e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}}$$

  - Parameters $u(s)$, $\nu_1(s)$ and $\nu_2(s)$ are determined by the rest of orthogonality conditions

- For uncoupled motion:

  $u = 0$, $\beta_{1y} = \beta_{2x} = 0$, and $\alpha_{1y} = \alpha_{2x} = 0$

- Solutions for $u(s)$, $\nu_1(s)$ and $\nu_2(s)$

$$u = \frac{-\kappa_x^{\,2}\kappa_y^{\,2} \pm \sqrt{\kappa_x^{\,2}\kappa_y^{\,2}\left(1+\frac{A_x^{\,2}-A_y^{\,2}}{\kappa_x^{\,2}-\kappa_y^{\,2}}\left(1-\kappa_x^{\,2}\kappa_y^{\,2}\right)\right)}}{1-\kappa_x^{\,2}\kappa_y^{\,2}}, \qquad e^{i\nu_+} \equiv e^{i(\nu_2+\nu_2)} = \frac{A_x + i\left(\kappa_x(1-u)+\kappa_x^{\,-1}u\right)}{A_y - i\left(\kappa_y(1-u)-\kappa_y^{\,-1}u\right)}, \qquad e^{i\nu_-} \equiv e^{i(\nu_2-\nu_2)} = \frac{A_x + i\left(\kappa_x(1-u)-\kappa_x^{\,-1}u\right)}{A_y + i\left(\kappa_y(1-u)-\kappa_y^{\,-1}u\right)},$$

$$\Rightarrow \qquad \nu_1 = n\pi + \frac{1}{2}(\nu_+ - \nu_-) \quad , \quad \nu_2 = m\pi + \frac{1}{2}(\nu_+ + \nu_-) \quad .$$

where

$$A_x = \kappa_x\alpha_{1x} - \kappa_x^{\,-1}\alpha_{2x}\ , \qquad A_y = \kappa_y\alpha_{2y} - \kappa_y^{\,-1}\alpha_{1y}\ , \qquad \kappa_x = \sqrt{\frac{\beta_{2x}}{\beta_{1x}}}, \quad \kappa_y = \sqrt{\frac{\beta_{1y}}{\beta_{2y}}}\ .$$

- Thus, for given α's and β's there are four solutions !!!
  - Two solutions for u and
  - Two solutions for $\nu_1(s)$ and $\nu_2(s)$ for each of two solutions for u
- Normally, $0 \le u \le 1$, and u can be considered as coupling strength
  - Percentage of y-motion in the x-plane eigen vector and x-motion in y-vector
- ν_- and ν_+ are determined modulo 2π,
  - that yields that ν_1 and ν_2 are determined modulo π.
  - Reason: the mirror reflection does not change β's and α's but changes relative signs for x and y components of eigen-vector $\Rightarrow$ change of ν_1 and ν_2 by π.

$$\begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y e^{i\nu} \end{bmatrix} = \begin{cases} \xrightarrow{\nu=0} \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \end{bmatrix} \\ \xrightarrow{\nu=\pi} \begin{bmatrix} \mathbf{F}_x \\ -\mathbf{F}_y \end{bmatrix} \end{cases}$$

Derivatives of the Tunes and Beta-Functions

A differential trajectory displacement related to the first eigen-vector

$$x(s+ds) = x(s) + x'(s)ds = x(s) + \left(p_x(s) + \frac{R}{2} y \right) ds =$$

$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)} + \left[-\frac{i(1-u(s)) + \alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} + \frac{R}{2}\sqrt{\beta_{1y}(s)}\, e^{i\nu_1(s)} \right] ds \right) e^{-i(\mu_1(s)+\psi_1)} \right) \quad .$$

Alternatively, the particle position can be expressed through the beta-functions at the new coordinate *s + ds*:

$$x(s+ds) = \mathrm{Re}\left(\sqrt{\varepsilon_1 \beta_x(s+ds)} e^{-i(\mu_1(s+ds)+\psi)} \right) =$$

$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)} + \frac{d\beta_{1x}}{2\sqrt{\beta_{1x}(s)}} - i\sqrt{\beta_{1x}(s)} d\mu \right) e^{-i(\mu_1(s)+\psi)} \right) \quad .$$

- Comparing one obtains:

$$\frac{d\beta_{1x}}{ds} = -2\alpha_{1x} + R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \quad , \qquad \frac{d\beta_{1y}}{ds} = -2\alpha_{1y} - R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \quad ,$$

$$\frac{d\mu_1}{ds} = \frac{1-u}{\beta_{1x}} - \frac{R}{2}\sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\sin\nu_1 \quad , \qquad \frac{d\mu_1}{ds} - \frac{d\nu_1}{ds} = \frac{u}{\beta_{1y}} + \frac{R}{2}\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\sin\nu_1 \quad ,$$

$$\frac{d\beta_{2y}}{ds} = -2\alpha_{2y} - R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 \quad , \qquad \frac{d\beta_{2x}}{ds} = -2\alpha_{2x} + R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 \quad ,$$

$$\frac{d\mu_2}{ds} = \frac{1-u}{\beta_{2y}} + \frac{R}{2}\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\sin\nu_2 \quad , \qquad \frac{d\mu_2}{ds} - \frac{d\nu_2}{ds} = \frac{u}{\beta_{2x}} - \frac{R}{2}\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\sin\nu_2 \quad .$$

Main Properties of Mais-Ripken parameterization

- For given eigen-vector there are unique set of generalized Twiss parameters
- For given α's and βs there four solutions and at least two of three parameters $u(s)$, $\nu_1(s)$ and $\nu_2(s)$ have two be known
- Single particle motion

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s)e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s)e^{-i(\psi_2+\mu_2(s))}\right)$$

$$x(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1\beta_{1x}(s)}e^{-i(\mu_1(s)+\psi_1)} + \sqrt{\varepsilon_2\beta_{2x}(s)}e^{-i(\mu_2(s)-\nu_2(s)+\psi_2)}\right)$$

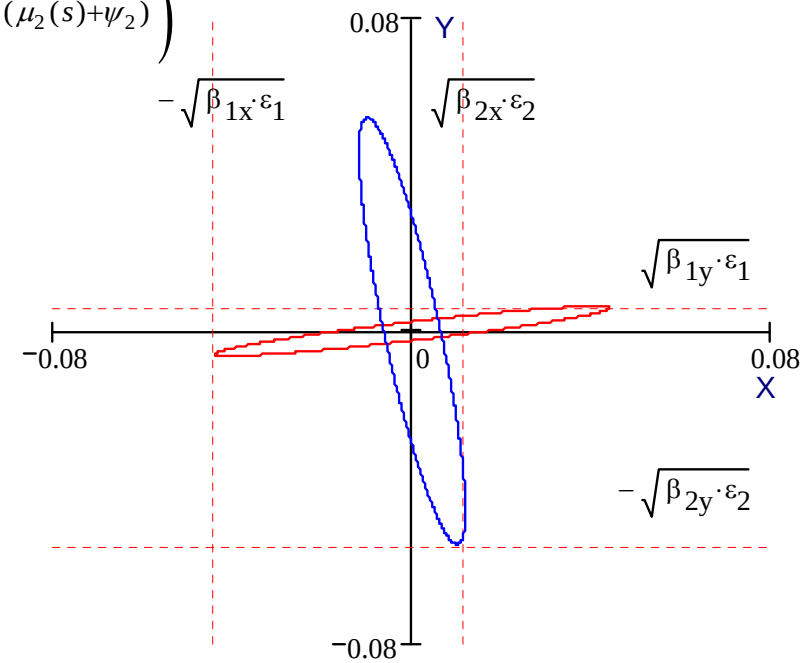
$$\Rightarrow \quad y(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1\beta_{1y}(s)}e^{-i(\mu_1(s)-\nu_1(s)+\psi_1)} + \sqrt{\varepsilon_2\beta_{2x}(s)}e^{-i(\mu_2(s)+\psi_2)}\right)$$

- Beam sizes

$$\sigma_x = \sqrt{\varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}}\ ,$$

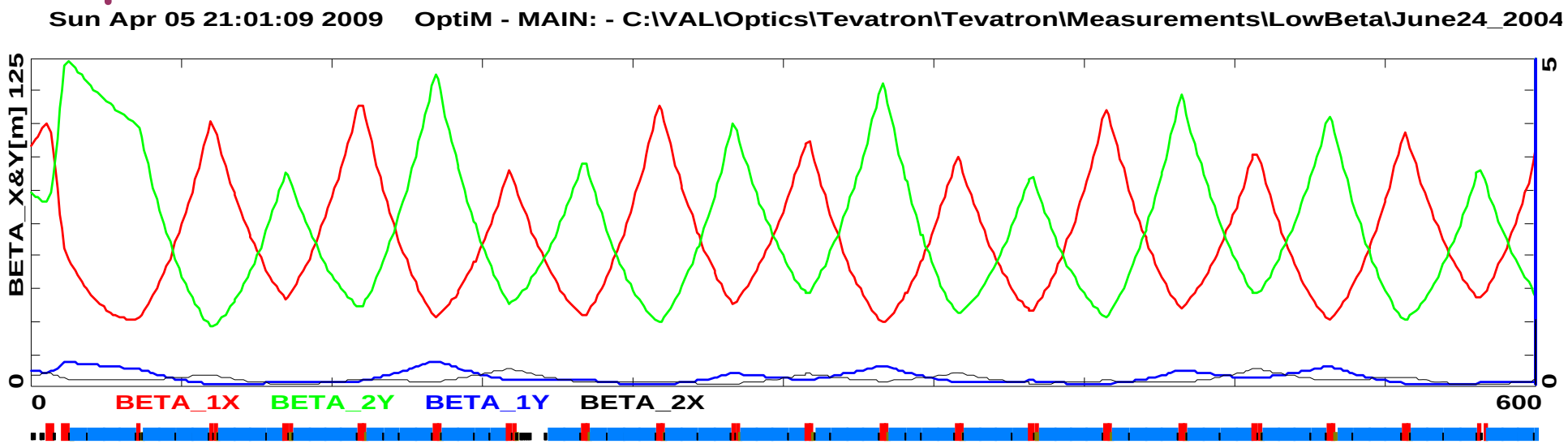
$$\sigma_y = \sqrt{\varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}}\ ,$$

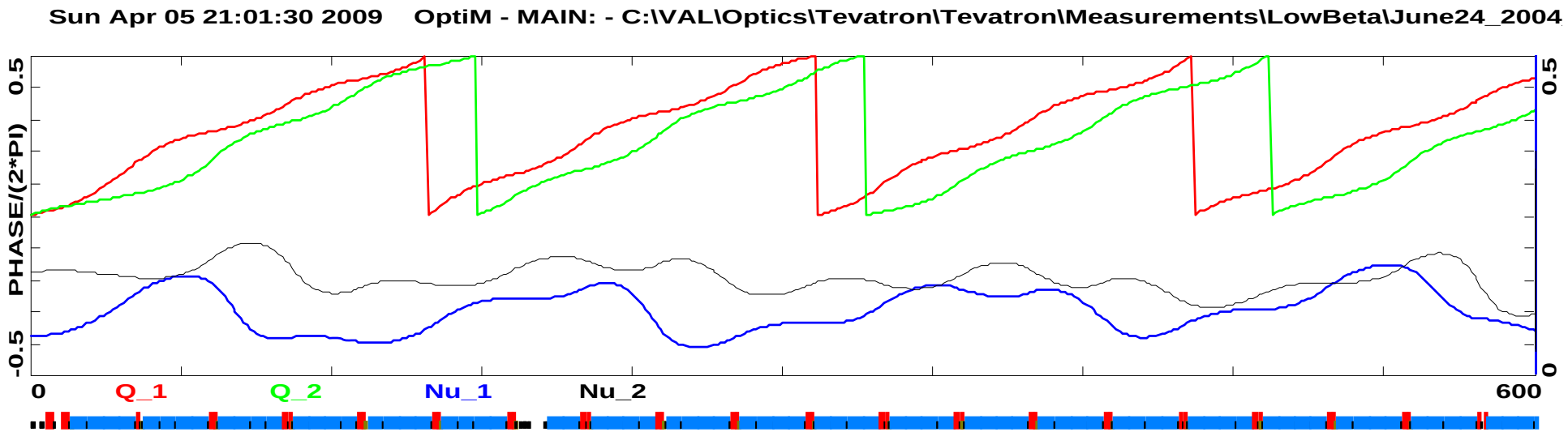
$$\langle xy\rangle \equiv \sigma_{xy}{}^2 = \varepsilon_1\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 + \varepsilon_2\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2$$



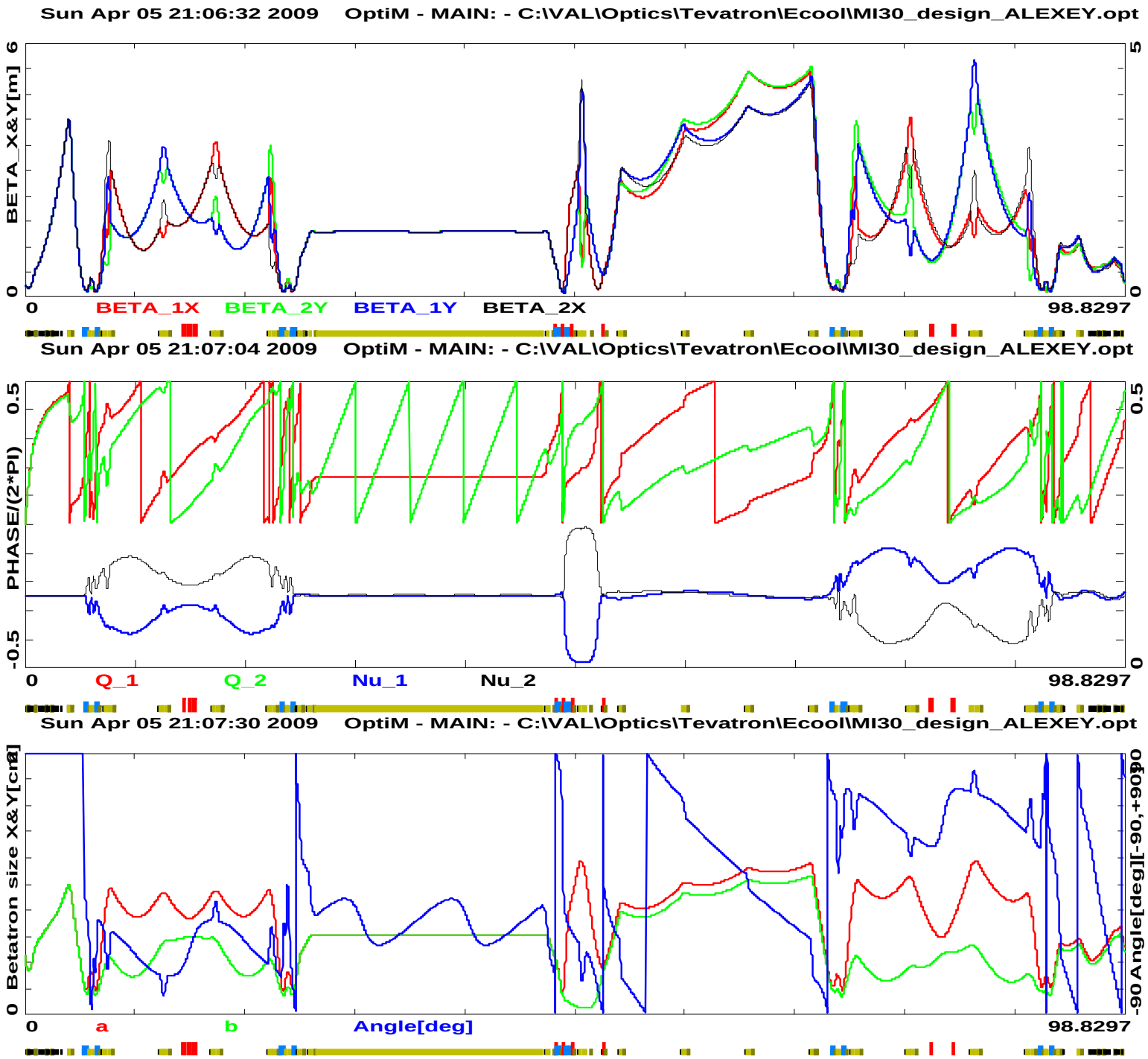
- In vicinity of coupling resonance, $\nu_x \sim \nu_y$
  - $\beta_{1x} + \beta_{2x} \approx \beta_{x0}$
  - $\beta_{1y} + \beta_{2y} \approx \beta_{y0}$
  - Coupling weakly affects beam size projections for $\varepsilon_{1x} \approx \varepsilon_x$

Examples of Generalized Beta-functions





Coupled beta-function in Tevatron (600 m in sector F)



Coupled beta-function in the FNAL Electron cooler

Edwards -Teng Parametrization

- The parametrization is based on a canonical transform R which reduces a 4×4 transfer matrix to its normal modes form $\tilde{\mathbf{M}} = \tilde{\mathbf{R}}\hat{\mathbf{M}}\tilde{\mathbf{R}}^{-1}$

  where $\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{P} & \mathbf{p} \\ \mathbf{q} & \mathbf{Q} \end{bmatrix}$ $\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$

- Teng suggested to parametrize a symplectic matrix $\tilde{\mathbf{R}}$ as follows:

$$\tilde{\mathbf{R}} = \begin{bmatrix} \mathbf{E}\cos\phi & -\mathbf{D}^{-1}\sin\phi \\ \mathbf{D}\sin\phi & \mathbf{E}\cos\phi \end{bmatrix}$$

  where **E** is the unit 2×2 matrix, and **D** is a 2×2 symplectic matrix.

- <u>10 parameters:</u>
  - $\tilde{\mathbf{R}}$ is parametrized by four parameters: D_{11}, D_{12}, D_{22} and ϕ.
  - ***A*** and **B** are parametrized by 3 parameters each β, α and μ.
- Eigen-vectors of the normal modes form
  - Eigen vector definition, $\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{M}}\tilde{\mathbf{R}}\hat{\mathbf{v}}_i = \lambda_i \hat{\mathbf{v}}_i$,
  - can be rewritten as $\tilde{\mathbf{M}}\tilde{\mathbf{v}}_i = \lambda_i \tilde{\mathbf{v}}_i$

    where the vector $\tilde{\mathbf{v}}_i = \tilde{\mathbf{R}}\hat{\mathbf{v}}_i$ is the eigen-vector of matrix $\tilde{\mathbf{M}}$.

Equations to find Teng's parameters

$$\begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -d\sin\phi & b\sin\phi \\ 0 & \cos\phi & c\sin\phi & -a\sin\phi \\ a\sin\phi & b\sin\phi & \cos\phi & 0 \\ c\sin\phi & d\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ \sqrt{\beta_2} \\ -\dfrac{i+\alpha_2}{\sqrt{\beta_2}} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -d\sin\phi & b\sin\phi \\ 0 & \cos\phi & c\sin\phi & -a\sin\phi \\ a\sin\phi & b\sin\phi & \cos\phi & 0 \\ c\sin\phi & d\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \sqrt{\beta_{2x}}e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

Eight scalar equations and eight parameters (no betatron tunes)

Solutions for Teng's parameters

$$\sin\phi = \pm\sqrt{u}$$

$$\beta_1 = \frac{\beta_{1x}}{1-u} \quad , \quad \alpha_1 = \frac{\alpha_{1x}}{1-u} \quad ,$$

$$\beta_2 = \frac{\beta_{2y}}{1-u} \quad , \quad \alpha_2 = \frac{\alpha_{2y}}{1-u} \quad .$$

$$a\tan\phi = \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\,\frac{\alpha_{2x}\sin\nu_2 + u\cos\nu_2}{1-u} \quad ,$$

$$b\tan\phi = \sqrt{\beta_{1x}\beta_{1y}}\,\frac{\sin\nu_1}{1-u} \quad ,$$

$$c\tan\phi = \frac{\cos\nu_2\left(\alpha_{2x}(1-u) - \alpha_{2y}\,u\right) - \sin\nu_2\left(u(1-u) + \alpha_{2x}\alpha_{2y}\right)}{(1-u)\sqrt{\beta_{2x}\beta_{2y}}} ,$$

$$d\tan\phi = -\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\,\frac{u\cos\nu_1 + \alpha_{1y}\sin\nu_1}{1-u} \quad ,$$

- Four solutions for Teng's rotation angle ϕ but unique values for β-functions.

Betatron phase advance in Teng's representation

- The betatron motion in the normal modes representation is

$$\tilde{\mathbf{x}}(s) = \tilde{\mathbf{M}}(0,s)\tilde{\mathbf{x}}(0)$$

where

$$\tilde{\mathbf{M}}(0,s) = \tilde{\mathbf{R}}(s)\hat{\mathbf{M}}(0,s)\tilde{\mathbf{R}}^{-1}(0)$$

- Phase advance of the betatron motion is determined by a standard recipe

$$\tilde{\mathbf{v}}_i(s)e^{-i\mu_i(s)} = \tilde{\mathbf{M}}(0,s)\tilde{\mathbf{v}}_i(0)$$

Using the definition of matrix $\tilde{\mathbf{M}}(0,s)$ one can obtain from the above equation

$$\hat{\mathbf{v}}_i(s)e^{-i\mu_i(s)} = \tilde{\mathbf{R}}(s)^{-1}\tilde{\mathbf{M}}(0,s)\tilde{\mathbf{R}}(0)\hat{\mathbf{v}}_i(0) = \hat{\mathbf{M}}(0,s)\hat{\mathbf{v}}_i(0)$$

That coincides with betatron phase advance determined by generalized Twiss functions

$\Rightarrow$ **The betatron phase advances for both parametrizations are equal**

Transfer matrix between two points

- V-matrices of two points are related by transfer matrix

$$\hat{\mathbf{V}}_2\,\mathbf{S} = \hat{\mathbf{M}}_{12}\,\hat{\mathbf{V}}_1$$

  - Here matrix **S** removes the betatron phase advance and restores the eigen-vectors to their normal form

$$\mathbf{S} = \begin{bmatrix} \cos\Delta\mu_1 & \sin\Delta\mu_1 & 0 & 0 \\ -\sin\Delta\mu_1 & \cos\Delta\mu_1 & 0 & 0 \\ 0 & 0 & \cos\Delta\mu_2 & \sin\Delta\mu_2 \\ 0 & 0 & -\sin\Delta\mu_2 & \cos\Delta\mu_2 \end{bmatrix}$$

  - Multiplying this equation on both sides by, $\hat{\mathbf{V}}_1^{-1} = -\mathbf{U}\hat{\mathbf{V}}_1^{T}\mathbf{U}$, results in

$$\boxed{\hat{\mathbf{M}}_{12} = -\hat{\mathbf{V}}_2\mathbf{S}\mathbf{U}\hat{\mathbf{V}}_1^{\mathrm{T}}\mathbf{U}}$$

Derbenev's vortex-to-plane transform

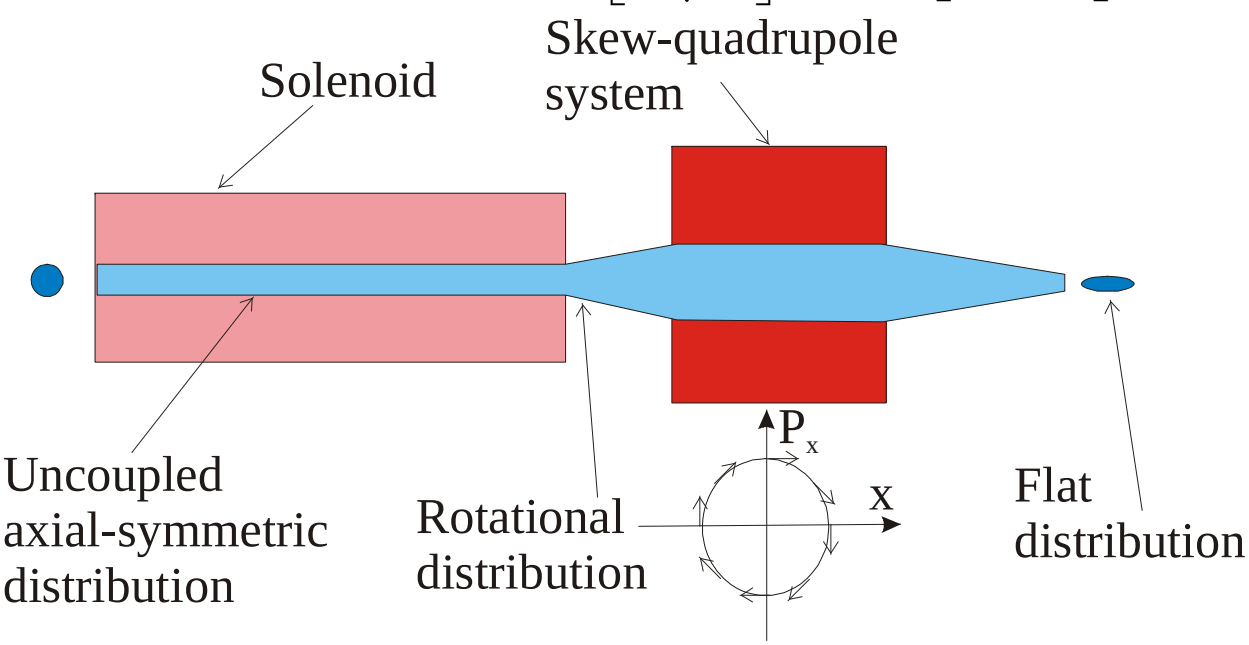
- Eigen-vectors of decoupled motion in the coordinate system rotated by 45 deg

$$\begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ 0 \\ 0 \end{bmatrix} \xrightarrow[\text{rotation}]{45\,\text{deg.}} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \end{bmatrix} \xrightarrow[\alpha_1=2\alpha]{\beta_1=2\beta} \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

- Rotational eigen-vectors

$$\begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix} \qquad \begin{bmatrix} i\mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

- Focusing system with 45 deg difference for X and Y betatron phase advances transforms from one to another distribution



$$\begin{bmatrix} \mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix} \xrightarrow[\mu_y=\mu+\pi/2]{\mu_x=\mu} e^{i\mu} \begin{bmatrix} \mathbf{F}_2 \\ \mathbf{F}_2 e^{i\pi/2} \end{bmatrix} = e^{i\mu} \begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix}$$

- Mode emittances are conserved in the vertex-to-plane transform

Tune Shifts due to Focusing Perturbations

Perturbation theory

- Unperturbed motion: $\hat{\mathbf{M}}\hat{\mathbf{v}}_j = \lambda_j \hat{\mathbf{v}}_j\,, \quad j = 1,\ldots 4$
- Perturbed motion: $\left(\hat{\mathbf{M}} + \Delta\hat{\mathbf{M}}\right)\tilde{\mathbf{v}}_j = \left(\lambda_j + \Delta\lambda_j\right)\tilde{\mathbf{v}}_j\,, \quad \tilde{\mathbf{v}}_j = \hat{\mathbf{v}}_j + \sum_{i=1}^{4} \varepsilon_{ij}\hat{\mathbf{v}}_i\,, \quad \varepsilon_{ij} << 1\,, \quad \varepsilon_{ii} = 0$
- Linearizing:

$$\left(\hat{\mathbf{M}} + \Delta\ddot{\mathbf{M}}\right)\left(\hat{\mathbf{v}}_j + \sum_{i=1}^{4}\varepsilon_{ij}\hat{\mathbf{v}}_i\right) = \left(\lambda_j + \Delta\lambda_j\right)\left(\hat{\mathbf{v}}_j + \sum_{i=1}^{4}\varepsilon_{ij}\hat{\mathbf{v}}_i\right),$$

$$\ddot{\mathbf{M}}\sum_{i=1}^{4}\varepsilon_{ij}\hat{\mathbf{v}}_i + \Delta\mathbf{M}\hat{\mathbf{v}}_j = \lambda_j\sum_{i=1}^{4}\varepsilon_{ij}\hat{\mathbf{v}}_i + \Delta\lambda_j\hat{\mathbf{v}}_j$$

$$\left(\ddot{\mathbf{M}} - \lambda_j\mathbf{I}\right)\sum_{i=1}^{4}\varepsilon_{ij}\hat{\mathbf{v}}_i = \left(\Delta\lambda_j\mathbf{I} - \Delta\ddot{\mathbf{M}}\right)\ddot{\mathbf{v}}_j$$

$$\sum_{i=1}^{4}\left(\lambda_i - \lambda_j\right)\varepsilon_{ij}\hat{\mathbf{v}}_i = \left(\Delta\lambda_j\mathbf{I} - \Delta\hat{\mathbf{M}}\right)\hat{\mathbf{v}}_j$$

- Eigen-values and eigen-vectors are complex conjugated. Reindexing $\left[\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_2 \quad \hat{\mathbf{v}}_3 \quad \hat{\mathbf{v}}_4\right] \rightarrow \left[\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_1^* \quad \hat{\mathbf{v}}_2 \quad \hat{\mathbf{v}}_2^*\right]$ results in two matrix equations

$$\mathbf{V}_c\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 - \lambda_1^* & 0 & 0 \\ 0 & 0 & \lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 - \lambda_2^* \end{bmatrix}\begin{bmatrix} \Delta\lambda_1 \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{41} \end{bmatrix} = \Delta\hat{\mathbf{M}}\hat{\mathbf{v}}_1\,, \quad \mathbf{V}_c\begin{bmatrix} \lambda_2 - \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1^* & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 - \lambda_2^* \end{bmatrix}\begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \Delta\lambda_2 \\ \varepsilon_{42} \end{bmatrix} = \Delta\hat{\mathbf{M}}\hat{\mathbf{v}}_2\,,$$

where: $\mathbf{V}_c = \left[\hat{\mathbf{v}}_1 \quad \hat{\mathbf{v}}_1^* \quad \hat{\mathbf{v}}_2 \quad \hat{\mathbf{v}}_2^*\right]$

Tune Shifts due to Focusing Perturbations (2)

Perturbation theory (continue)

- V consists of symplectic vectors and its inversion is: $\mathbf{V}_c^{-1} = -\frac{1}{2i}\mathbf{U}\mathbf{V}_c^T\mathbf{U}$
- Substituting one finally obtains

$$\begin{bmatrix} \Delta\lambda_1 \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{41} \end{bmatrix} = -\frac{1}{2i}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (\lambda_1 - \lambda_1^*)^{-1} & 0 & 0 \\ 0 & 0 & (\lambda_1 - \lambda_2)^{-1} & 0 \\ 0 & 0 & 0 & (\lambda_1 - \lambda_2^*)^{-1} \end{bmatrix} \mathbf{U}\,\mathbf{V}_c^T\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_1$$

$$\begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \Delta\lambda_2 \\ \varepsilon_{42} \end{bmatrix} = -\frac{1}{2i}\begin{bmatrix} (\lambda_2 - \lambda_1)^{-1} & 0 & 0 & 0 \\ 0 & (\lambda_2 - \lambda_1^*)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\lambda_2 - \lambda_2^*)^{-1} \end{bmatrix} \mathbf{U}\,\mathbf{V}_c^T\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_2$$

- Multiplication both sides by $[1 \quad 0 \quad 0 \quad 0]$ results in correction factors for eigen-values

$$\Delta\lambda_1 = -\frac{1}{2i}\,\hat{\mathbf{v}}_1^+\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_1$$

$$\Delta\lambda_2 = -\frac{1}{2i}\,\hat{\mathbf{v}}_2^+\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_2$$

Tune Shifts due to Focusing Perturbations (3)

Tune shifts due to quadrupole and skew-quadrupole errors

- Hamiltonian $\Delta H \propto \Phi_x x^2 + 2\Phi_s xy + \Phi_y y^2$
- Transfer matrix: $\hat{\mathbf{M}}_q = \mathbf{I} + \Delta\hat{\mathbf{M}}_q$, $\Delta\hat{\mathbf{M}}_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\Phi_x & 0 & -\Phi_s & 0 \\ 0 & 0 & 0 & 0 \\ -\Phi_s & 0 & -\Phi_y & 0 \end{bmatrix}$
- Eigen-values corrections: $\hat{\mathbf{M}} + \Delta\hat{\mathbf{M}} = \left(\mathbf{I} + \Delta\hat{\mathbf{M}}_q\right)\hat{\mathbf{M}}$ => $\Delta\hat{\mathbf{M}} = \Delta\hat{\mathbf{M}}_q\hat{\mathbf{M}}$

$$\Delta\lambda_1 = -\frac{1}{2i}\ \ddot{\mathbf{v}}_1^{+}\mathbf{U}\left(\Delta\hat{\mathbf{M}}_q\hat{\mathbf{M}}\right)\hat{\mathbf{v}}_1 = -\frac{\lambda_1}{2i}\ \hat{\mathbf{v}}_1^{+}\mathbf{U}\,\Delta\hat{\mathbf{M}}_q\,\hat{\mathbf{v}}_1$$

$$\Delta\lambda_2 = -\frac{1}{2i}\ \hat{\mathbf{v}}_2^{+}\mathbf{U}\left(\Delta\hat{\mathbf{M}}_q\hat{\mathbf{M}}\right)\hat{\mathbf{v}}_2 = -\frac{\lambda_2}{2i}\ \hat{\mathbf{v}}_2^{+}\mathbf{U}\,\Delta\hat{\mathbf{M}}_q\,\hat{\mathbf{v}}_2$$

- Tune shifts

$$\Delta Q_i = \frac{i}{2\pi}\frac{\Delta\lambda_i}{\lambda_i} \longrightarrow$$

$$\boxed{\begin{aligned} \Delta Q_1 &= -\frac{1}{4\pi}\ \hat{\mathbf{v}}_1^{+}\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_1 \\ \Delta Q_2 &= -\frac{1}{4\pi}\ \hat{\mathbf{v}}_2^{+}\mathbf{U}\,\Delta\hat{\mathbf{M}}\,\hat{\mathbf{v}}_2 \end{aligned}}$$

- Performing multiplications one finally obtains

$$\boxed{\begin{aligned} \Delta Q_1 &= \frac{1}{4\pi}\left(\Phi_x\beta_{1x} + 2\Phi_s\sqrt{\beta_{1x}\beta_{1y}}\cos(\nu_1) + \Phi_y\beta_{1y}\right) \\ \Delta Q_2 &= \frac{1}{4\pi}\left(\Phi_x\beta_{2x} + 2\Phi_s\sqrt{\beta_{2x}\beta_{2y}}\cos(\nu_2) + \Phi_y\beta_{2y}\right) \end{aligned}}$$

Conclusions

- The same as for uncoupled motion
  - Mode emittances and total emittance are conserved
  - Twiss parameters and emittances determine the beam ellipsoid in 4D phase space and vice-versa
  - Twiss parameters of two points and betatron phase advances determine corresponding eigen-vectors and transfer matrix between these points
- There are 8 independent parameters which determine the eigen-vectors
  - General structure of eigen-vectors and motion is closely coupled with the uncoupled case
- Both the extended Mais-Ripken and the Edwards-Teng parameterizations have problems to obtain unique solutions
  - Unique solution for eigen-vectors (Mais-Ripken)
  - Unique solution for Twiss-parameters (Edwards-Teng)
- Symplecticity of the motion enables to build effective perturbation theory

Appendix A
Explicit Expressions for Transfer Matrix, Bilinear Form and Matrix of Second Order Moments

- Transfer matrix elements

$$\hat{M}_{11} = (1-u)\cos\mu_1 + \alpha_{1x}\sin\mu_1 + u\cos\mu_2 + \alpha_{2x}\sin\mu_2 \quad , \tag{A.1.1}$$

$$\hat{M}_{12} = \beta_{1x}\sin\mu_1 + \beta_{2x}\sin\mu_2 \quad , \tag{A.1.2}$$

$$\hat{M}_{13} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[\alpha_{1y}\sin(\mu_1+\nu_1) + u\cos(\mu_1+\nu_1)\right] + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[\alpha_{2y}\sin(\mu_2-\nu_2) + (1-u)\cos(\mu_2-\nu_2)\right] \quad , \tag{A.1.3}$$

$$\hat{M}_{14} = \sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1+\nu_1) + \sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2-\nu_2) \quad , \tag{A.1.4}$$

$$\hat{M}_{21} = -\frac{(1-u)^2 + \alpha_{1x}^2}{\beta_{1x}}\sin\mu_1 - \frac{u^2 + \alpha_{2x}^2}{\beta_{2x}}\sin\mu_2 \quad , \tag{A.1.5}$$

$$\hat{M}_{22} = (1-u)\cos\mu_1 + u\cos\mu_2 - \alpha_{1x}\sin\mu_1 - \alpha_{2x}\sin\mu_2 \quad , \tag{A.1.6}$$

$$\hat{M}_{23} = \frac{\left[(1-u)\alpha_{1y} - u\alpha_{1x}\right]\cos(\mu_1+\nu_1) - \left[\alpha_{1x}\alpha_{1y} + u(1-u)\right]\sin(\mu_1+\nu_1)}{\sqrt{\beta_{1x}\beta_{1y}}} + \frac{\left[u\alpha_{2y} - (1-u)\alpha_{2x}\right]\cos(\mu_2-\nu_2) - \left[\alpha_{2x}\alpha_{2y} + u(1-u)\right]\sin(\mu_2-\nu_2)}{\sqrt{\beta_{2x}\beta_{2y}}} \quad , \tag{A.1.7}$$

$$\hat{M}_{24} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[(1-u)\cos(\mu_1+\nu_1) - \alpha_{1x}\sin(\mu_1+\nu_1)\right] + \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[u\cos(\mu_2-\nu_2) - \alpha_{2x}\sin(\mu_2-\nu_2)\right] \quad , \tag{A.1.8}$$

$$\hat{M}_{31} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[\alpha_{1x}\sin(\mu_1-\nu_1)+(1-u)\cos(\mu_1-\nu_1)\right]+\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[\alpha_{2x}\sin(\mu_2+\nu_2)+u\cos(\mu_2+\nu_2)\right] \quad , \qquad (A.1.9)$$

$$\hat{M}_{32} = \sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1-\nu_1)+\sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2+\nu_2) \quad , \qquad (A.1.10)$$

$$\hat{M}_{33} = u\cos\mu_1+(1-u)\cos\mu_2+\alpha_{2y}\sin\mu_2+\alpha_{1y}\sin\mu_1 \quad , \qquad (A.1.11)$$

$$\hat{M}_{34} = \beta_{1y}\sin\mu_1+\beta_{2y}\sin\mu_2 \quad , \qquad (A.1.12)$$

$$\hat{M}_{41} = \frac{\left[\alpha_{1x}u-(1-u)\alpha_{1y}\right]\cos(\mu_1-\nu_1)-\left[\alpha_{1x}\alpha_{1y}+u(1-u)\right]\sin(\mu_1-\nu_1)}{\sqrt{\beta_{1x}\beta_{1y}}}+ \frac{\left[(1-u)\alpha_{2x}-u\alpha_{2y}\right]\cos(\mu_2+\nu_2)-\left[\alpha_{2x}\alpha_{2y}+u(1-u)\right]\sin(\mu_2+\nu_2)}{\sqrt{\beta_{2x}\beta_{2y}}} \quad , \qquad (A.1.13)$$

$$\hat{M}_{42} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[u\cos(\mu_1-\nu_1)-\alpha_{1y}\sin(\mu_1-\nu_1)\right]+\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[(1-u)\cos(\mu_2+\nu_2)-\alpha_{2y}\sin(\mu_2+\nu_2)\right] \quad , \qquad (A.1.14)$$

$$\hat{M}_{43} = -\frac{u^2+\alpha_{1y}^2}{\beta_{1y}}\sin\mu_1-\frac{(1-u)^2+\alpha_{2y}^2}{\beta_{2y}}\sin\mu_2 \quad , \qquad (A.1.15)$$

$$\hat{M}_{44} = u\cos\mu_1+(1-u)\cos\mu_2-\alpha_{1y}\sin\mu_1-\alpha_{2y}\sin\mu_2 \quad . \qquad (A.1.16)$$

- Elements of bilinear form describing the particle ellipsoid in 4D space:

$$\hat{\Xi}_{11} = \frac{(1-u)^2 + \alpha_{1x}{}^2}{\varepsilon_1 \beta_{1x}} + \frac{u^2 + \alpha_{2x}{}^2}{\varepsilon_2 \beta_{2x}} , \qquad (A.2.1)$$

$$\hat{\Xi}_{22} = \frac{\beta_{1x}}{\varepsilon_1} + \frac{\beta_{2x}}{\varepsilon_2} , \qquad (A.2.2)$$

$$\hat{\Xi}_{33} = \frac{u^2 + \alpha_{1y}{}^2}{\varepsilon_1 \beta_{1y}} + \frac{(1-u)^2 + \alpha_{2y}{}^2}{\varepsilon_2 \beta_{2y}} , \qquad (A.2.3)$$

$$\hat{\Xi}_{44} = \frac{\beta_{1y}}{\varepsilon_1} + \frac{\beta_{2y}}{\varepsilon_2} , \qquad (A.2.4)$$

$$\hat{\Xi}_{12} = \hat{\Xi}_{21} = \frac{\alpha_{1x}}{\varepsilon_1} + \frac{\alpha_{2x}}{\varepsilon_2} , \qquad (A.2.5)$$

$$\hat{\Xi}_{34} = \hat{\Xi}_{43} = \frac{\alpha_{1y}}{\varepsilon_1} + \frac{\alpha_{2y}}{\varepsilon_2} , \qquad (A.2.6)$$

$$\hat{\Xi}_{13} = \hat{\Xi}_{31} = \frac{\left[\alpha_{1x}\alpha_{1y} + u(1-u)\right]\cos\nu_1 + \left[\alpha_{1y}(1-u) - \alpha_{1x}u\right]\sin\nu_1}{\varepsilon_1\sqrt{\beta_{1x}\beta_{1y}}} + \frac{\left[\alpha_{2x}\alpha_{2y} + u(1-u)\right]\cos\nu_2 + \left[\alpha_{2x}(1-u) - \alpha_{2y}u\right]\sin\nu_2}{\varepsilon_2\sqrt{\beta_{2x}\beta_{2y}}} , \qquad (A.2.7)$$

$$\hat{\Xi}_{14} = \hat{\Xi}_{41} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\,\frac{\alpha_{1x}\cos\nu_1 + (1-u)\sin\nu_1}{\varepsilon_1} + \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\,\frac{\alpha_{2x}\cos\nu_2 - u\sin\nu_2}{\varepsilon_2} , \qquad (A.2.8)$$

$$\hat{\Xi}_{23} = \hat{\Xi}_{32} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\frac{\alpha_{1y}\cos\nu_1 - u\sin\nu_1}{\varepsilon_1} + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\frac{\alpha_{2y}\cos\nu_2 + (1-u)\sin\nu_2}{\varepsilon_2} \quad , \qquad (A.2.9)$$

$$\hat{\Xi}_{24} = \hat{\Xi}_{42} = \frac{\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1}{\varepsilon_1} + \frac{\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2}{\varepsilon_2} \quad . \qquad (A.2.10)$$

- Second order moments:

$$\hat{\mathbf{X}}_{11} = \varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x} \ , \qquad (A.3.1)$$

$$\hat{\mathbf{X}}_{12} = \hat{\mathbf{X}}_{21} = -\varepsilon_1\alpha_{1x} - \varepsilon_2\alpha_{2x} \quad , \qquad (A.3.2)$$

$$\hat{\mathbf{X}}_{22} = \varepsilon_1\frac{(1-u)^2 + \alpha_{1x}{}^2}{\beta_{1x}} + \varepsilon_2\frac{u^2 + \alpha_{2x}{}^2}{\beta_{2x}} \quad , \qquad (A.3.3)$$

$$\hat{\mathbf{X}}_{33} = \varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y} \quad , \qquad (A.3.4)$$

$$\hat{\mathbf{X}}_{34} = \hat{\mathbf{X}}_{43} = -\varepsilon_1\alpha_{1y} - \varepsilon_2\alpha_{2y} \quad , \qquad (A.3.5)$$

$$\hat{\mathbf{X}}_{44} = \varepsilon_1\frac{u^2 + \alpha_{1y}{}^2}{\beta_{1y}} + \varepsilon_2\frac{(1-u)^2 + \alpha_{2y}{}^2}{\beta_{2y}} \quad , \qquad (A.3.6)$$

$$\hat{\mathbf{X}}_{13} = \hat{\mathbf{X}}_{31} = \varepsilon_1\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 + \varepsilon_2\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 \quad , \qquad (A.3.7)$$

$$\hat{\mathbf{X}}_{14} = \hat{\mathbf{X}}_{41} = \varepsilon_1\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left(u\sin\nu_1 - \alpha_{1y}\cos\nu_1\right) - \varepsilon_2\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left((1-u)\sin\nu_2 + \alpha_{2y}\cos\nu_2\right) \ , \qquad (A.3.8)$$

$$\hat{\mathbf{X}}_{23} = \hat{\mathbf{X}}_{32} = -\varepsilon_1\sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left((1-u)\sin\nu_1 + \alpha_{1x}\cos\nu_1\right) + \varepsilon_2\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left(u\sin\nu_2 - \alpha_{2x}\cos\nu_2\right) \ , \qquad (A.3.9)$$

$$\hat{\mathbf{X}}_{24} = \hat{\mathbf{X}}_{42} = \varepsilon_1 \frac{\left(\alpha_{1y}(1-u)-\alpha_{1x}u\right)\sin\nu_1 + \left(u(1-u)+\alpha_{1x}\alpha_{1y}\right)\cos\nu_1}{\sqrt{\beta_{1x}\beta_{1y}}} + \varepsilon_2 \frac{\left(\alpha_{2x}(1-u)-\alpha_{2y}u\right)\sin\nu_2 + \left(u(1-u)+\alpha_{2x}\alpha_{2y}\right)\cos\nu_2}{\sqrt{\beta_{2x}\beta_{2y}}} \quad . \tag{A.3.10}$$

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