Measuring Electron Cloud Density with Trapped Modes

Cheng-Yang Tan

Accelerator Division/Tevatron

ABSTRACT: This started out as a back of the envelope (BOE) calculation to see if a phase shift can be obtained with trapped modes for measuring electron cloud density. Unfortunately, the BOE has grown to more than ten pages! Since this is a BOE, I am not rigourous and have put in caveats in the calculation. However, I believe the final result is correct that there is an amplification factor of $2R/1-R^2$ where R is the reflection coefficient of the phase shift. S. de Santis of LBNL[1] has actually measured a phase shift with a standing wave and so at least to zeroth order, this method works.

THEORY

Instead of using a travelling wave mode which is above cut off for measuring the phase shift from the electron cloud density, I will see if it is possible to measure the phase shift with a trapped mode instead. The advantage is that the trapped mode will have a larger phase shift than a travelling mode because the trapped mode will traverse the electron cloud more than once, but because of the low Q of the trapped mode, I'm not sure whether it is better than the "pass through once" travelling wave mode method. So here is the "back of the envelope" calculation to see if this method can even work.

Let me assume that I can calculate the wave in the "cavity" as a superposition of plane waves going in the $+z$ and $-z$ direction. The goal is then to calculate the amplitude and phase of each plane wave and then sum them all. See Figure 1 shows the idea. **Note: the boundary conditions must be such that L is a multiple of the excitation wavelength or else the wave is not sustained in the "cavity", i.e. $kL = 2\pi n$ where $n \in \mathbb{Z}$. Technically, there is a phase which is the cavity phase response which is NON-ZERO, but for simplicity I am setting this to zero, i.e. For a real measurement, the phase that is shifted is the difference between zero electron cloud and with electron cloud**.

Let me assume that the source generates a plane wave going in the $+z$ direction $\psi_{\rightarrow,1}$

$$\psi_{\rightarrow,1}(t,z) = Ae^{i(\omega t - kz)} \tag{1}$$

After passing through the electron cloud, the plane wave gets phase shifted by the amount ϕ at $z = L$ and its amplitude is reduced by $e^{-\alpha L}$ because of the finite surface conducivity of the waveguide. It is reflected at this location

$$\psi_{\leftarrow,1}(t,L) = Ae^{-\alpha L}Re^{i\phi}e^{i(\omega t + kL)} \tag{2}$$

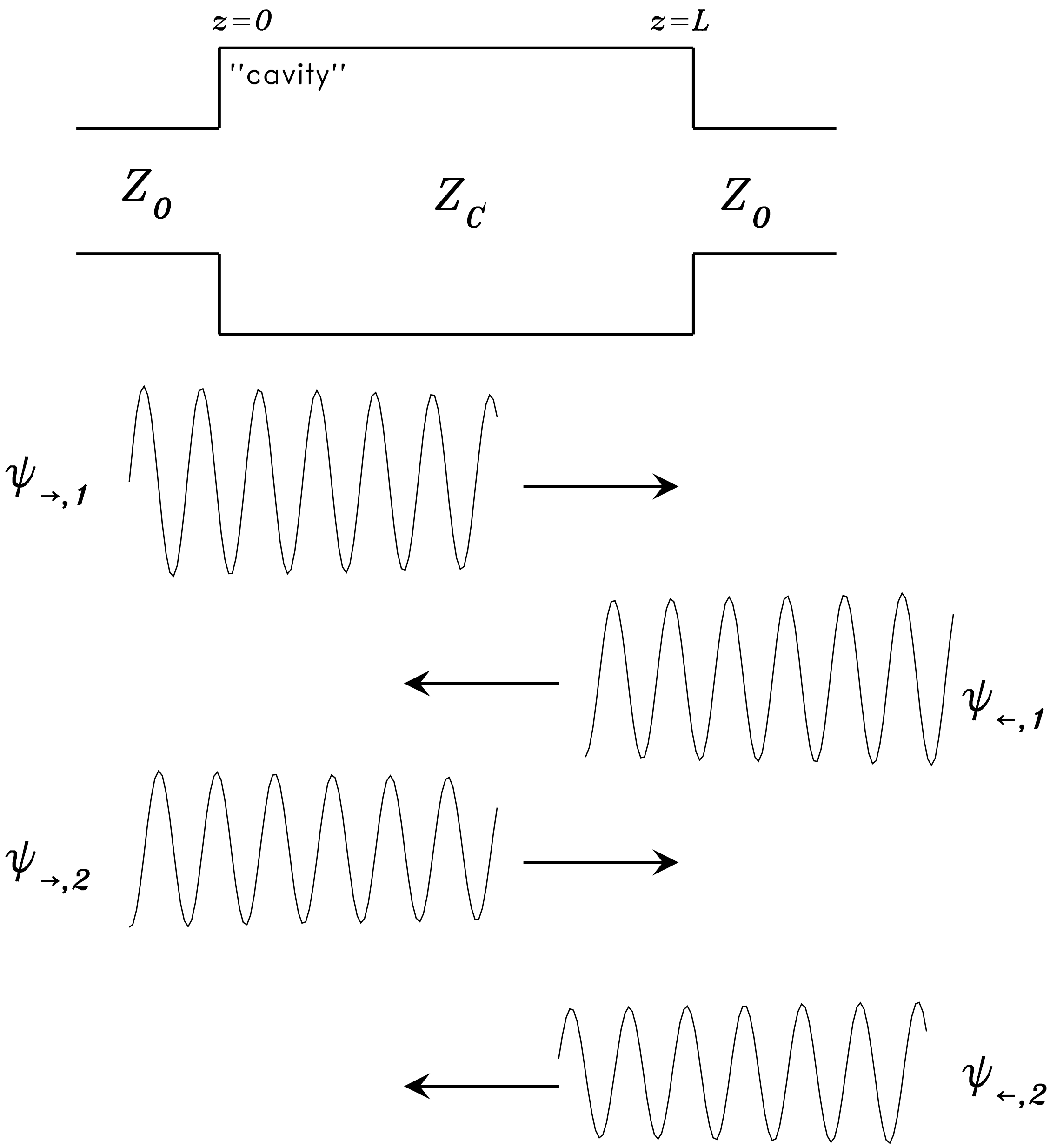

Figure 1 The “cavity” is formed with one large beam pipe connected to two smaller beam pipes at $z = 0$ and $z = L$. The impedances are Z_C for the “cavity” and Z_O for the smaller pipes. $\psi_{\rightarrow,n}$ and $\psi_{\leftarrow,n}$ are the plane waves which are propagating with attenuation in the “cavity” which I will sum to get the measured ψ. For the waves to be trapped, the frequency of the wave must be above the cutoff in the “cavity” but below the smaller pipes.

where $R \in \mathbb{C}$ is the reflection coefficient. Once this plane wave gets back to $z = 0$, it gets phase shifted by ϕ from the electron cloud and attenuated again by $e^{-\alpha L}$

$$\psi_{\leftarrow,1}(t,0) = Ae^{-2\alpha L}Re^{i2\phi}e^{i\omega t} \tag{3}$$

Therefore, I can superposition these two forward and backward plane waves at $z = 0$ to get

$$\left.\begin{aligned} \psi_1 &= \psi_{\rightarrow,1} + \psi_{\leftarrow,1} \\ &= A\Big(1 + e^{-2\alpha L}Re^{i2\phi}\Big)e^{i\omega t} \end{aligned}\right\} \tag{4}$$

To construct the next $+z$ going plane wave, I reflect the $\psi_{\leftarrow,1}$ so that it now propagates to the right, taking into account the reflection coefficient

$$\psi_{\rightarrow,2}(t,z) = Ae^{-\alpha(2L+z)}R^2e^{i2\phi}e^{i(\omega t - kz)} \tag{5}$$

Once this gets to $z = L$, it has gained phase through the electron cloud, and once again is reflected at $z = L$. The reflected plane wave arising from $\psi_{\rightarrow,2}$ at $z = L$ is

$$\psi_{\leftarrow,2} = Ae^{-3\alpha L}R^3e^{i3\phi}e^{i(\omega t + kL)} \tag{6}$$

and then when it gets to $z = 0$, I add the phase from the electron cloud to get

$$\psi_{\leftarrow,2} = Ae^{-4\alpha L}R^3e^{i4\phi}e^{i\omega t} \tag{6}$$

Summing the forward and reflected waves from this turn, I get

$$\left.\begin{aligned} \psi_2 &= \psi_{\rightarrow,2} + \psi_{\leftarrow,2} \\ &= Ae^{-2\alpha L}R^2e^{i2\phi}\Big(1 + e^{-2\alpha L}Re^{i2\phi}\Big)e^{i\omega t} \end{aligned}\right\} \tag{7}$$

And again, $\psi_{\rightarrow,3}$ is created by the reflection of $\psi_{\leftarrow,2}$

$$\psi_{\rightarrow,3}(t,z) = Ae^{-\alpha(4L+z)}R^4e^{i4\phi}e^{i(\omega t - kz)} \tag{8}$$

And once more reflecting at $z = L$ and gaining phase from the electron cloud, I get

$$\psi_3 = Ae^{-4\alpha L}R^4e^{i4\phi}\Big(1 + e^{-2\alpha L}Re^{i2\phi}\Big)e^{i\omega t} \tag{9}$$

and *ad infinitum*, I can construct all the plane waves at $z = 0$ and $z = L$ which together form the trapped wave in the "cavity".

Therefore the trapped wave measured at $z = 0$ is found by summing all the ψ's to get

$$\left.\begin{aligned} \psi &= A\Big(1 + e^{-2\alpha L} R e^{i2\phi}\Big) \sum_{k=0}^{N} e^{-2k\alpha L} R^{2k} e^{i2k\phi} e^{i\omega t} \\ &= A\Big(1 + e^{-2\alpha L} R e^{i2\phi}\Big) \frac{1 - e^{-2(N+1)\alpha L} R^{2(N+1)} e^{i2(N+1)\phi}}{1 - e^{-2\alpha L} R^2 e^{i2\phi}} e^{i\omega t} \end{aligned}\right\} \tag{10}$$

In order to extract out the phase shift from the electron cloud, I have to write ψ in phase amplitude form, but R is a complex number and so I really need to know it.

The characteristic impedance of a circular waveguide is given by

$$Z_{\text{TM}} = \frac{\beta}{\omega\epsilon} \qquad Z_{\text{TE}} = \frac{\omega\mu}{\beta} \tag{11}$$

where $\beta = k\sqrt{1 - \left(\frac{\omega_{cm,n}}{\omega}\right)^2}$ and $k^2 = \omega^2\mu\epsilon$. What I want is a travelling wave in the "cavity" which means the frequency must be above cutoff in it, while at the ends at $z = 0$ and $z = L$, the frequency is below cutoff for there to be a reflection. This means that Z is real in the "cavity" and is purely imaginary outside. See Figure 1. Thus

$$R = \frac{Z_O - Z_C}{Z_O + Z_C} \in \mathbb{C} \tag{12}$$

with Z_O purely imaginary and Z_C a real number. (Note: I have checked may sources that (12) is indeed correct, see Harrington pg. 55, Ramo *et al* pg. 220, Wikipedia and even a demo `http://www.bg.ic.ac.uk/research/intro_to_wia/wia-6-1.html`. However, it bothers me that R is not negative for the *second case* that I am considering because does this mean that there isn't a standing wave in the cavity? Pain pg. 112 gives $(Z_C - Z_O)/(Z_C + Z_O)$ which seems to make more sense, but nobody else does).

Reflection Coefficent

Let me assume that the mode that will be used is TE_{11}. The cut off frequency $\omega_{c_{11}}$ is

$$\omega_{c_{11}} = 2\pi \times \frac{0.293c}{a} \tag{13}$$

because $c = 1/\sqrt{\mu\epsilon}$ and a is the radius of the waveguide. From (11)

$$Z_{\text{TE}} = \frac{\eta}{\sqrt{1 - \left(\frac{\omega_{c_{11}}}{\omega}\right)^2}} \tag{14}$$

where $\eta = \sqrt{\mu/\epsilon} = 377\ \Omega$ is the impedance of free space. Putting in some numbers, let's say I have a 6" OD waveguide (or 7.62 cm radius) connected to a 10 cm OD waveguide (or 5 cm radius), the cutoff frequencies are:

$$\left.\begin{aligned} \omega_{c_{11}}(a = 7.62) &= 2\pi \times (1.15 \times 10^9)\ \text{s}^{-1} \qquad \text{for the "cavity"} \\ \omega_{c_{11}}(a = 5) &= 2\pi \times (1.76 \times 10^9)\ \text{s}^{-1} \qquad \text{for the pipes} \end{aligned}\right\} \tag{15}$$

So for this method to work, I have to excite above $\omega_{c_{11}}(a = 7.62)$ to get the travelling waves going but below $\omega_{c_{11}}(a = 5)$ to get the waves trapped.

Therefore, by using (12) and from the cutoff frequencies calculated above, I can calculate R to be

$$R(f) = \frac{\sqrt{1 - \frac{1.3225}{f^2}} - \sqrt{1 - \frac{3.0976}{f^2}}}{\sqrt{1 - \frac{1.3225}{f^2}} + \sqrt{1 - \frac{3.0976}{f^2}}} = \exp\left[i2\tan^{-1}\left(\frac{\sqrt{\frac{3.0976}{f^2} - 1}}{\sqrt{1 - \frac{1.3225}{f^2}}}\right)\right] \equiv e^{i\theta_R(f)} \tag{16}$$

for the range 1.15 GHz $< f <$ 1.76 GHz. The reason why $|R| = 1$ is because Z_C is real and Z_O is purely imaginary. In fact, I can plot θ_R for $a = 7.62$ cm "cavity" and 5 cm pipes and show it in Figure 2.

Substituting (16) into (10), I get

$$\psi = A\left(1 + e^{-2\alpha L} e^{i\theta_R} e^{i2\phi}\right) \frac{1 - e^{-2(N+1)\alpha L} e^{i2(N+1)\theta_R} e^{i2(N+1)\phi}}{1 - e^{-2\alpha L} e^{i2\theta_R} e^{i2\phi}} e^{i\omega t} \tag{17}$$

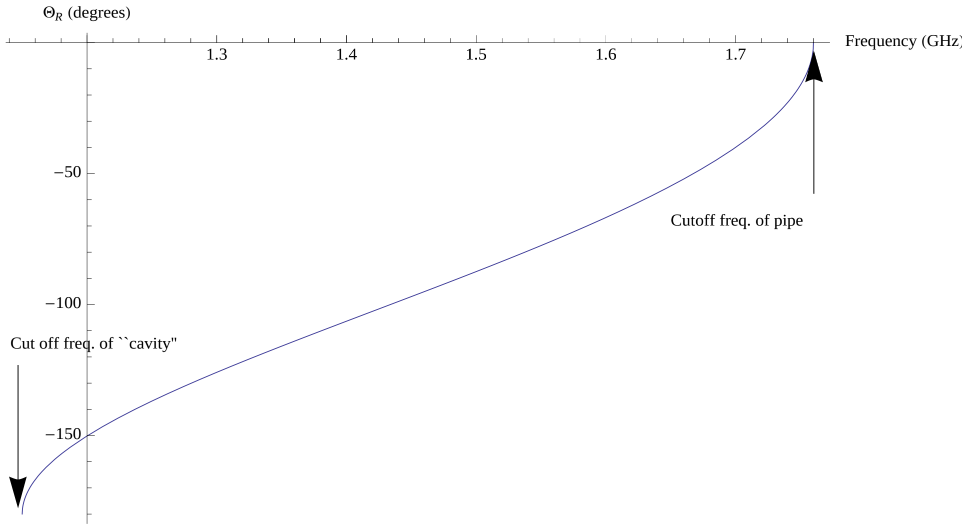

Figure 2 This plot shows θ_R between the cutoff frequencies of the "cavity" and the pipe.

Attenuation Constant α

The attenuation constant α for TE_{11} is given by the formula

$$\alpha = \frac{R_s}{a\eta} \frac{\left(\frac{\omega_c}{\omega}\right)^2 + 0.420}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \tag{18}$$

where the impedance of free space $\eta = \sqrt{\mu/\epsilon} = 377\ \Omega$ and R_s is the surface resistivity which is proportional to $\sqrt{\omega}$ and is given by

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{19}$$

where σ is the bulk resistivity. For stainless steel $\sigma \sim 1.3 \times 10^6\ (\Omega \mathrm{m})^{-1}$. This means that R_s in ohms is

$$R_s = 7 \times 10^{-7}\sqrt{\omega} = 4.4 \times 10^{-6}\sqrt{f} \tag{20}$$

which can be compared to copper which is $(2.6 \times 10^{-7}\sqrt{f})\ \Omega$. Surface resistivity of steel is pretty lousy!

Substituting R_s for stainless steel into (13) for $a = 7.62$ cm of the "cavity", I have

$$\alpha = \frac{2.03 \times 10^{-9} + 6.43 \times 10^{-10} f^2}{\sqrt{1 + 1.3225/f^2}\, f^{3/2}} \tag{21}$$

for f in GHz. When I evaluate α at just below the cutoff frequency of the pipe at 1.755 GHz, I get $\alpha = 2.3 \times 10^{-9}$ cm^{-1}, which is not very large and so after $15 \times L = 1500$ cm for $L = 100$ cm, $e^{-\alpha 15 L} \approx 1$ but less than one.

Calculating ψ

So, for $N \to \infty$, I have $e^{-2(N+1)\alpha L} \to 0$ and $e^{-2\alpha L} \approx 1$ for stainless steel and so ψ from (17) is

$$\psi = A \left(\frac{1 + e^{i\theta_R} e^{i2\phi}}{1 - e^{i2\theta_R} e^{i2\phi}} \right) e^{i\omega t} \tag{22}$$

and $R(1.755 \text{ GHz}) = e^{-0.2i}$ or $\theta_R \approx -11°$. This angle is small enough so that the term in the parenthesis can be approximated as

$$\frac{1 + e^{i\theta_R} e^{i2\phi}}{1 - e^{i2\theta_R} e^{i2\phi}} \approx \frac{1 + (1 + i\theta_R)(1 + i2\phi)}{1 - (1 + i2\theta_R)(1 + i2\phi)} \approx \frac{2 + i(\theta_R + 2\phi)}{-2i(\theta_R + \phi)} \tag{23}$$

The argument of the above is

$$\text{Arg}\left(\frac{2 + i(\theta_R + 2\phi)}{-2i(\theta_R + \phi)} \right) \approx \frac{\theta_R}{2} + \phi - \text{sgn}(\theta_R + \phi)\frac{\pi}{2} \tag{24}$$

which means that there is **NO** amplification whatsoever.

When I try other values of θ_R like for $\theta_R = \pi/2$, I have

$$\frac{1 + e^{i\theta_R} e^{i2\phi}}{1 - e^{i2\theta_R} e^{i2\phi}} \approx \frac{1 + i(1 + i2\phi)}{1 + (1 + i2\phi)} = \frac{(1 - 2\phi) + i}{2 + i2\phi} \tag{25}$$

The argument of the above is

$$\begin{aligned} \text{Arg}\left(\frac{1 + i(1 + i2\phi)}{1 + (1 + i2\phi)} \right) &= \tan^{-1}\left(\frac{1}{1 - 2\phi} \right) - \tan^{-1}\phi \\ &\approx \tan^{-1}(1 + 2\phi) - \phi \\ &= \frac{\pi}{4} + \phi - \phi = \pi/4 \end{aligned}$$

which again shows that having an $R = e^{i\theta_R}$ does **NOT** work!

A Second Possibility

Therefore, a purely trapped mode will not work. Suppose instead, I have a travelling mode in both the "cavity" and the pipe, then R will be completely real because $Z_{\rm TE}$ is real from (14).

The reflection coefficient R is

$$R(f) = \frac{\sqrt{1 - \frac{1.3225}{f^2}} - \sqrt{1 - \frac{3.0976}{f^2}}}{\sqrt{1 - \frac{1.3225}{f^2}} + \sqrt{1 - \frac{3.0976}{f^2}}} > 0 \in \mathbb{R} \tag{26}$$

for the range $f > 1.76$ GHz. See Figure 3 which shows the that the R decreases rapidly as the frequency is increased from the cutoff frequency of the beam pipe.

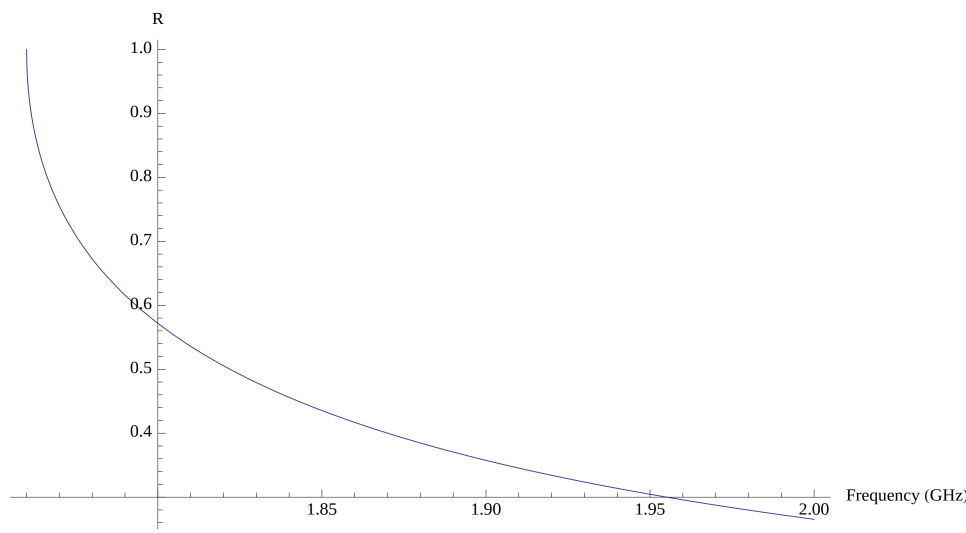

Figure 3 This plot shows R for frequencies above cutoff frequency of the pipe. R is real and greater than zero and decreases rapidly away from the cutoff frequency of the pipe.

When I apply the approximation that $e^{-2\alpha L} \approx 1$ to ψ in (10), I have

$$\psi = A \left(\frac{1 + Re^{i2\phi}}{1 - R^2 e^{i2\phi}} \right) e^{i\omega t} \tag{27}$$

Concentrating only on the term in the parenthesis, I have

$$\left.\begin{aligned}
1+Re^{i2\phi} &= (1+R\cos 2\phi)+iR\sin 2\phi \\
&= \sqrt{(1+R\cos 2\phi)^2+R^2\sin^2 2\phi}\exp\left[\tan^{-1}\left(\frac{R\sin 2\phi}{1+R\cos 2\phi}\right)\right] \\
1-R^2e^{i2\phi} &= (1-R^2\cos 2\phi)-iR^2\sin 2\phi \\
&= \sqrt{(1-R^2\cos 2\phi)^2+R^4\sin^2 2\phi}\exp\left[\tan^{-1}\left(\frac{-R^2\sin 2\phi}{1-R^2\cos 2\phi}\right)\right]
\end{aligned}\right\} \tag{28}$$

Therefore, the phase shift $\Delta\phi$ is

$$\left.\begin{aligned}
\Delta\phi &= \tan^{-1}\left(\frac{R\sin 2\phi}{1+R\cos 2\phi}\right)+\tan^{-1}\left(\frac{R^2\sin 2\phi}{1-R^2\cos 2\phi}\right) \\
&= \tan^{-1}\left[\frac{R(1+R)\sin 2\phi}{(1-R)(1+R+R^2+R\cos 2\phi)}\right] \\
&\approx \frac{2R}{1-R^2}\phi \quad \text{for } \Delta\phi \ll 1
\end{aligned}\right\} \tag{29}$$

Clearly, the "amplification factor" is $2R/1-R^2$ and will enhance the electron cloud phase shift ϕ. **NOTE: This approximation only true for very small phase shifts** $\ll 1^\circ$.

I can calculate the required R for the requirement that $\Delta\phi = 15\phi$ because I need $15\times$ for a 1 m long beam pipe because I know I can see a phase shift for 15 m long beam pipe, I have for $\phi \ll 1$

$$15\phi = \frac{2R}{1-R^2}\phi \qquad \Rightarrow R = 0.94 \tag{30}$$

And with the above, I can calculate the required excitation for the above R to be 1.76048 GHz which is just 480 kHz above the cutoff frequency of the pipe.

REFERENCES

[1] S. de Santis, *The TE Wave Transmission Method for Electron Cloud Measurements at CESR-TA*, PAC2009.