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CBP AFRD LBNL

Radiation & Acceleration

Tutorial

- Optics
- Diffraction
- Formation Length
- Volume of coherence
- Weizsäcker-Williams method of equivalent photons
- Spectrum

  a. Synchrotron radiation

  b. Cherenkov radiation

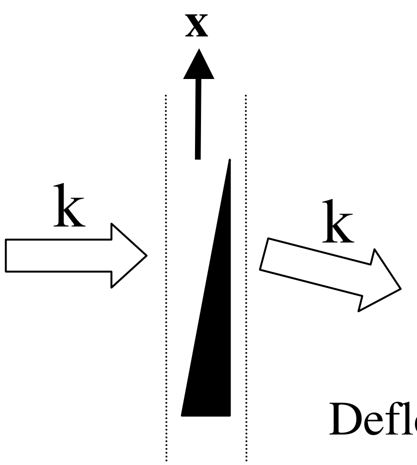
  c. Transition radiation
- Fluctuations
- Linear Acceleration / deceleration
- Nonlinear interaction

  a. Ponderomotive acceleration in vacuum

  b. FEL small signal small gain

Light Optics

Prism



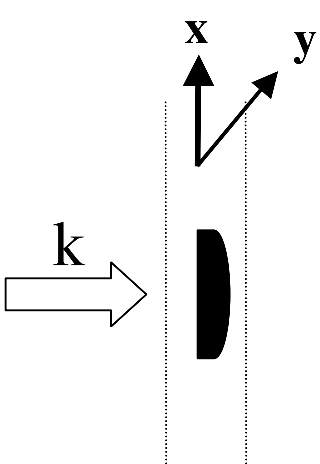
$$e^{i\int n(r)\vec{k}\,d\vec{r} - i\int \omega\, dt}$$

$$e^{i(n-1)\alpha k x - i\omega t}$$

Deflection angle = $(n-1)\,\alpha$

n , α

Lens



$$e^{i\int n(r)\vec{k}\,d\vec{r} - i\int \omega\, dt}$$

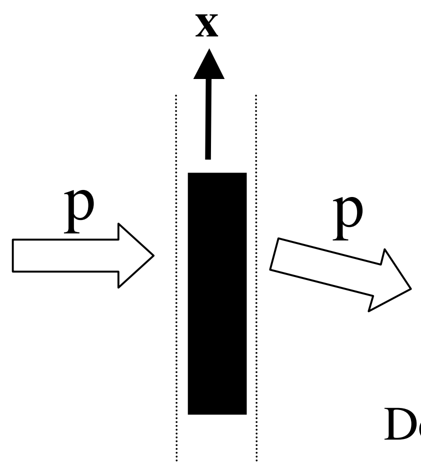
$$e^{i(n-1)\,k\,\frac{x^2+y^2}{2R} - i\omega t}$$

$$\frac{1}{F_x} = \frac{1}{F_y} = (n-1)\,\frac{1}{R}$$

n , R

Charge Particle Optics

Bend magnet



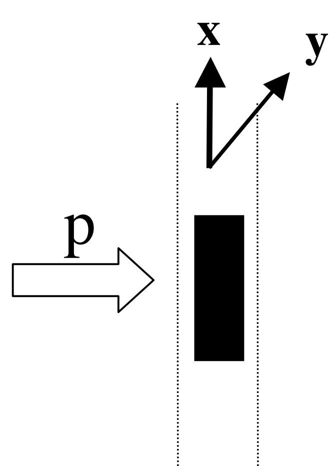
$$e^{\frac{i}{\hbar}\int\left(\vec{p}-\frac{e}{c}\vec{A}\right)d\vec{r} - \frac{i}{\hbar}\int(\mathcal{E}-e\varphi)\,dt}$$

$$e^{\frac{i}{\hbar}\frac{eBL}{pc}\,px - \frac{i}{\hbar}\mathcal{E}\,t}$$

Deflection angle = ***eBL/cp***

B , L

Quadrupole lens

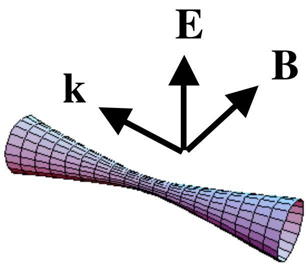


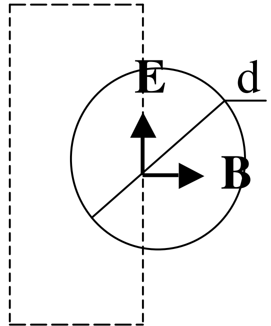
$$e^{\frac{i}{\hbar}\int\left(\vec{p}-\frac{e}{c}\vec{A}\right)d\vec{r} - \frac{i}{\hbar}\int(\mathcal{E}-e\varphi)\,dt}$$

$$e^{\frac{i}{\hbar}\frac{eGL}{2pc}\,p\,(x^2-y^2) - \frac{i}{\hbar}\mathcal{E}\,t}$$

$$\frac{1}{F_x} = -\frac{1}{F_y} = \frac{eGL}{cP}$$

G , L





$$\oint \vec{E}\, d\vec{l} = \frac{\partial}{c\,\partial t}\int \vec{B}\, d\vec{A}$$

$$E\, d \approx B\, k\, d^2\, \theta_{dif}$$

$$\theta_{dif} \approx \frac{1}{k\, d}$$

Transverse coherence $$d_\perp = \frac{1}{k\,\theta}$$

Longitudinal coherence $$l_\parallel = \frac{c}{\Delta\omega}$$

Volume of coherence $$V_c = d_\perp^2\, l_\parallel$$

Degeneracy parameter = number of photons per mode =
= number of photons in volume of coherence

Degeneracy parameter = number of particle per mode =
= number of particle in volume of coherence

Spin 1 (photons)

$$\delta = \frac{1}{e^{\frac{h\nu}{kT}} - 1} \ll 1$$ **for thermal sources of radiation in visible range**

$$\delta \sim \alpha\, N_e\, \omega\, \tau_b \sim 10^3$$ **for synchrotron sources of radiation in visible range**

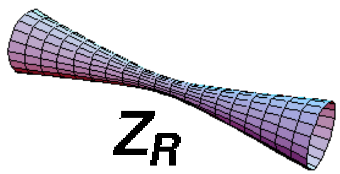
$$\delta \sim N_{ph} \sim 3\ 10^{18}$$ **for 1 J laser in visible range**

Spin 1/2 (electrons)

$$\delta = 2$$ **for electrons in Cu (maximum possible for unpolarized electrons)**

$$\delta \sim N_e \frac{\lambda\!\!\!^{-}{}_c^3}{\varepsilon_x\, \varepsilon_y\, \varepsilon_z} \sim 2\ 10^{-11}$$ **for electrons from RF photo guns**

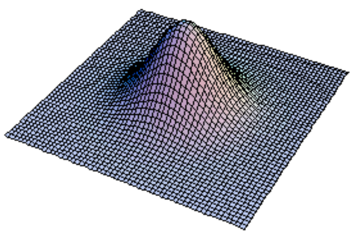
$$\delta \sim 10^{-6}$$ **for electrons from needle cathode**



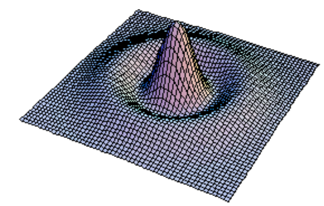
$$\theta_{dif} \approx \frac{1}{k\,d}$$

$$d \sim \Theta_{dif}\, Z_R$$

$$Z_R \approx \frac{d}{\theta_{dif}} \approx \frac{1}{k\theta^2_{dif}}$$

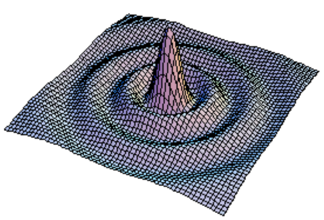


$$\epsilon = \sqrt{<x'^2> \;\; <x^2> - <x'\,x>^2}$$

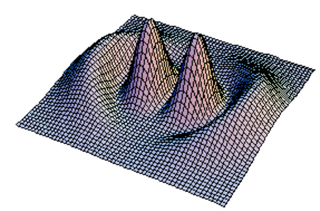


$$\epsilon \approx \theta_{dif}\, d \approx \frac{1}{k}$$

for the high order (m) of optical beam emittance $\epsilon \approx \frac{m}{k}$



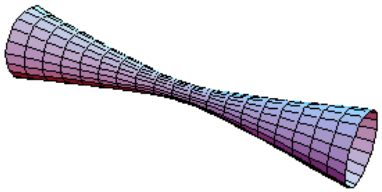
Liouville's Theorem



Conservation of emittance

Gaussian beam approximation

Light optics



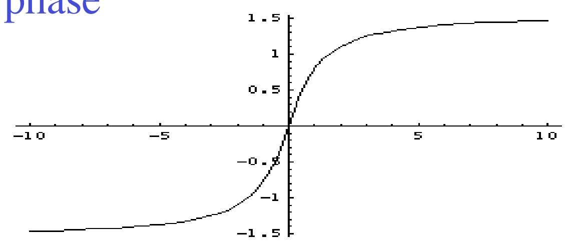
$$\varepsilon_R = \frac{\lambda}{4\pi} = \frac{1}{2k}$$

$$\sigma_{trR}^2 = \varepsilon_R Z_R$$

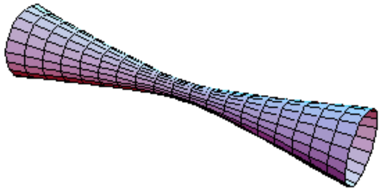
$$\sigma_{\theta R}^2 = \frac{\varepsilon_R}{Z_R}$$

$$\sigma_{trR}^2(z) = \sigma_{trR}^2(0)\left[1 + \frac{z^2}{z_R^2}\right]$$

Guoy phase



Accelerator optics



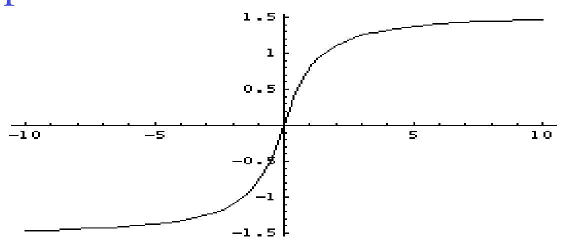
$$\varepsilon_b \gg \lambda_c$$

$$\sigma_{trb}^2 = \varepsilon_b \beta$$

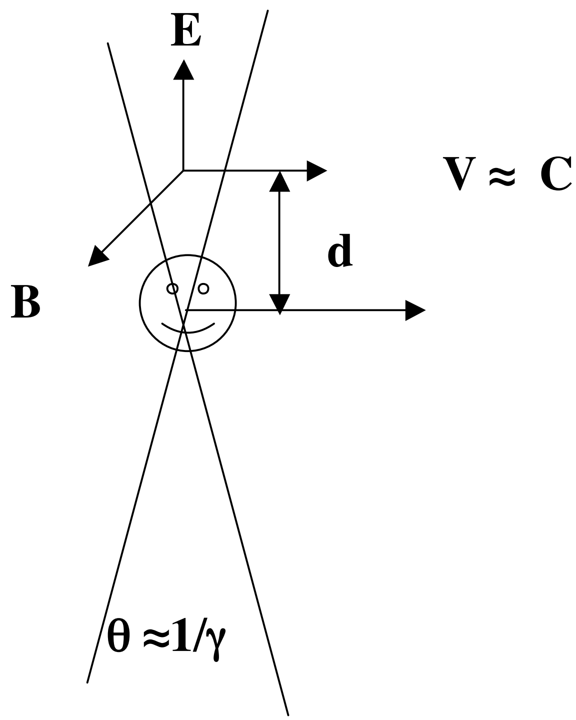
$$\sigma_{\theta b}^2 = \frac{\varepsilon_b}{\beta}$$

$$\sigma_{trb}^2(z) = \sigma_{trb}^2(0)\left[1 + \frac{z^2}{\beta^2}\right]$$

Betatron phase



Weizsäcker-Williams method of equivalent photons

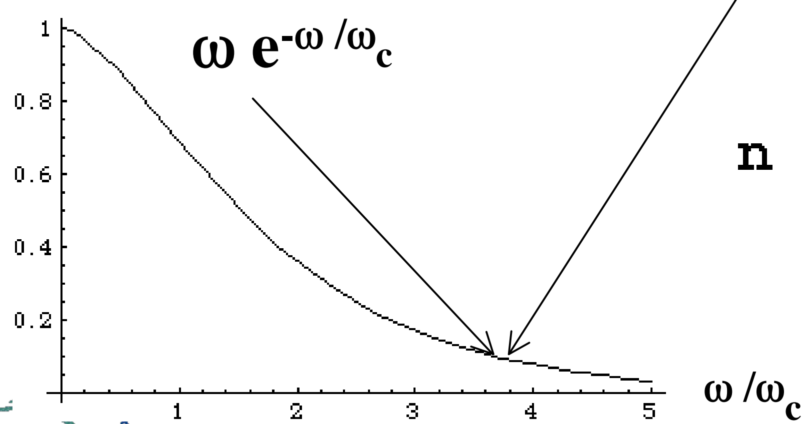


In the rest frame $\vec{E} = \frac{e}{r^3}\,\vec{r}$

Transverse field $E_\perp = \frac{e\,d}{(d^2 + z^2)^{3/2}}$

In Lab. frame $E_\perp = \frac{\gamma\, e\, d}{(d^2 + (\gamma c t)^2)^{3/2}}$

$$|E_\perp(\omega, d)|^2 = \frac{e^2}{\pi^2 d^2 c^2}\left(\frac{\omega d}{\gamma c}\right)^2 K_1^2\left(\frac{\omega d}{\gamma c}\right)$$

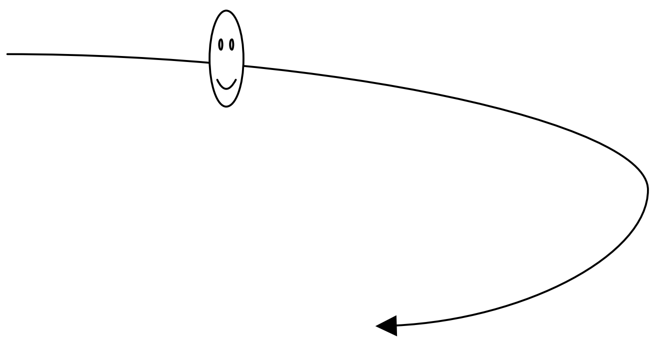


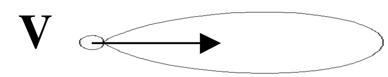
$$n(\omega)\, d\omega = \frac{2}{\pi}\,\alpha \ln\left(\frac{\gamma m c^2}{\hbar\omega}\right)\frac{d\omega}{\omega}$$

$\omega_c = 2\,\gamma\, c\,/d$

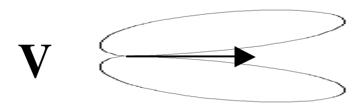
Virtual photons became a real:

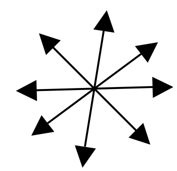
1. If charge particle receive a kick to delay from virtual photons of its cloud
In vacuum in a case $\gamma >> 1$ the only practical way it is a transverse kick



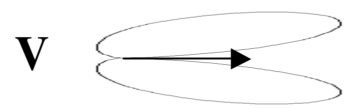


2. In medium if **V > c/n** ⇨ Cherenkov radiation





3. Transition radiation (non homogenous medium)

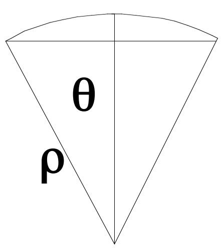


Formation length and Transfers coherence
Radiation of charge particles

Formation length $\Longleftrightarrow$ **Rayleigh range**

Transfers coherence size $\Longleftrightarrow$ **Waist**

1. Synchrotron radiation



$$\frac{k\,\rho\,\theta}{\beta} - 2\,k\,\rho\,\mathrm{Sin}\,\frac{\theta}{2} \approx 1$$

$$\frac{\theta}{2\,\gamma^2} + \frac{\theta^3}{24} = \frac{1}{k\,\rho}$$

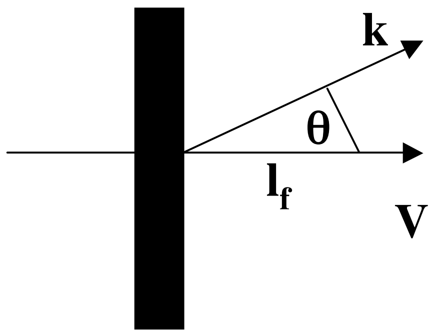
for $\Theta >> 1/\gamma$

$$\Theta \sim (1/k\rho)^{1/3}$$

$$k l_f \approx k\rho\,\theta \approx 2.9\ (k\,\rho)^{2/3}$$

Formation length

2. Transition radiation



$$k l_f = \frac{1}{\left(\frac{1}{\beta} - \cos\theta\right)}$$

3. Cherenkov radiation

$$\cos\theta_{\check{c}} = \frac{1}{n\,\beta}$$

$$k l_f = \frac{1}{\sin^2\theta_{\check{c}}}$$

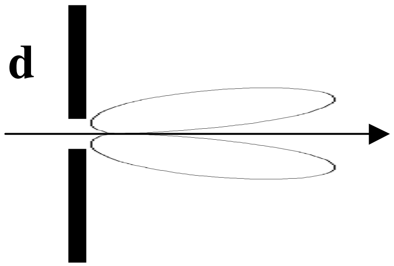
Spectral Intensity

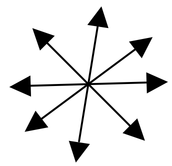
On formation length virtual photons became real !

$$dn_{ph} \approx \alpha \frac{d\omega}{\omega}$$ In one transfers mode and $$\frac{d^2 n_{ph}}{dx} \approx \frac{\alpha}{l_f} \frac{d\omega}{\omega}$$

Estimation	Exact solution
1. Synchrotron radiation	
$\frac{dI}{d\omega} \approx .39 \frac{e^2}{c} \omega^{1/3} \Omega^{2/3}$	$\frac{dI}{d\omega} = .52 \frac{e^2}{c} \omega^{1/3} \Omega^{2/3}$
2. Cerenkov radiation	
$\frac{dI}{d\omega} = \frac{e^2}{c} \omega \, \mathrm{Sin}^2 \theta_{\check{c}}$	$\frac{dI}{d\omega} = \frac{e^2}{c} \omega \, \mathrm{Sin}^2 \theta_{\check{c}}$

3. Transition radiation





$$dn_{ph} \approx \alpha \frac{d\omega}{\omega} \qquad k = \frac{1}{d\,\theta} \qquad k_{min} = \frac{1}{d} \;(\theta \approx 1)$$

$$d\mathcal{E} \approx e^2\, dk \qquad k_{max} \approx \frac{\gamma}{d} \left(\theta \approx \frac{1}{\gamma}\right)$$

$$\mathcal{E} \approx \gamma \frac{e^2}{d} \approx \alpha\, \hbar\omega_{max}$$

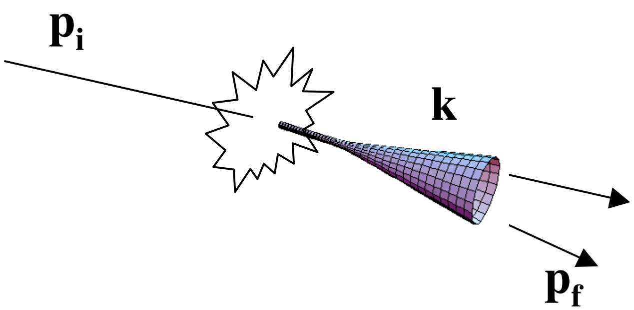
In a limit d -> 0 we still have a hole

$$d_{min} \approx \frac{\lambda_p}{2}$$

Total radiated energy in forward direction

$$\mathcal{E} \approx \alpha \frac{\gamma \hbar \omega_p}{3}$$

Superposition and radiation of beam



$$E_s(t, r) = \sum e_s\left(t - \tau_i, \vec{r} - \vec{r}_i\right)$$

$$E_s\left(\omega, \vec{k}\right) = e_s\left(\omega, \vec{k}\right) \sum e^{i\vec{k}\vec{r}_i - i\omega\tau_i}$$

If $k a \theta << 1$ $\qquad a << d_c$

$$E_s(\omega, \theta) = e_s(\omega, \theta) \sum e^{-i\omega\tau_i} = e_s(\omega, \theta)\, \rho(\omega)$$

Incoherent radiation $\omega\, \tau_b >> 1$

amplitude $E_s(\omega, \theta) \sim N^{1/2}$

intensity $\sim \left|Es(\omega, \theta)\right|^2 \sim N$

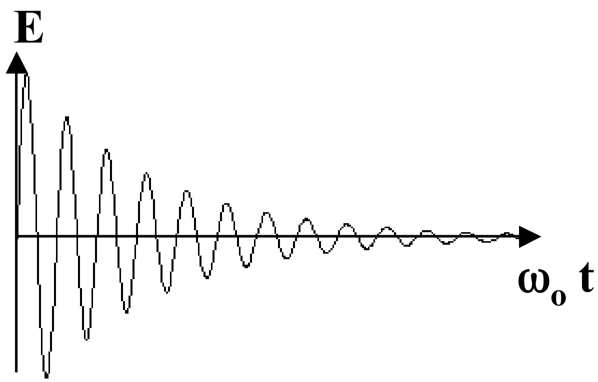
Coherent radiation $\omega\, \tau_b << 1$

amplitude $E_s(\omega, \theta) \sim N$

intensity $\sim \left|Es(\omega, \theta)\right|^2 \sim N^2$

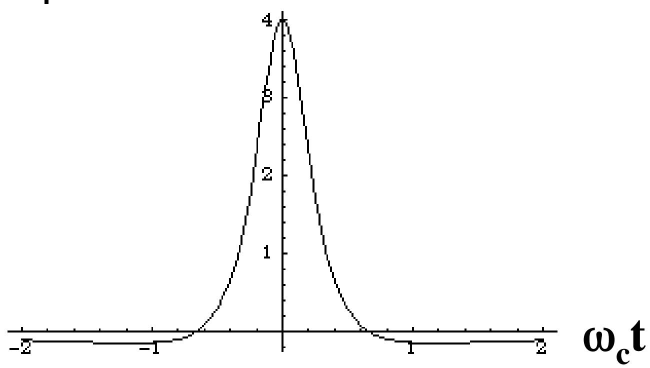
Fluctuation Properties of Electromagnetic Field

Examples of radiated field



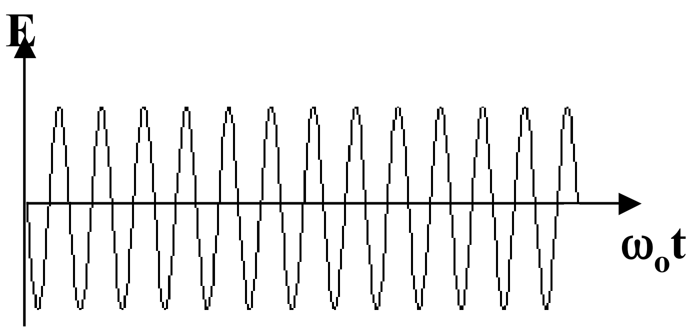
Electric field of spontaneous radiation of a single atom or electric field excited by single electron passing through resonator cavity or response of an oscillator to a kick and…

$$E_{max} \sim \beta\, e/R\, \lambda_0$$



Electric field of spontaneous radiation of a single electron passing through bend magnet (synchrotron radiation)

$$E_{max} = 4\, e\, \gamma^4 / R\rho \sim \gamma\beta\, e/R\, \lambda_c$$

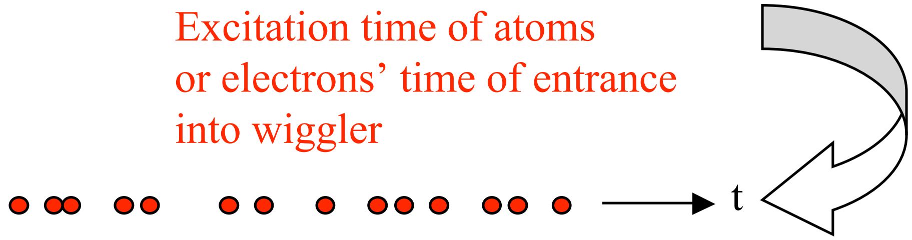


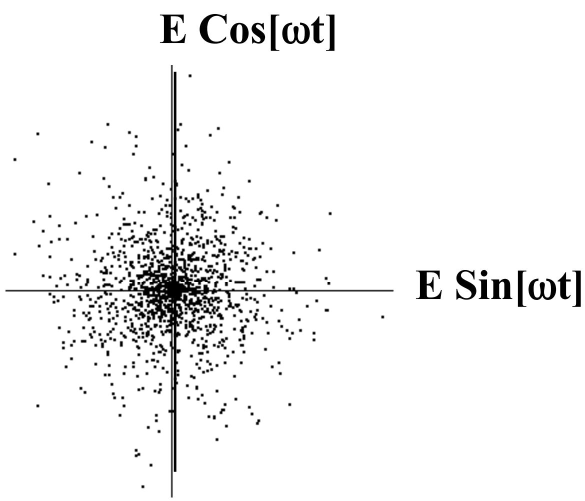
Electric field of spontaneous radiation of a single electron passing through a wiggler

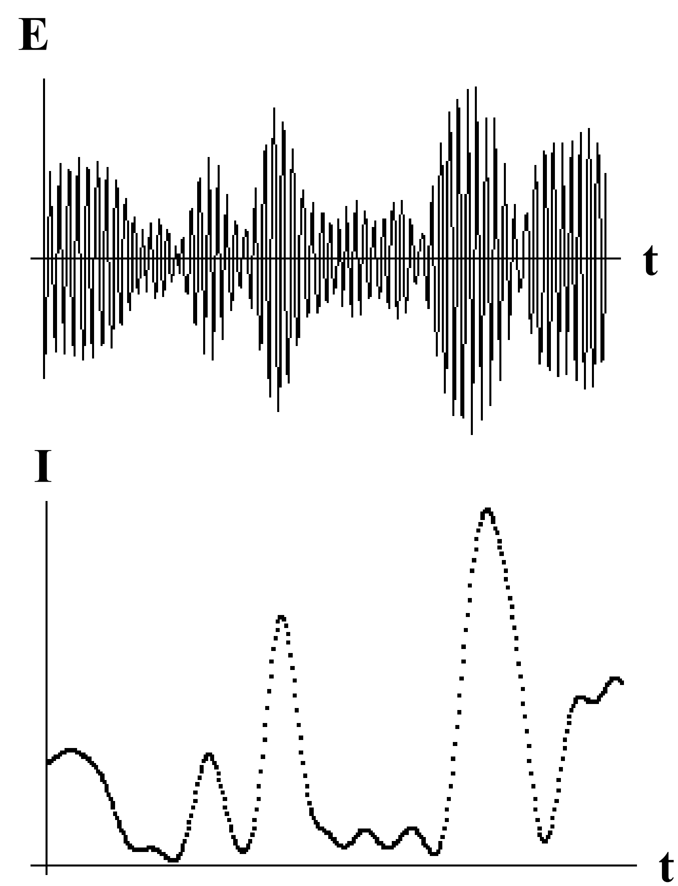
$$E_{max} = \frac{1}{(1+K^2)^2}\; 4\, e\, \gamma^4 / R\rho \sim \gamma\beta\, e/R\, \lambda_0$$

Chaotic Electromagnetic Field

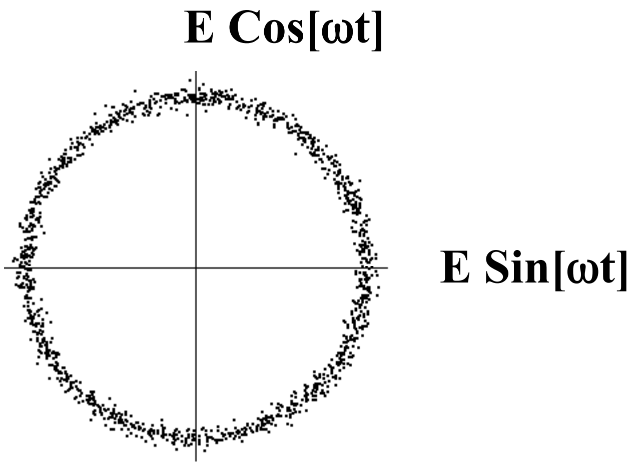
Excitation time of atoms
or electrons' time of entrance
into wiggler

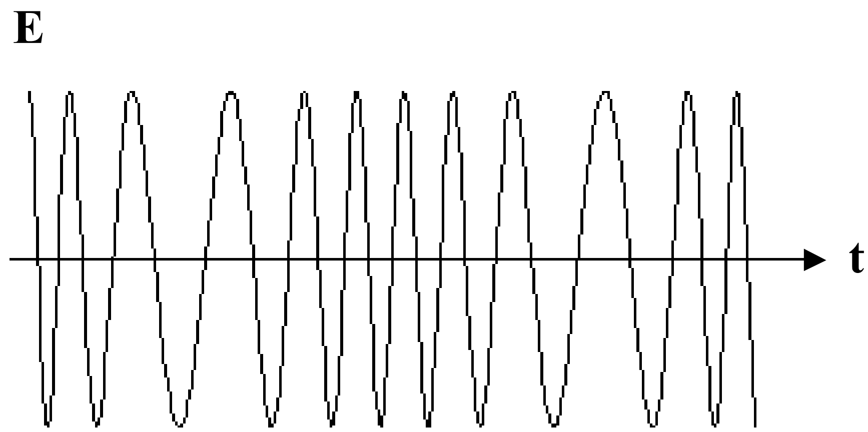


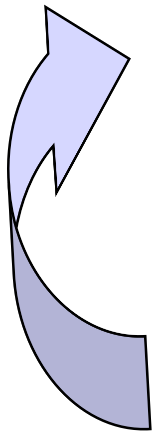




Coherent Electromagnetic Field

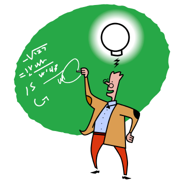






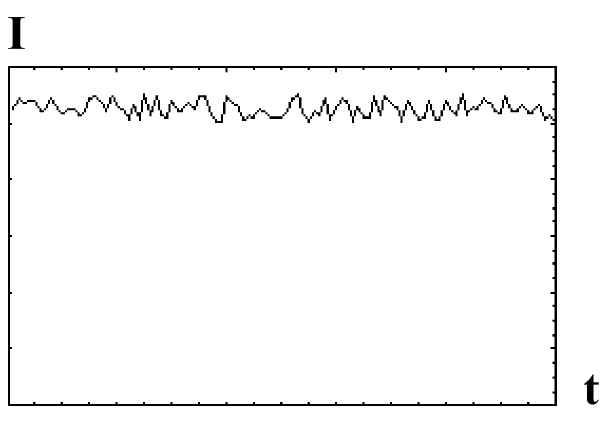
Media with inverse population (atomic or electron beam in wiggler) can amplify signal (and + positive feedback = generator)

Max, you are telling lies !



If you are right, why do we not observe 100% fluctuations in the signal from fluorescent lamp on time scale 10-20 ns (life time of transitions)? Photo detector and oscilloscope have right time resolution.

Light radiated from area >> d^2 (d ~ transverse coherence size ~λ/θ , λ - wave length and θ - observation angle) is incoherent, thus intensities add and therefore observer will see very small fluctuations.
We can observe 100% fluctuations, but we need to install diaphragms that will transmit light only from area of coherence.



When we try to observe 100% fluctuations,
we are force to go to small solid angle
and thus small intensity and gives rise to the question

When do quantum phenomena become important ?

Fields can be treated classically if degeneracy parameter $\delta \gg 1$

Degeneracy parameter

Transverse coherence size	$d \sim \lambda/\theta$
Longitudinal coherence size	$l \sim c / \Delta\omega$
Volume of coherence	$d^2\, l$
Degeneracy parameter	δ

δ = number of photons in volume of coherence

δ = number of photons per mode

1. Thermal source $\quad \delta = \dfrac{1}{\mathrm{Exp}[h\nu/kT] - 1} \approx 10^{-2}$ $\quad$ Sunlight

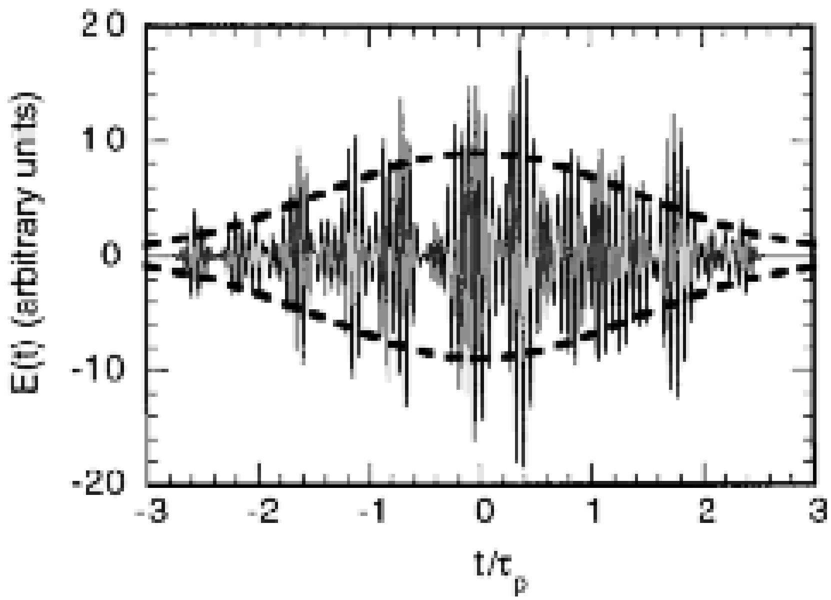
2. Synchrotron radiation $\quad \delta \approx \alpha N \lambda / (c\tau F) \quad \approx 10^4/F$

3. Wiggler radiation $\quad \delta \approx 2 M_W \alpha N \lambda / (c\tau F) \quad$ for $K_W > 1$

F = number of transverse coherence modes in beam area

4. Laser ~1 mJ visible light $\quad \approx 10^{15}$

Can information about longitudinal charge distribution be extracted from incoherent radiation ?

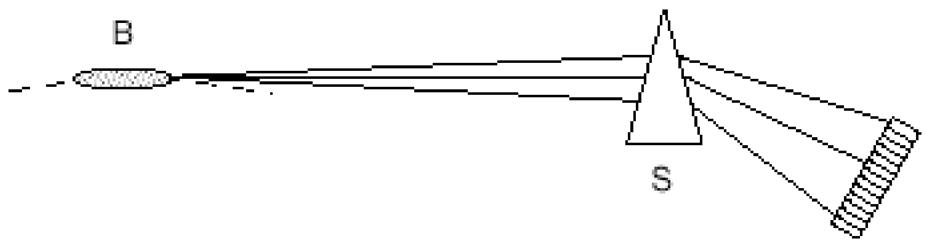


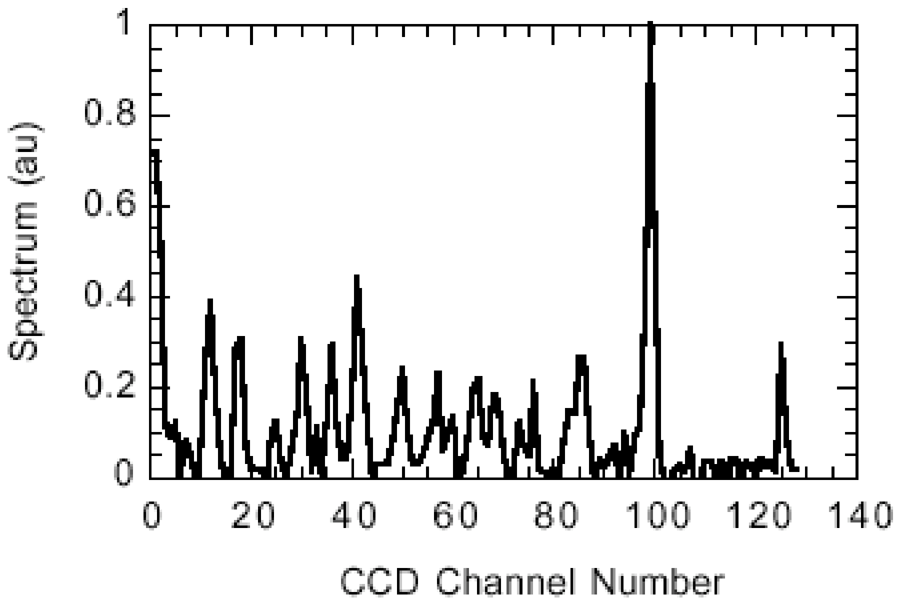
In time domain

Time domain picture of incoherent radiation produced by Gaussian longitudinal charge distribution and filtered with $\omega/\Delta\omega = 10$ and $\tau \approx 10/\Delta\omega$

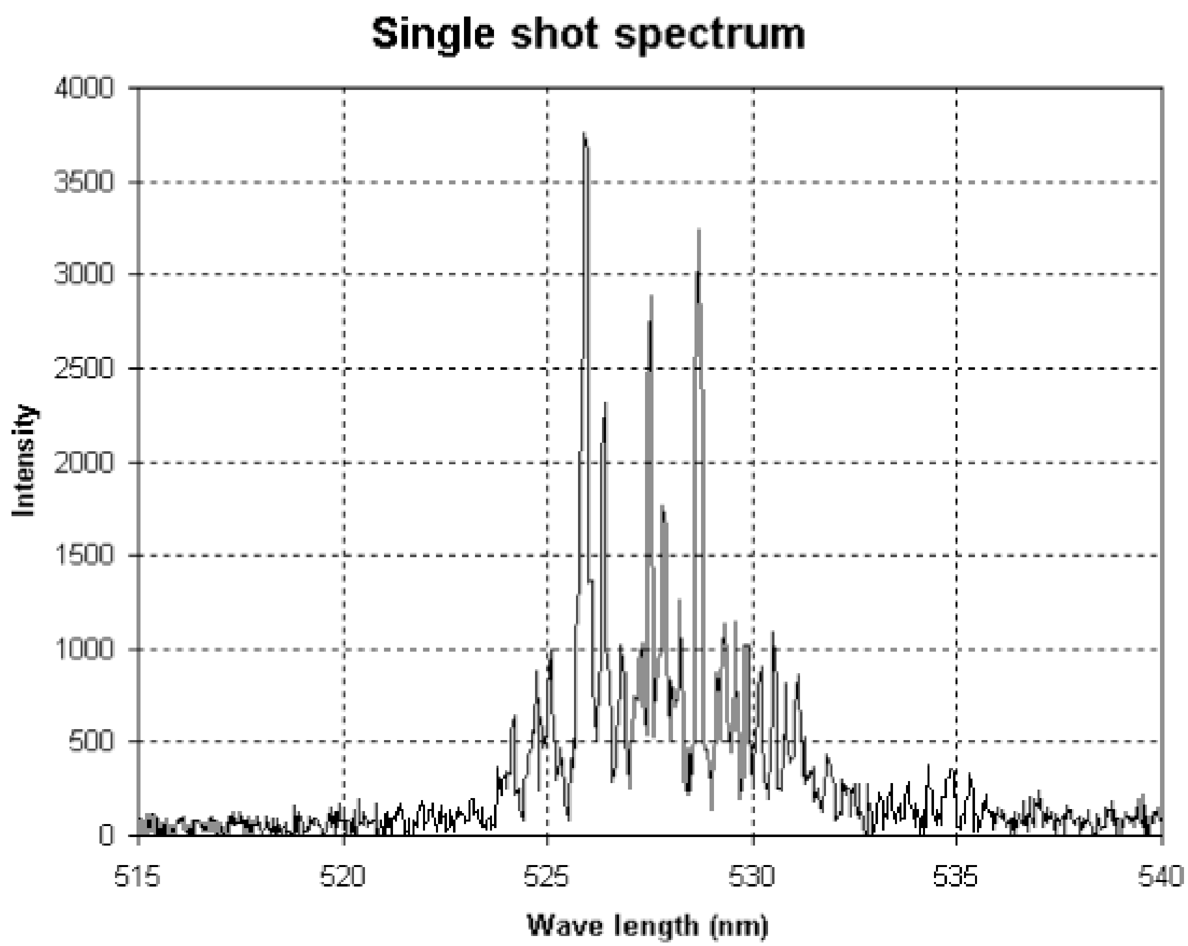
Pulse-to-pulse fluctuations in intensity under these conditions will be ~ $1/\sqrt{M}$, where M is the number of groups. In this case M ≈10 and fluctuations ~ 30%. This does not depend on the number of electrons if the degeneracy parameter is large. Pulse length can be recovered from known filter bandwidth and measured fluctuations

Frequency Domain



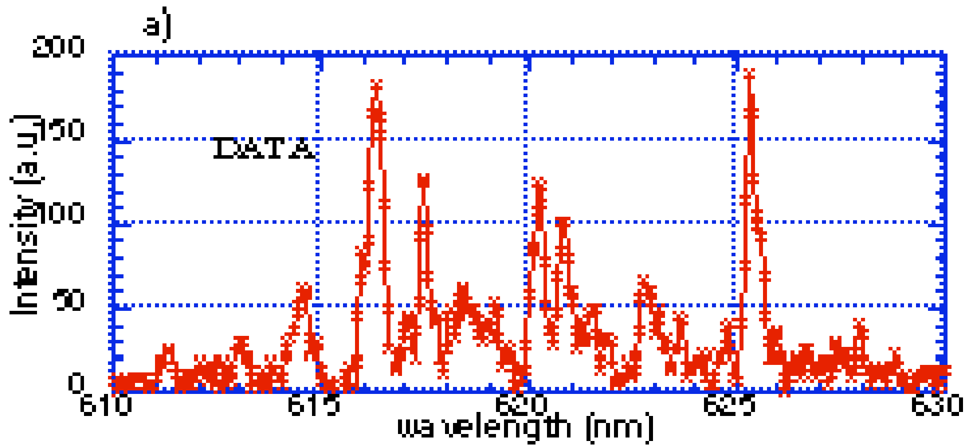


Spectral fluctuations: narrow spikes with width $1/\tau_b$.

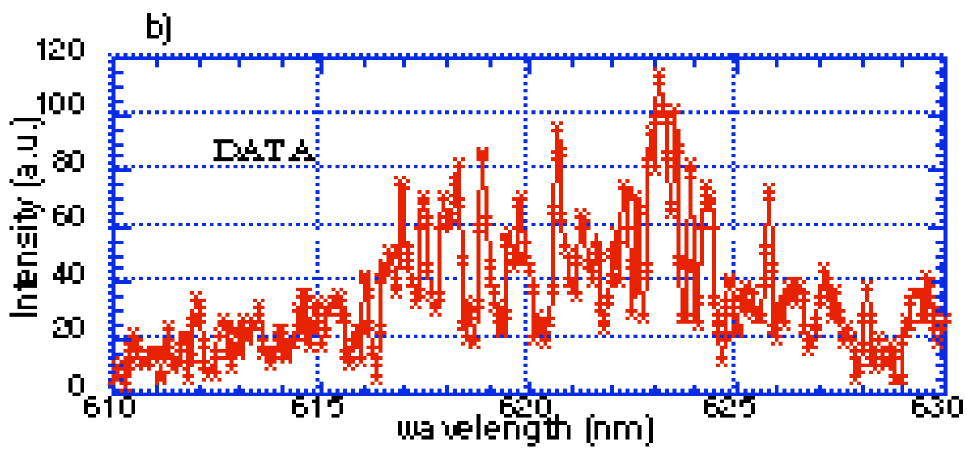


Experimental data taken at Argon National Lab (V. Sajaev)

Probability distribution for spikes



for beam size smaller than transverse coherence size and/or resolution of spectrometer $\Delta\omega < 1/\tau_b$, follows the Poisson distribution

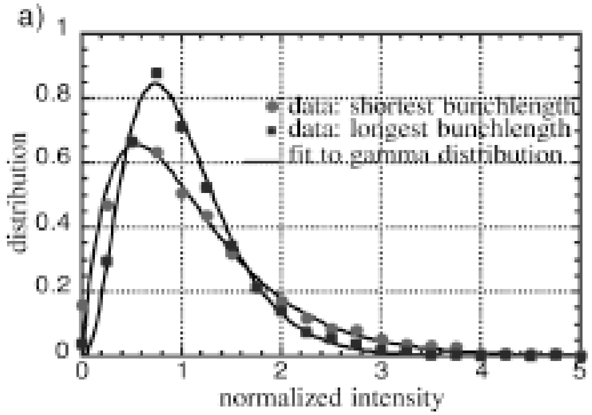


for beam size larger than transverse coherence size and/or resolution of spectrometer $\Delta\omega > 1/\tau_b$, follows the Gamma distribution

Experimental data taken at Brookhaven National Lab (P. Catravs)

Surviving in case of poor resolution

Average spike width gives information about time duration of radiation when resolution of spectrometer is much better than inverse length of radiation pulse. In the opposite case, it is still possible to extract information about time duration of radiation. Signal obtained in the frequency band equal to resolution of the spectrometer will be the sum of several independent events (spikes in frequency domain and from different transverse coherence regions) each distributed according Poisson statistic. Resulting distribution will follow Gamma distribution.



Measured distribution of spectral intensity fluctuations for 1.5 ps and 4.5 ps bunch lengths are plotted along with gamma distribution fit.

$$f(x;k) = \frac{x^{k-1}k^k}{\Gamma(k)} e^{-kx}$$

where x = I/<I> , and k is number of independent unresolved spikes

The same information can be extracted from the measurement of fluctuations relative to the base. For example if k =10, it will be 30% of relative fluctuations and, otherwise, if 30% fluctuations are measured then there were 10 independent spikes in spectrometer resolution width. One can find the number of spikes in frequency domain and time duration of radiation by using spatial transverse mode filter.

Can it be used at LCLS for measurement of X ray pulse duration?

degeneracy parameter $\delta \approx 2\, M_g\, \alpha\, N\, \lambda / (c\tau\, F)$ and F~ area of the beam / λ^2

for fixed linear density and size of the electron beam $\delta \sim \lambda^3$

but if beam emittance ~ $\lambda/4\pi$; F~1 and $\delta \sim \lambda$

for LCLS (M_g~200) $\lambda \approx .15$ nm; $\delta \approx$ gain x10^4 are good enough even for gain ≈ 1. Fluctuation signal can be used for tuning LCLS.

One can measure in time domain, then number of spikes ~ 10^3 - 10^4 and fluctuations are ~ 1 - 3%
(this can be used for rough estimation of x-ray pulse length)
Or, for a one-shot measurement, can use x ray spectrometer with width

$\Delta\omega/\omega \sim 10^{-3}$ and resolution

$\delta\omega/\omega \sim 10^{-6}$

spectrometer resolution X ray frequency

It looks possible !

14 June 1996

Dr. Gennady Stupakov
SLAC
P.O. Box 4349
Stanford, CA 94309

Re: Fluctuational interferometry for measurement of short pulses of incoherent radiation

By: M.S. Zolotorev and G.V. Stupakov

LQ5948

Dear Dr. Stupakov:

The above manuscript has been reviewed by our referee(s).

On the basis of the resulting report(s), it is our judgment that the paper is unacceptable for publication in Physical Review Letters. We enclose comments from the criticism that led to our decision.

Yours sincerely,

Reinhardt B. Schuhmann

Reinhardt B. Schuhmann
Assistant Editor
Physical Review Letters

Referee A

"First, the authors' technique requires the measurement of rapid fluctuations. Unfortunately, these fluctuations occur on a time scale that is too fast to be measured. If such fluctuations can be measured then the pulse itself could be measured without the authors' technique. This is a catch-22 common to fast-pulse measurement schemes…

Finally, fast pulse measurement has experienced tremendous advances in the past few years, and full-characterization techniques are now available…Yet another technique for measuring the pulse intensity and/or field *autocorrelation* is not of great interest, and especially not worthy of PRL."

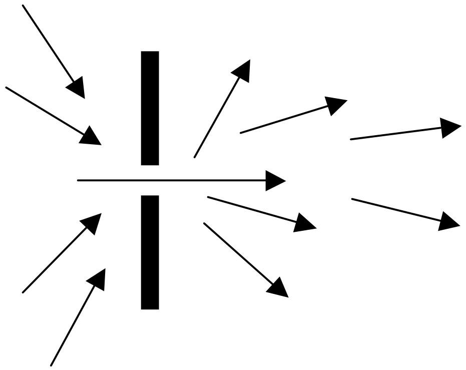
Referee B

"It is not obvious that the model (i) is of especially wide interest or importance, or that it applies to anything but almost completely incoherent radiation. The authors do not claim that it should, but the sources of such radiation are few and far between. I am not expert enough to know whether synchrotron radiation is of this type, but certainly laser-based optical sources (even ASE sources) do not typically produce such light…"

Referee C

"Generally, I agree with referee B. I find that the importance and broad interest criteria for publication in PRL are not met,…

Appropriate light sources are indeed few and far between…



Linear acceleration/ deceleration

If in the far field region one has fields from an external source and fields from spontaneous emission from particles

$$\mathbf{E} = \mathbf{E}_{ex}(\omega,\theta) + \mathbf{e}_s(\omega,\theta)$$

$$\mathcal{E} \sim \int |E_{ex}(\omega,\theta) + e_s(\omega,\theta)|^2 \, d\Omega d\omega$$

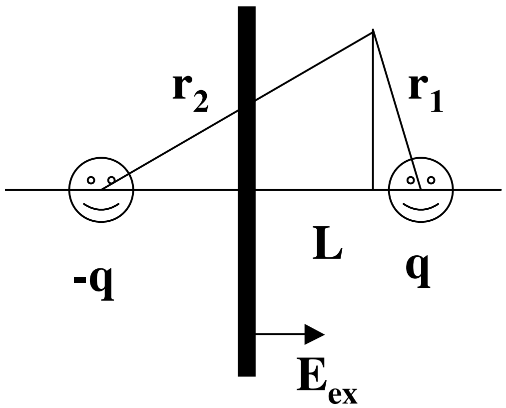
$$\mathcal{E} = \mathcal{E}_{ex} + \mathcal{E}_s + 2\eta\sqrt{\mathcal{E}_{ex}\mathcal{E}_s}\,\mathrm{Cos}\varphi$$

$$\boxed{\begin{aligned} E(\omega) &\sim \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt \\ E(0) &\sim \int_{-\infty}^{\infty} E(t)\, dt \equiv 0 \end{aligned}}$$

Were η is the overlap of external and spontaneous fields

Acceleration/ deceleration linear with external field always is interference of spontaneous radiation with external field. In vacuum, far away from matter (in far field region L>>Z_R) no spontaneous radiation from particles and

No acceleration linear with external field .



Electrostatic accelerator

$$\mathcal{E}_{int} = \frac{2}{8\pi}\int \vec{E}_{ex}\,\vec{E}_s\, d^3 r =$$

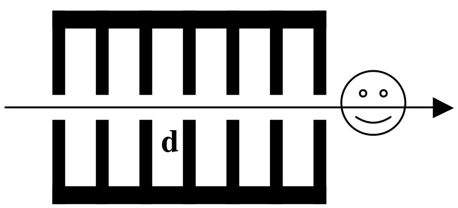
$$\frac{q\,\vec{E}_{ex}}{4\pi}\int\left(\frac{\vec{r}_1}{r_1^3} - \frac{\vec{r}_2}{r_2^3}\right) d^3 r = -qE_{ex}\,L$$

RF accelerator

$$E_s = E_{st}(1 - e^{ikd}) \sim 2E_{st}\,\mathrm{Sin}\,\frac{kd}{2}$$

$$\mathcal{E}_s = 4\,\mathcal{E}_{st}\,\mathrm{Sin}^2\,\frac{kd}{4} \approx \mathcal{E}_{st}\,k^2 d^2 \qquad \text{for } kd<<1$$

$$\mathcal{E}_{st} = \frac{q^2}{\pi^2}\,d\Omega d\omega$$



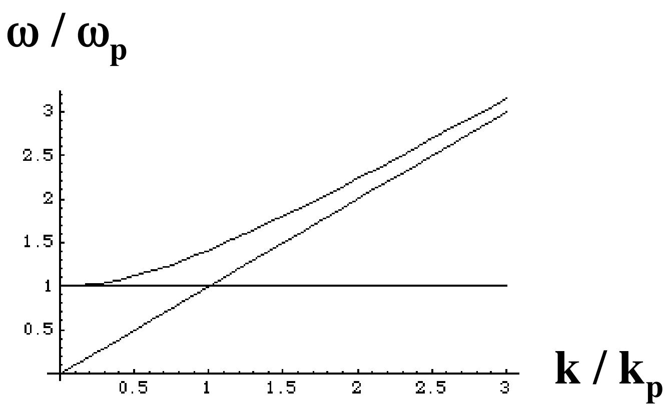
$$\mathcal{E}_{int} = 2\sqrt{\mathcal{E}_{ex}\,\mathcal{E}_s}\,\mathrm{Cos}\varphi$$

$$\approx 2\sqrt{\frac{E_{ex}^2}{8\pi}\,\frac{\mathrm{Vol}}{2}\,d^2 q^2 k^2\,\frac{d\Omega dk}{\pi^2}} = e\,E_{ex}\,d\sqrt{M}$$

$$M = \frac{2\,k^2\,dk\,d\Omega\,\mathrm{Vol}}{(2\pi)^3} \equiv 1$$

$$\mathcal{E}_{int} = q\,E_{ex}\,d$$

Plasma acceleration



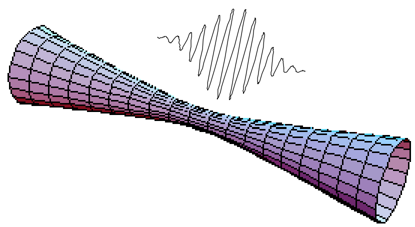
$$\omega_{\perp}^{2} = \omega_{p}^{2} + c^{2} k_{\perp}^{2}$$

$$\omega_{\parallel} = \omega_{p}$$

Moving charged particle in plasma radiate longitudinal Cherenkov wave

Interference of this radiation with plasma wave exited by external source result in acceleration of charge particle in plasma

Nonlinear acceleration in vacuum (ponderomotive)



Primary wave forced particle oscillate in transverse direction and as a result, secondary wave are radiated. Interference of this waves result in energy exchange between particle and wave (ponderomotive acceleration).

1. In a plane wave electron moved with velocity: $\beta_{tr} = \frac{a\, e^{i \frac{\omega t}{1+a^2/2}}}{1+a^2/2}$ **and** $\beta_l = \frac{a^2/2}{1+a^2/2}$

2. Radiated power in one special mode on frequency ω , Δω are $\mathcal{E}_s = \frac{a^2/2}{1+a^2/2}\, \alpha\, \hbar\, \omega$

3. Energy gain $2\sqrt{\mathcal{E}_I\, \mathcal{E}_s}\, \mathrm{Cos}[\varphi] \approx 2\sqrt{\mathcal{E}_I\, \mathcal{E}_s}\, \frac{1}{\omega\, \tau}$

Microbunch instability

$$\Delta E_i = 2\sqrt{E_L E_s}\ \mathrm{Cos}[\varphi_i] \ \text{->}\ 2\sqrt{E_L\left(E_s + \partial_{E,} E_s\, \Delta E_i\right)}\ \mathrm{Cos}[\varphi_i]$$

$$E_s = \frac{e^2}{c}\Delta\omega$$

$$g = \frac{\Delta E}{E_L} = N\, \partial_{E,} E_s = \frac{I}{\gamma I_A}$$

$$\frac{db}{dl} = \frac{\Delta\varphi}{l_f} = k l_f \eta \frac{\Delta E}{E}$$

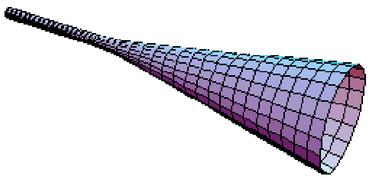
$$\frac{d}{dl}\left(\frac{\Delta E}{E}\right) = \frac{g}{l_f}$$

$$\frac{d^2 b}{dl^2} - \frac{g}{l_f} k \eta b = 0$$

$$\frac{1}{l_g{}^2} = \frac{I}{\gamma I_A}\ \frac{\eta k^2}{(k\rho)^{2/3}}$$

Very close to exact solution by G. Stupakov and S. Heifets

FEL small signal and small gain



If in the far field region one has fields from an external source and fields from spontaneous emission from particles

$$\vec{E}(\vec{r},t) = \vec{E}_L(\vec{r},t) + \sum \vec{E}_S(\vec{r},t-\tau_i)$$

$$\Delta\mathcal{E}_e = 2\int \vec{E}_L(\vec{r},t)\vec{E}_S(\vec{r},t-\tau_i)\,dtds \Rightarrow 2\int \vec{E}_L(\vec{r},\omega')\vec{E}^{*}s(\vec{r},\omega'-\omega_e)e^{i\omega'\tau_i}\,d\omega' ds$$

$$\vec{E}(\vec{r},\omega) = \vec{E}_L(\vec{r},\omega) + \{\vec{E}_S(\vec{r},\omega-\omega_e) + \frac{\partial\vec{E}_S(\vec{r},\omega-\omega_e)}{\partial\omega_e}\frac{\partial\omega_e}{\partial\mathcal{E}_e}\Delta\mathcal{E}_e\}\sum e^{-i\omega\tau_i} =$$

$$\vec{E}_L(\vec{r},\omega) + 2\frac{\partial\vec{E}_S(\vec{r},\omega-\omega_e)}{\partial\omega_e}\frac{\partial\omega_e}{\partial\mathcal{E}_e}\dot{N}_e\int e^{-i(\omega-\omega')\tau}d\tau\int \vec{E}_L(\vec{r},\omega')\vec{E}^{*}s(\vec{r},\omega'-\omega_e)\,d\omega' ds$$

$$\vec{E}(\vec{r},\omega) = \vec{E}_L(\vec{r},\omega)\{1 + 2\pi\dot{N}_e\frac{\partial\omega_e}{\partial\mathcal{E}_e}\frac{\partial}{\partial\omega_e}\int\left|\vec{E}_S(\vec{r},\omega-\omega_e)\right|^2 ds\}$$

$$g_o(\omega) = 2\pi\dot{N}_e\frac{\partial\omega_e}{\partial\mathcal{E}_e}\frac{\partial W_s(\omega-\omega_e)}{\partial\omega_e}; \qquad \int_{-\infty}^{\infty} W_s(\omega-\omega_e)\,d\omega = W_s$$

$$g_o(\omega) = 2\pi\dot{N}_e\frac{\partial W_s[\omega-\omega_e(\mathcal{E}_e)]}{\partial\mathcal{E}_e}$$