Optical Stochastic Cooling in Tevatron

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Objectives

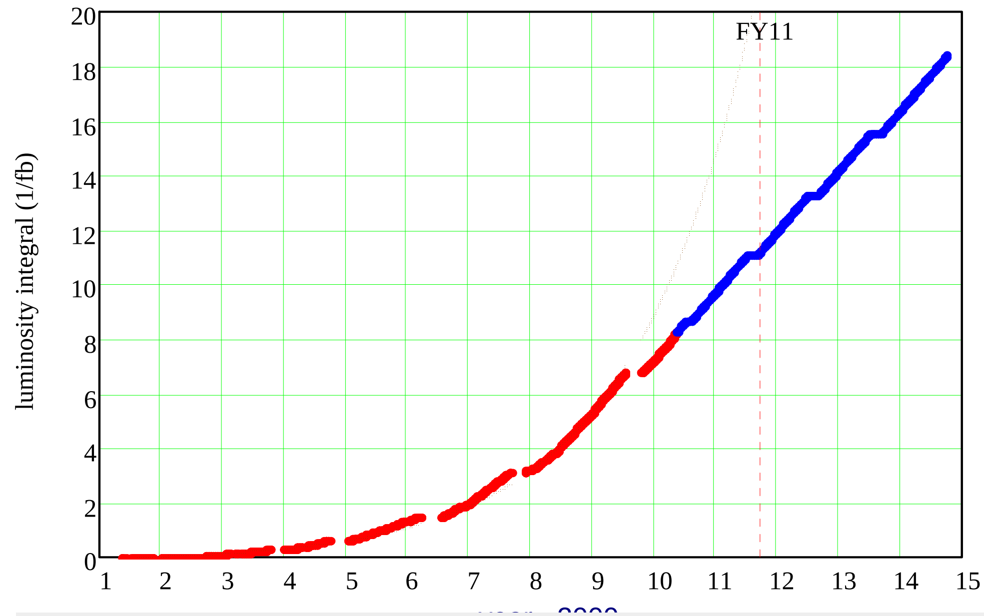
- Extension of Tevatron operation to 2014
- Are there luminosity upgrades?
- Can the Optical Stochastic Cooling (OSC) help?

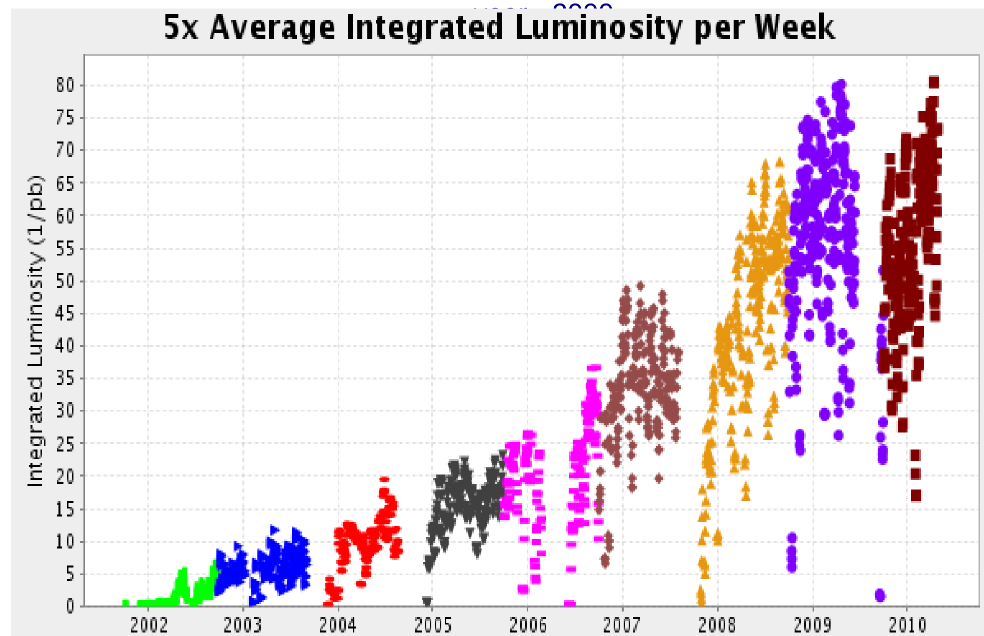
Outline

- Tevatron luminosity and its evolution
- Requirements to the cooling
- Optical stochastic cooling principles
- Damping rates computation and optimization
- Optimization and efficiency of laser kick
- Requirements to the laser power
- Conclusions

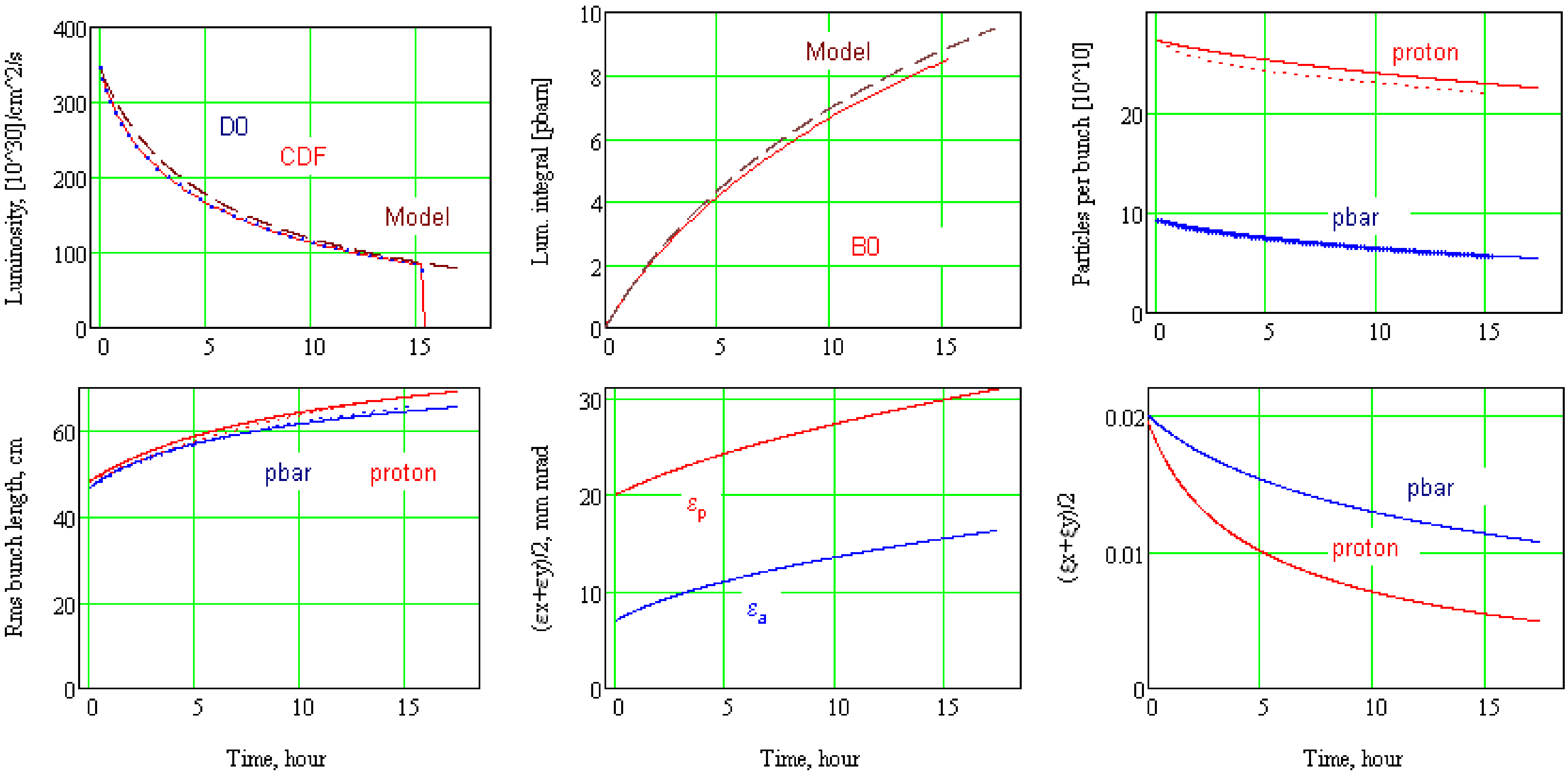
Tevatron Luminosity

- All planned luminosity upgrades are completed in the spring of 2009
- From Run II start to 2009 the luminosity integral was doubling every 17 months
- Since 2009 average luminosity stays the same ~51 pb^{-1}/week
- The average luminosity is limited by the IBS
  - Larger beam brightness results in faster luminosity decay
- It is impossible to make significant (~2 times) average luminosity increase with one exception - The beam cooling in Tevatron
  - 10-20% is still possible (new tunes, larger intensity beams)



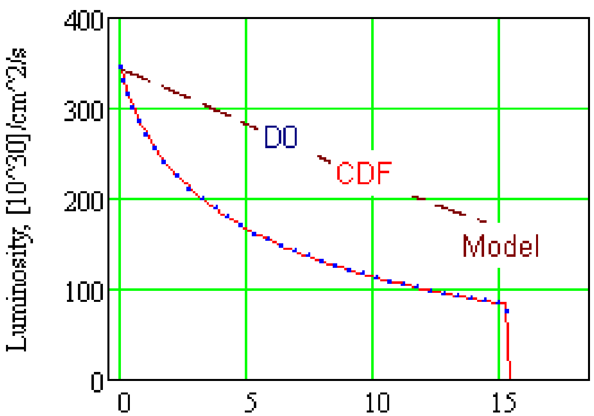


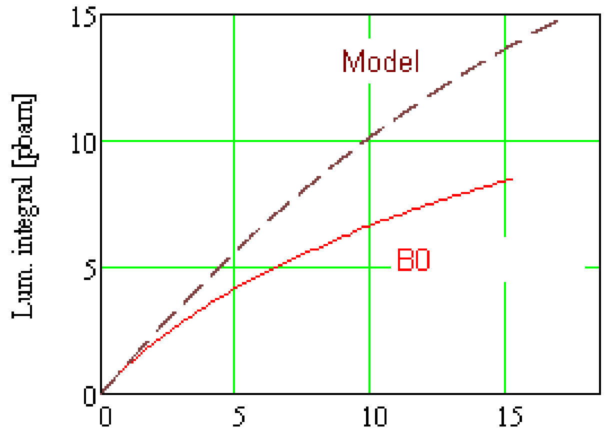
Luminosity Evolution for Present Stores (Store 6950)

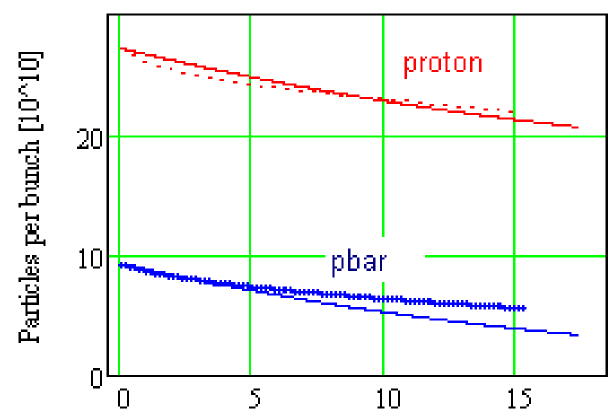


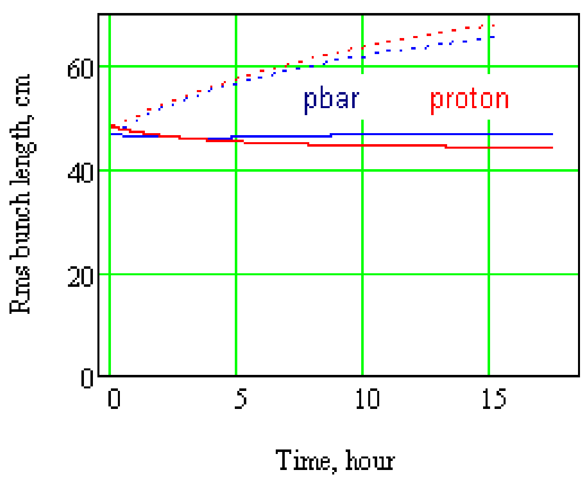
- About 10% of luminosity integral is lost due to beam-beam
- IBS is the main mechanism causing fast luminosity decrease
  - Presently, there are no means to reduce IBS in Tevatron
- About 40% of pbars are burned in luminosity
  - It is the second leading reason of luminosity decrease

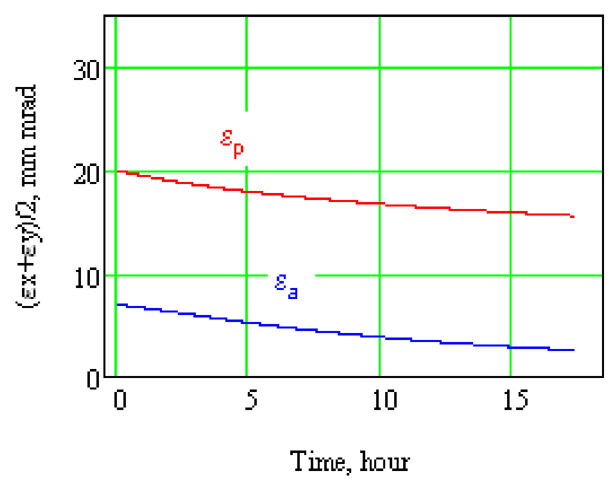
Luminosity Evolution with Moderate Cooling

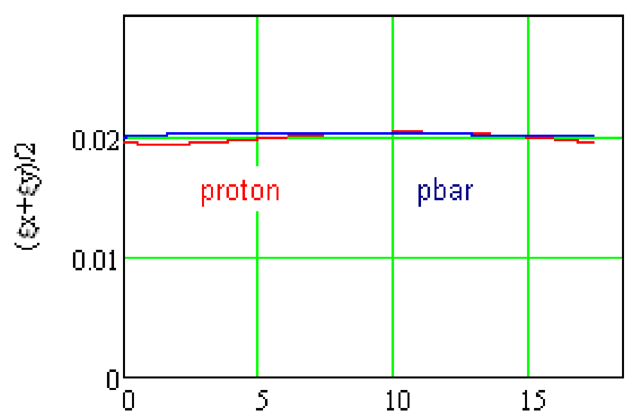




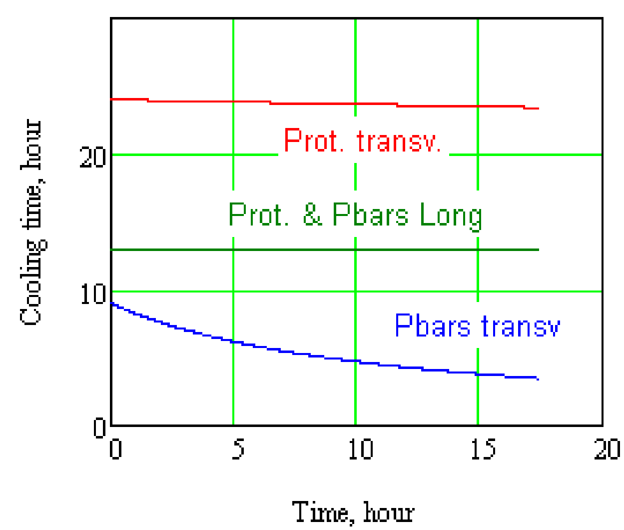




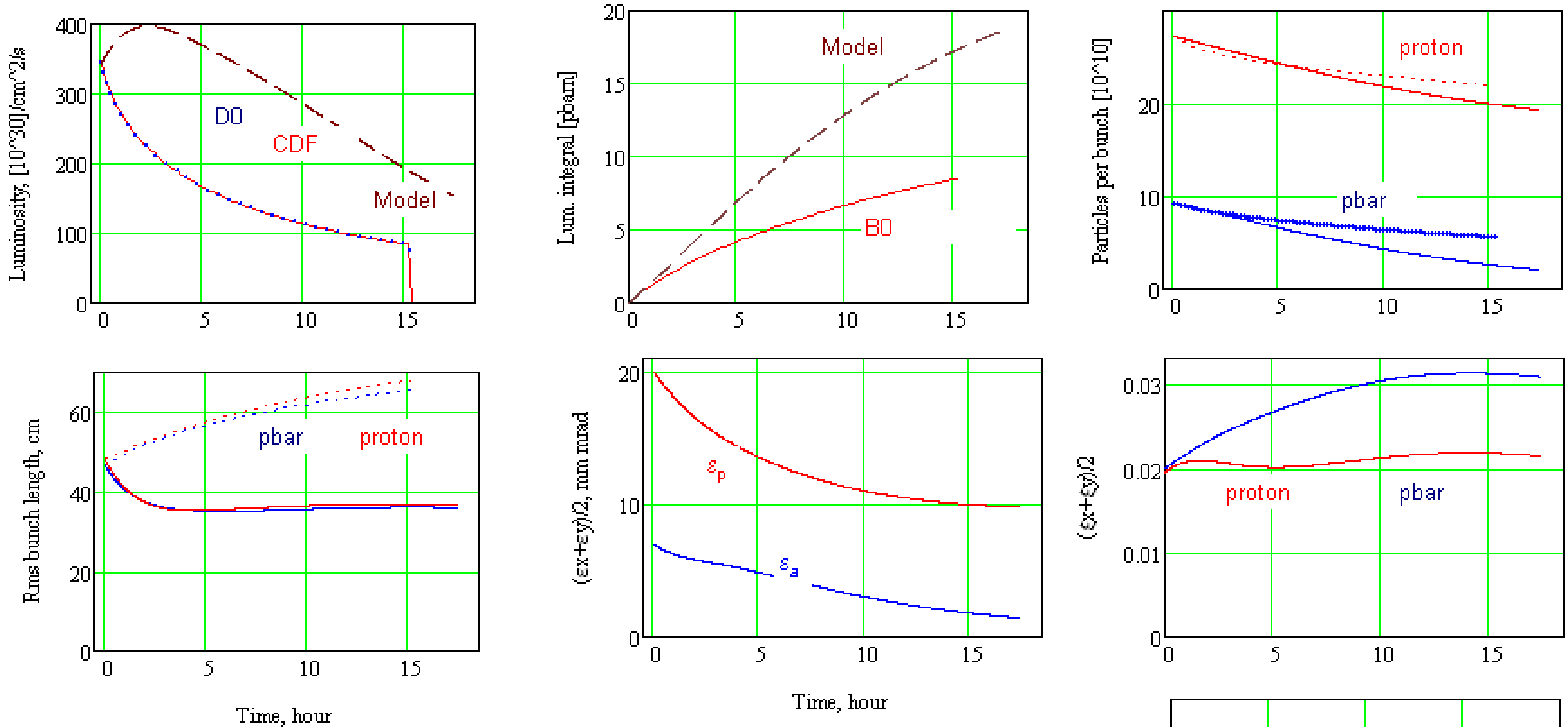




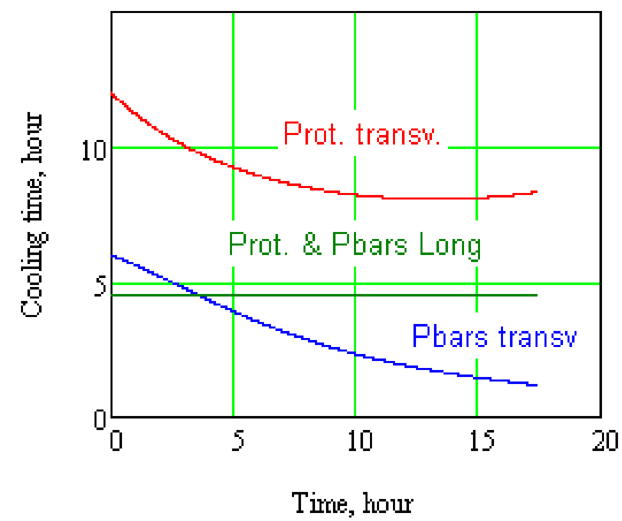
- Cooling rate is limited by ξ_{BB} of 0.02
- 1.58 times increase of luminosity integral
- 63% of pbars are burned in luminosity
- Much smaller luminosity variations



Luminosity Evolution with Aggressive Cooling



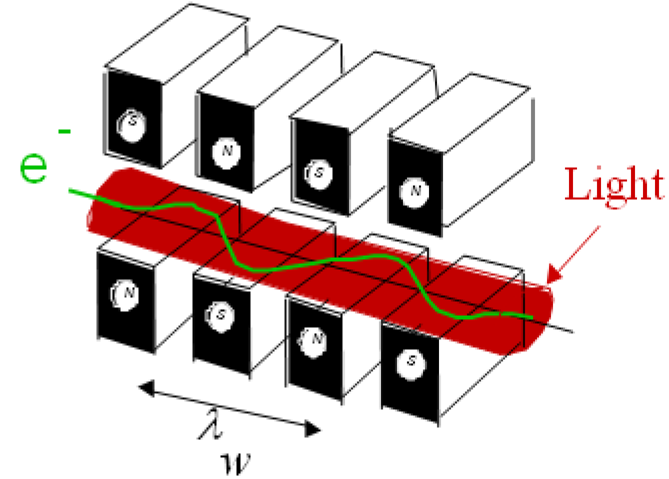
- Cooling rate is limited by peak luminosity of $4\cdot10^{32}$ and by ξ_{BB} = 0.03 for pbars
  - Requires tunes closer to half-integer(0.58→.52)
- 1.96 times increase in average luminosity
  - 78% of pbars are burned in luminosity



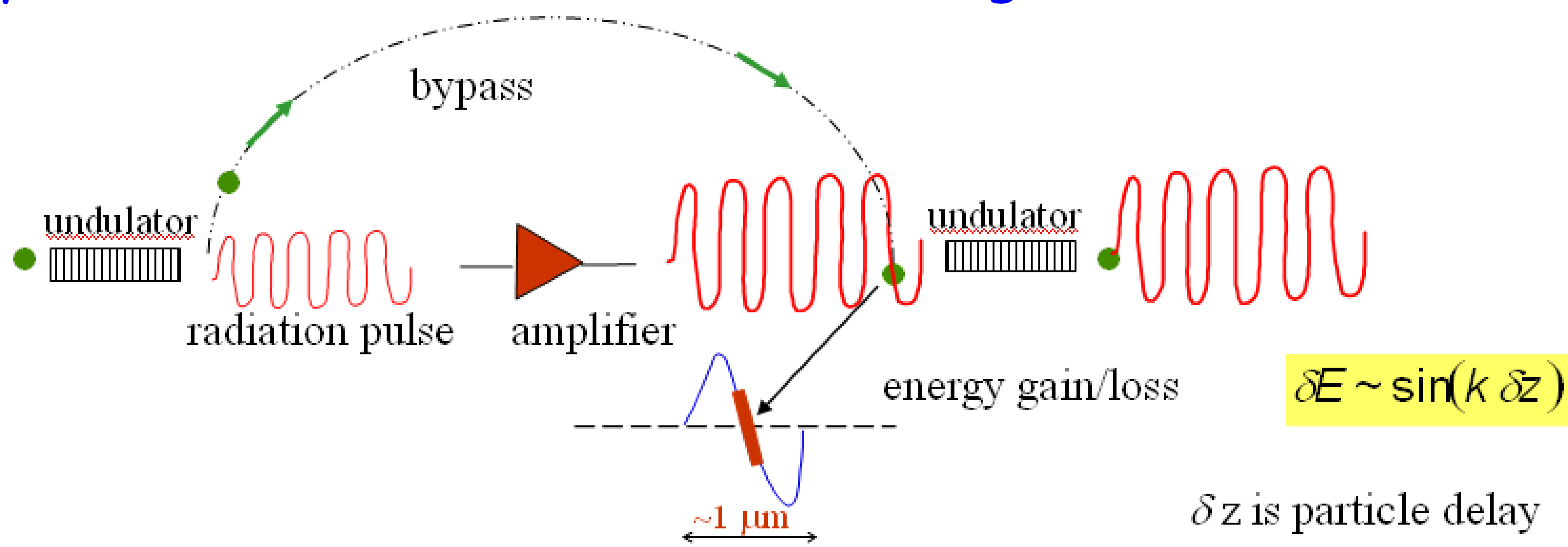
Requirements to the Beam Cooling

- Cooling time has to to be varied during the store independently for protons and pbars and transverse and longitudinal planes
  - Beam overcooling results in
    - Particle loss due to beam-beam (transverse overcooling)
    - Longitudinal instability (longitudinal overcooling)
- Simple estimate of required bandwidth based on (λ=2 *W*/ *N*) results in ~200 GHz
  - Well above bandwidth of normal stochastic cooling
  - Only optical stochastic cooling has sufficient bandwidth
- Cooling times (in amplitude):
  - Protons: L - 4.5 hour; $\perp$ - 8 hour
  - Antiprotons: L - 4.5 hour; $\perp$ - 1.2 hour
- Tevatron has considerable coupling and all transverse cooling can be applied in one plane
  - It requires doubling hor. cooling decrement:
    - I.e. for protons $\lambda_s = \lambda_x = 4.5$ hour

Optical Stochastic Cooling



- Suggested by Zolotorev, Zholents and Mikhailichenko (1994)
- Never tested experimentally
- OSC obeys the same principles as the microwave stochastic cooling, but exploits the superior bandwidth of optical amplifiers ~ 10^{14} Hz
- Undulator can be used as pickup & kicker
- Pick-up and Kicker should be installed at locations with nonzero dispersion to have both ⊥ and L cooling.



MIT-Bates Proposal for Tevatron (2007)

OSC and Tevatron Luminosity

How to increase luminosity (peak and integrated) ?

Record Comparison Before/After 2006 Shutdown

- Peak luminosity increased 62% (180 → 292 μb^{-1}/s)   1 μb^{-1}/s = 10^{30} cm^{-2} s^{-1}
- Weekly integrated luminosity increased ~75% (25 pb^{-1} → 45 pb^{-1})
- Monthly integrated luminosity increased ~95% (85 pb^{-1} → 167 pb^{-1})
- One hour antiproton stacking record -- 23.1 10^{10}/hr
- Antiproton accumulation for one week -- 2800 10^{10}

$$L = \frac{f N_p N_a}{2\pi(\varepsilon_p + \varepsilon_a)\beta^*} H\left(\sigma_z / \beta^*\right)$$

Big progress this year

- *N = bunch intensity, f = collision frequency*
- *ε = transverse emittance (size), σ_z = bunch length*
- *H = "hour glass" factor (<1, accounts for beam size over finite bunch length)*

R. Moore - FNAL   HCP 2007   6

f = colliding frequency = 47.7 KHz ×36
β^* = beta function at IR = 35 cm
H = hourglass factor = .60 - .75

OSC provides possible "damping" to the 1 TeV p & pbar beams.

Damping on:

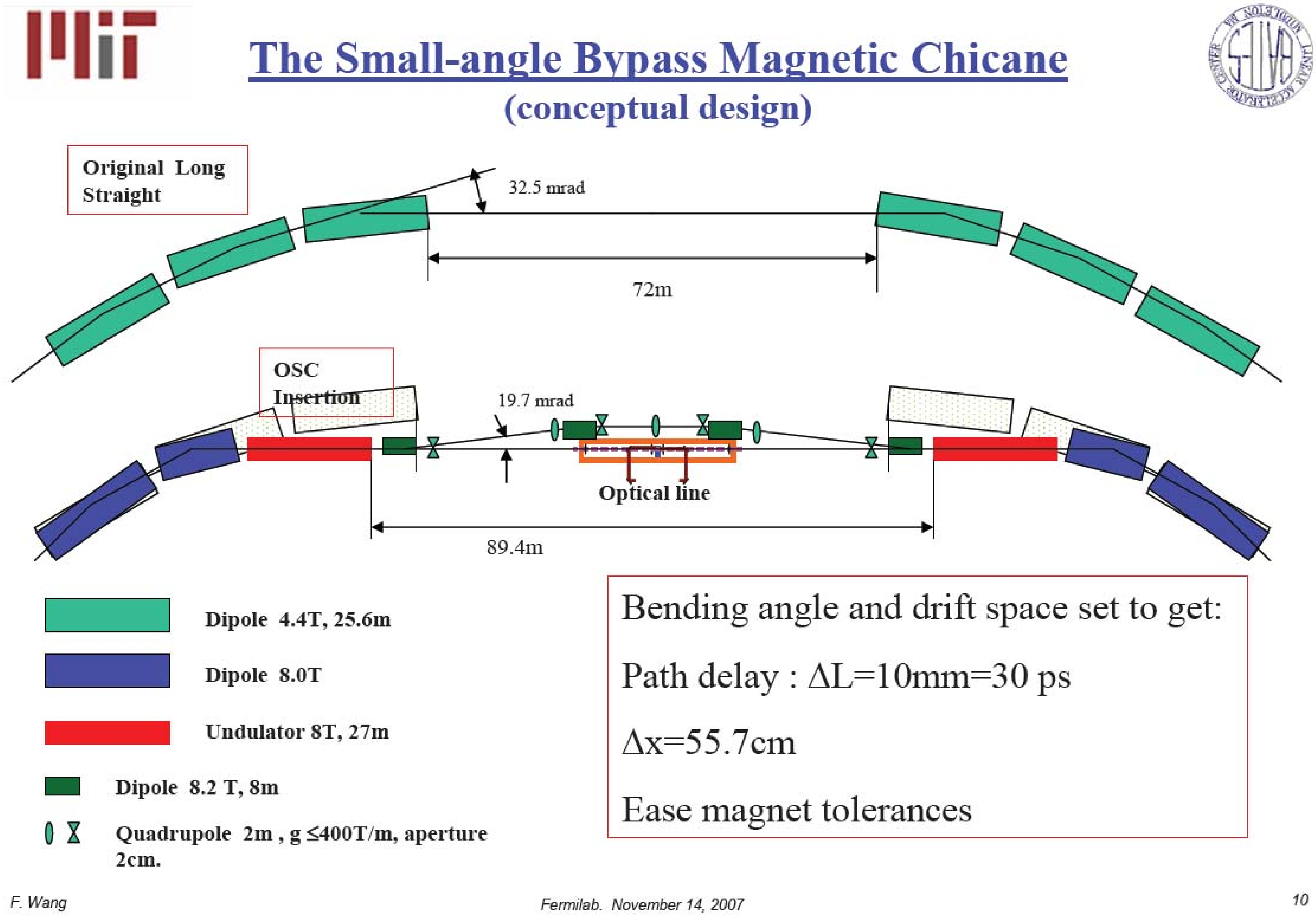
ε_p, ε_a, σ_z

Could reduce

N_p, N_a losses

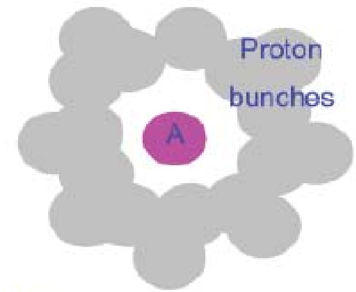
F. Wang   Fermilab. November 14, 2007   3

MIT-Bates Proposal (continue)



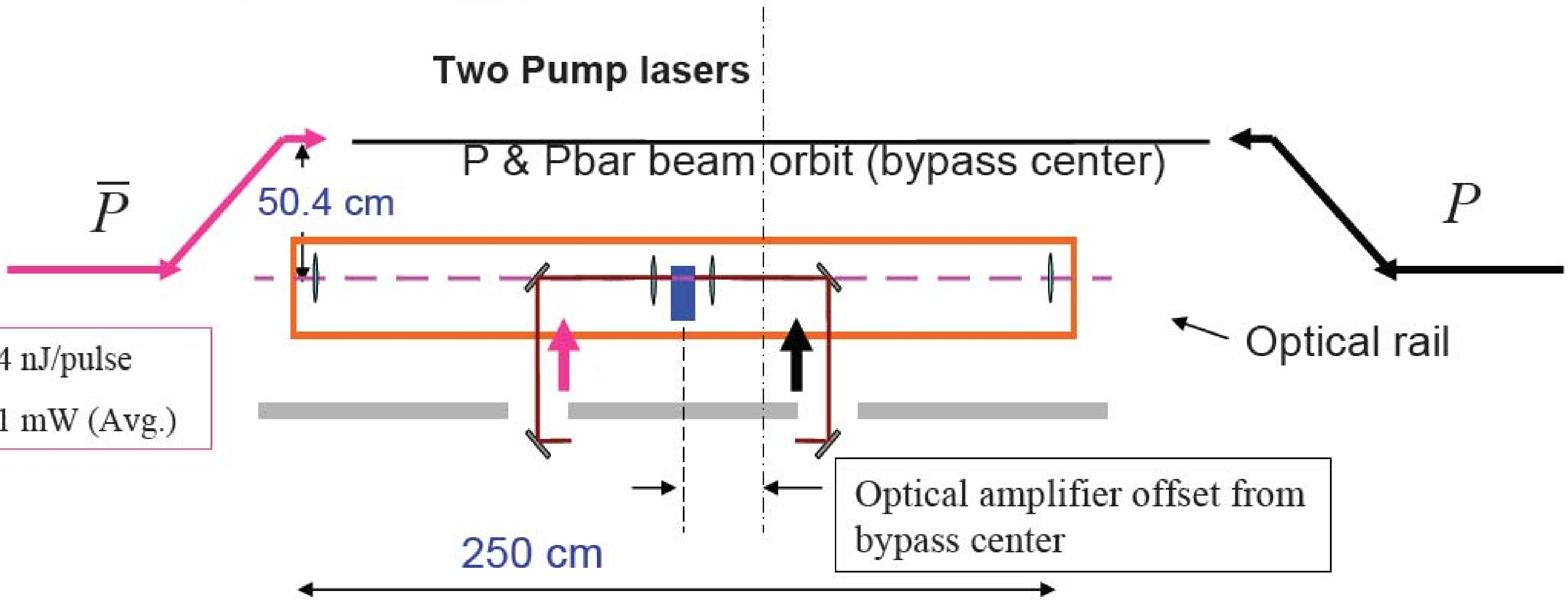
MIT-Bates Proposal (continue)

Cooling p and pbar Beams: Cooling Separately



- One cooling insertion for both p and pbar.
- Special circumstance: two beams in one ring.

Timing two pump lasers to cool p and pbar separately. For equal cooling time, cooling rate of each beam will be reduced to half .



MIT-Bates Proposal (continue)

Cooling Estimates

	Tevatron	Bates
Gamma	1045 (980 GeV)	587 (0.3GeV)
Bunch length (m)	0.57	0.025
Particle/bunch	2.5E11	1.0E8
Bunch number	36	12
Laser λ (μm)	**1.98**	**2.06**
Undulator period (m) / length (m)	2.7/27	0.2/2
Undulator parameter K	1.1	3.5
Undulator radiation/pulse (pJ)	222	0.13
Average radiation power (μW)	381	2.5
Optical power limit (W)	**20/200**	5 (Not a limit)
Optical power gain	4.84E4 / 5.25E5	1750
Laser output/pulse (μJ)	11.6 / 116	0.23 nJ
Damping time (hours)×2	**2×2 / 0.6×2**	**0.14 second**

Questions to be Answered

- Do we have a fast way (2-3 years) of OSC implementation in Tevatron?
- What is the optimal optics and how to get to it?
- What is the optimal wiggler?
- What is the laser power?

Damping Rates

- The optics design will be significantly simplified if the damping rates can be expressed through beta-functions, dispersions and their derivatives
- The sequence is
  - Express transfer matrices (6x6) through Twiss-parameters at kicker and pickup
  - Find eigen-values and eigen-vectors of the ring without cooling
  - Using perturbation theory find damping decrements
  - Determine the cooling range in amplitudes
    - Correction factors for the finite amplitude particles

Transfer Matrix Parameterization

- Vertical plane is uncoupled and we omit it in further equations
- Matrix from point 1 to point 2

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p / p \end{bmatrix}$$

$$M_{11} = \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu)$$

$$M_{22} = \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2 \sin\mu)$$

$$M_{12} = \sqrt{\beta_1 \beta_2} \sin\mu$$

$$M_{21} = \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos\mu - \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin\mu$$

- M_{16} & M_{26} can be expressed through dispersion

$$\begin{bmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_1' \\ 1 \end{bmatrix} = \begin{bmatrix} D_2 \\ D_2' \\ 1 \end{bmatrix}$$

=>

$$M_{16} = D_2 - M_{11} D_1 - M_{12} D_1'$$
$$M_{26} = D_2' - M_{21} D_1 - M_{22} D_1'$$

- Symplecticity ($\mathbf{M}^T \mathbf{U} \mathbf{M} = \mathbf{U}$) binds up M_{51},M_{52} and M_{16},M_{26} =>

$$M_{51} = M_{21} M_{16} - M_{11} M_{26}$$
$$M_{52} = M_{22} M_{16} - M_{12} M_{26}$$

- M_{56} is related to the partial slip factor, $\eta_{1\to 2}$

=> All matrix elements can be expressed through $\beta, \alpha, D, D', \eta_{1\to 2}$

Transfer Matrix Parameterization (continue)

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to M_{56}

$$\Delta s_{1\to 2} \equiv 2\pi R \eta_1 \frac{\Delta p}{p} = M_{51} D_1 \frac{\Delta p}{p} + M_{52} D_1' \frac{\Delta p}{p} + M_{56} \frac{\Delta p}{p} + \frac{1}{\gamma^2} \frac{\Delta p}{p}$$

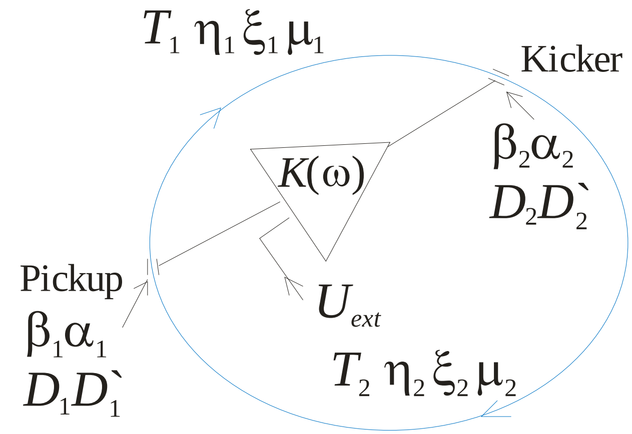
  - Further we assume that $\mathrm{v} = c$, i.e. $1/\gamma^2 = 0$ and $\eta_1 = -\alpha_{1\to 2}$.

- That results in

$$\eta_1 = \frac{M_{51} D_1 + M_{52} D_1' + M_{56}}{2\pi R}$$

  - Note that M_{56} sign is positive if a particle with positive Δp moves faster than the reference particle

Damping Rates of Optical Stochastic Cooling



Longitudinal kick

$$\frac{\delta p}{p} = \kappa \Delta s = \kappa \left(M_{151} x_1 + M_{152} \theta_{x\,1} + M_{156} \frac{\Delta p}{p} \right)$$

Or in the matrix form: $\boldsymbol{\delta}\mathbf{x}_2 = \mathbf{M}_c \mathbf{x}_1$

$$\mathbf{M}_c = \kappa \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{151} & M_{152} & 0 & M_{156} \end{bmatrix}$$

$\mathbf{M}_1$ - pickup-to-kicker matrix

$\mathbf{M}_2$ - kicker-to-pickup matrix

$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$ – ring matrix

$\mu = \mu_1 + \mu_2$

Find the total ring matrix related to kicker

$$\mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

$$\mathbf{M}_{tot}\mathbf{x}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{x}_2 + \boldsymbol{\delta}\mathbf{x}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{x}_2 + \mathbf{M}_c\mathbf{x}_1 = \left(\mathbf{M}_1\mathbf{M}_2 + \mathbf{M}_c\mathbf{M}_2\right)\mathbf{x}_2$$

⇨ $\mathbf{M}_{tot} = \mathbf{M} + \Delta\mathbf{M}_c$ where $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\ , \quad \Delta\mathbf{M} = \mathbf{M}_c\mathbf{M}_2$

Perturbation theory yields that the tune shifts are:

$$\delta Q_k = \frac{1}{4\pi} \mathbf{v}_k^{+} \mathbf{U}\ \mathbf{M}_c\ \mathbf{U}\, \mathbf{M}_1^{T} \mathbf{U}\, \mathbf{v}_k = \frac{\kappa}{4\pi} \mathbf{v}_k^{+} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{126} & -M_{116} & 0 & M_{156} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{v}_k$$

Damping Rates of Optical Stochastic Cooling (continue)

- Expressing matrix elements and eigen-vectors through Twiss parameters one obtains the cooling rates

$$\lambda_1 = -\frac{\kappa}{2}\left[D_1 D_2 \frac{(1+\alpha_1\alpha_2)\sin\mu_1 + (\alpha_2-\alpha_1)\cos\mu_1}{\sqrt{\beta_1\beta_2}} - D_1' D_2 \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu_1 - \alpha_2\cos\mu_1) \right.$$
$$\left. + D_1 D_2' \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_1 + \alpha_1\sin\mu_1) + D_1' D_2' \sqrt{\beta_1\beta_2}\sin\mu_1 \right]$$

$$\lambda_2 = -\frac{\kappa}{2} M_{1_{56}} - \lambda_1 = -\pi\kappa R\eta_1$$

The bottom equation can be directly obtained from the definition of the partial slip factor.

- The above equations yield that the sum of the decrements is

$$\lambda_1 + \lambda_2 = -\frac{\kappa}{2} M_{1_{56}}$$

Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

$$\sigma_{\Delta s}{}^2 = \int \left(M_{151} x + M_{152} \theta_x + M_{156} \tilde{p} \right)^2 f\left(x, \theta_x, \tilde{p} \right) dx d\theta_x d\tilde{p} \ , \quad \tilde{p} = \frac{\Delta p}{p}$$

  - Performing integration one obtains for Gaussian distribution

$$\sigma_{\Delta s}{}^2 = \sigma_{\Delta s \varepsilon}{}^2 + \sigma_{\Delta s p}{}^2$$

$$\sigma_{\Delta s \varepsilon}{}^2 = \varepsilon \left(\beta_p M_{151}{}^2 - 2\alpha_p M_{151} M_{152} + \gamma_p M_{152}{}^2 \right)$$

$$\sigma_{\Delta s p}{}^2 = \sigma_p{}^2 \left(M_{151} D_p + M_{152} D'_p + M_{156} \right)^2$$

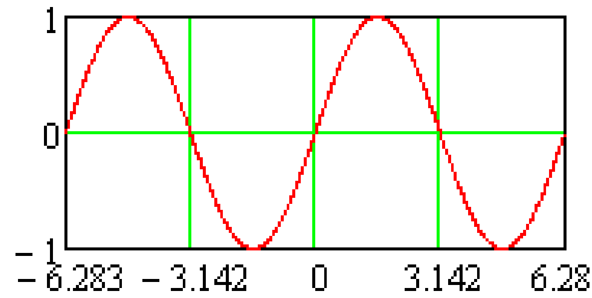
  - Both $\Delta p/p$ and ε contribute to the lengthening

- Expressing matrix elements through Twiss parameters and assuming all derivatives (D & β) equal to zero† one obtains

$$\sigma_{\Delta s}{}^2 = \varepsilon \left(\frac{D_k{}^2}{\beta_k} + \frac{D_p{}^2}{\beta_p} - \frac{2 D_k D_p}{\sqrt{\beta_k \beta_p}} \cos \mu_1 \right) + \sigma_p{}^2 \left(M_{156} - \frac{D_k D_p}{\sqrt{\beta_k \beta_p}} \sin \mu_1 \right)$$

† See complete expression in backup viewgraphs

Cooling Range



- The cooling force depends on Δs nonlinearly

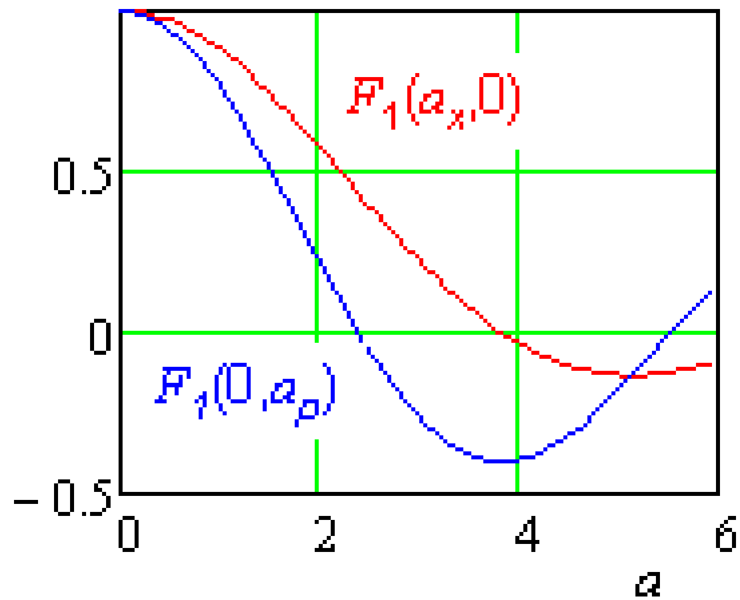
$$\frac{\delta p}{p} = \frac{\Delta E_{\max}}{E}\sin(k\delta s) = \frac{\Delta E_{\max}}{E}\sin\left(a_x \sin(\psi_x) + a_p \sin(\psi_p)\right)$$

where a_x & a_p are the lengthening amplitudes due to ⊥ and L motions measured in units of laser phase ($a = k\,\delta s$)

- The form-factor for damping rate of longitudinal cooling for particle with amplitudes a_x & a_p

$$F_2(a_x, a_p) = \frac{2}{a_p}\oint \sin\left(a_x \sin\psi_x + a_p \sin\psi_p\right)\sin\psi_p \frac{d\psi_x}{2\pi}\frac{d\psi_p}{2\pi}$$

⇨ $$F_2(a_x, a_p) = \frac{2}{a_p} J_0(a_x) J_1(a_p)$$



- Similar for transverse motion

⇨ $$F_1(a_x, a_p) = \frac{2}{a_x} J_0(a_p) J_1(a_x)$$

- Damping requires both lengthening amplitudes be smaller μ_0≈2.405

Cooling of the Gaussian beam

- Averaging the cooling form-factors for Gaussian distribution can be presented in the following form

$$F_{1G}(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = \frac{1}{2k^2\sigma_{\Delta s\varepsilon}{}^2}\int_0^\infty a_x{}^2 F_1(a_x, a_p)\exp\left(-\frac{a_x{}^2}{2k^2\sigma_{\Delta s\varepsilon}{}^2} - \frac{a_p{}^2}{2k^2\sigma_{\Delta sp}{}^2}\right)\frac{a_x da_x a_p da_p}{k^4\sigma_{\Delta s\varepsilon}{}^2\sigma_{\Delta sp}{}^2}$$

  - ♦ Integration yields

$$F_{1G}(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = F_{2G}(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = \exp\left(-\frac{k^2\sigma_{\Delta sp}{}^2}{2}\right)\exp\left(-\frac{k^2\sigma_{\Delta s\varepsilon}{}^2}{2}\right)$$

- Good beam lifetime requires the cooling force to be positive for large amplitude particles
- Assuming that cooling becomes zero at 4σ for both planes

⇨ $k\sigma_{\Delta sp} = k\sigma_{\Delta s\varepsilon} = \mu_0/4 \approx 0.6$

⇨ Nonlinearity of cooling force results in the cooling force reduction by factor $F_{1G}(\mu_0/4, \mu_0/4) = F_{2G}(\mu_0/4, \mu_0/4) \approx 0.697$

Cooling Parameters Optimization

- Eqs. for the damping rates and the sample lengthening at pickup-to-kicker travel are simplified if $\alpha_p = \alpha_k = D'_p = D'_k = 0$

$$\lambda_1 = -\frac{\kappa}{2}\frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1$$

$$\lambda_2 = -\frac{\kappa}{2}\left[M_{156} - \frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1\right]$$

⇨

$$\sigma_{\Delta s}{}^2 = \varepsilon\left(\frac{D_k{}^2}{\beta_k} + \frac{D_p{}^2}{\beta_p} - \frac{2D_k D_p}{\sqrt{\beta_k\beta_p}}\cos\mu_1\right) + \sigma_p{}^2\left(M_{156} - \frac{D_k D_p}{\sqrt{\beta_k\beta_p}}\sin\mu_1\right)$$

- One can see that for fixed decrements a minimization of sample lengthening requires $D_k{}^2/\beta_k = D_p{}^2/\beta_p$

⇨ Ratio of cooling decrements bounds up D^2/β and M_{156}:

$$\frac{D^2}{\beta}\sin\mu_1 = \frac{\lambda_1}{\lambda_1+\lambda_2}M_{156}$$

⇨

$$\sigma_{\Delta s}{}^2 = 2\varepsilon M_{156}\frac{\lambda_1}{\lambda_1+\lambda_2}\tan\frac{\mu_1}{2} + \sigma_p{}^2 M_{156}{}^2\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$$

- Requirements $k\sigma_{\Delta s\varepsilon} = k\sigma_{\Delta sp} = \mu_0 / n_\sigma \xrightarrow{n_\sigma=4} 0.601$ yields

$$2\varepsilon M_{156} \frac{\lambda_1}{\lambda_1+\lambda_2} \tan\frac{\mu_1}{2} = \sigma_p{}^2 M_{156}{}^2 \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 = \left(\frac{\lambda_w}{2\pi}\frac{\mu_0}{n_\sigma}\right)^2 , \quad \lambda_w = \frac{2\pi}{k}$$

The solution is

$$\tan\frac{\mu_1}{2} = \frac{\mu_0}{n_\sigma}\frac{\sigma_p \lambda_w}{4\pi\varepsilon}\frac{\lambda_2}{\lambda_1}$$

$$M_{156} = \frac{\mu_0}{n_\sigma}\frac{\lambda_w}{2\pi\sigma_p}\frac{\lambda_1+\lambda_2}{\lambda_2}$$

$$\frac{D^2}{\beta} = \frac{\varepsilon}{\sigma_p{}^2}\left(\frac{\lambda_1{}^2}{\lambda_2{}^2} + \frac{\sigma_p{}^2}{4\varepsilon^2}\left(\frac{\mu_0}{n_\sigma}\frac{\lambda_w}{2\pi}\right)^2\right)$$

- For λ_w = 12 μm, ε_{n95} = 20 mm mrad, $\sigma_p = 1.2\cdot10^{-4}$, n_σ = 4 and $\lambda_1 = \lambda_2$ one obtains the optimal parameters
  - $\mu_{1\_opt}/2\pi = 6.88\cdot10^{-3}$
  - M_{156} = 1.91 cm
  - D^2/β = 22.1 cm
    (β = 50 m, D = 3.3 m)
  - ♦ Tough requirements on the betatron phase advance ($\Delta\nu_1 \sim 10^{-3}$)
    - Hardly possible for λ_w = 2 μm ($\Delta\nu_1 \sim 2\cdot10^{-4}$)

Combinations of Optics Parameters for Optimal Cooling

$D_1=D_2=D$, $D^2/\beta=22.1$ cm, $\delta\nu_1=6.88\cdot10^{-3}$

$\nu_1=\mu_1/2\pi$	M_{156} [cm]
n- $\delta\nu_1$	-1.91
n+ $\delta\nu_1$	1.91

$D_1=-D_2=D$, $D^2/\beta=22$ cm, $\delta\nu_1=6.88\cdot10^{-3}$

$\nu_1=\mu_1/2\pi$	M_{156} [cm]
n+1/2- $\delta\nu_1$	-1.91
n+1/2+ $\delta\nu_1$	1.91

Requirements to the System Stability

- The major limitations on system stability come from
  - Relative change path length for the beam and the light
    - Cooling force $\propto F(a,\delta a) \equiv \frac{2}{a} J_1(a)\cos(\delta a)\,, \quad \delta a = k\delta L$
      - o Reduces the force but does not change cooling acceptance

      $\Rightarrow$ k δL < 0.5, i.e. δL ~ 1 μm (λ_w=12 μm, 10% force reduction)
    - No additional requirements for high frequency jitter
  - Changes of cooling rates due to optics variations

$$\lambda_1 = -\frac{\kappa}{2}\left(D_2 M_{1_{2,6}} - D_2' M_{1_{1,6}}\right)$$

$$\lambda_2 = -\frac{\kappa}{2}\left(- D_2 M_{1_{2,6}} + D_2' M_{1_{1,6}} + M_{1_{5,6}}\right)$$

- External (changes in kicker dispersion)
  - o ΔD/D<5-10% - Is not expected to be a problem
- Internal (pickup-to-kicker transport matrix)
  - o Looks extremely sensitive: $\Delta\nu_1 \sim 10^{-3}$ is required
  - o Additional insight is needed

Longitudinal Kick by E.-M. Wave

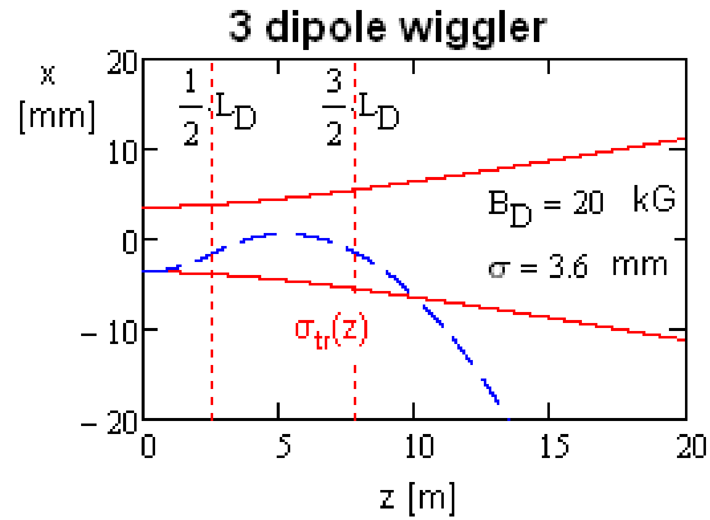
- Electric field of e.-m. wave focused at z=0 to the rms size $\sigma_\perp$

$$E_x(x,y,z,t) = \mathrm{Re}\left(E_0 e^{i(\omega t - kz)} \frac{\sigma_\perp^2}{\sigma^2(z)} \exp\left(-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} \right) \right)$$

$$E_y(x,y,z,t) = 0$$

$$E_z(x,y,z,t) = \mathrm{Re}\left(iE_0 e^{i(\omega t - kz)} \frac{\sigma_\perp^2 x}{k\sigma^4(z)} \exp\left(-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} \right) \right)$$

$$E_0 = \sqrt{\frac{8P}{c\sigma_\perp^2}}, \quad \sigma^2(z) = \sigma_\perp^2 - i\frac{z}{k}, \quad k = \frac{2\pi}{\lambda_w}$$



- The beam is deflected in the x-plane by wiggler magnetic field
  - That results in the beam energy change $\Delta E = e\int (\mathbf{E}\cdot\mathbf{v})dt$

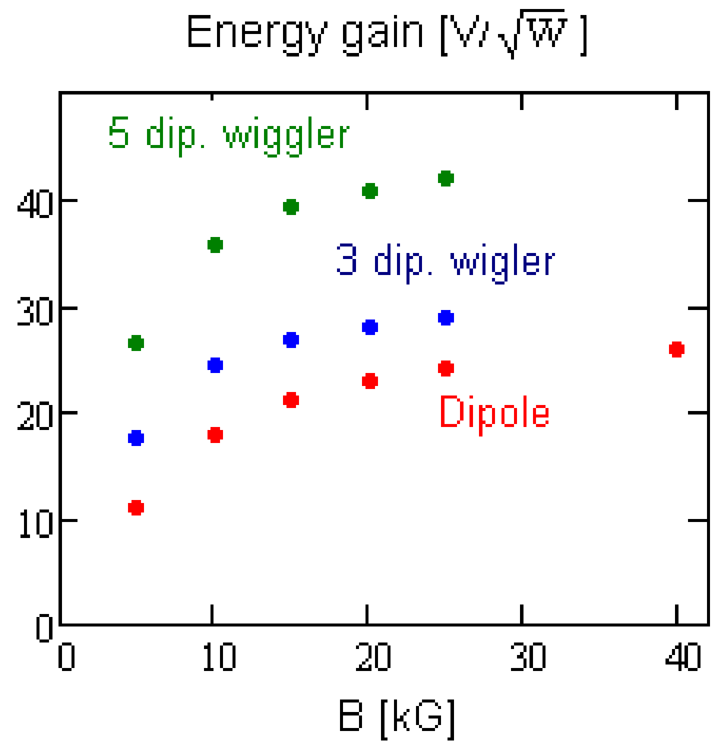
⇨
$$\Delta E = eE_0 \int \mathrm{Re}\left\{ \left(\frac{dx}{dz}\frac{\sigma_\perp^2}{\sigma^2(z)} + \frac{i\sigma_\perp^2 x}{k\sigma^4(z)} \right) \exp\left[-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} + ik\left(\frac{z}{2\gamma^2} + \frac{1}{2}\int_0^z \left(\frac{dx}{dz'}\right)^2 dz' \right) + i\psi \right] \right\} dz$$

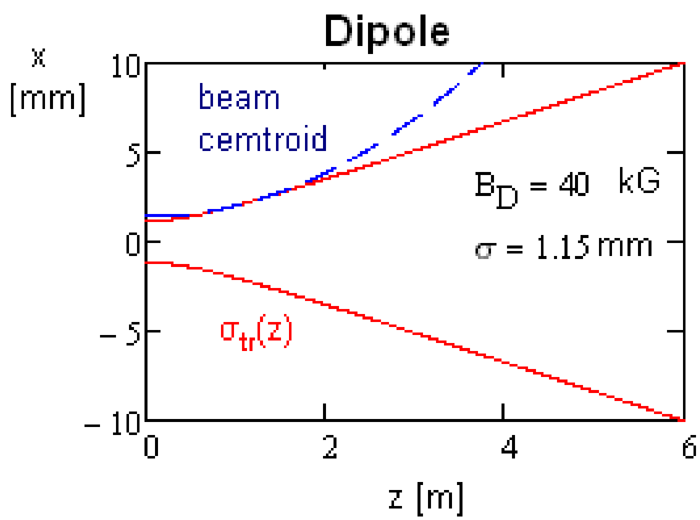
where ψ is the accelerating phase (ΔE = 0 for ψ = 0)

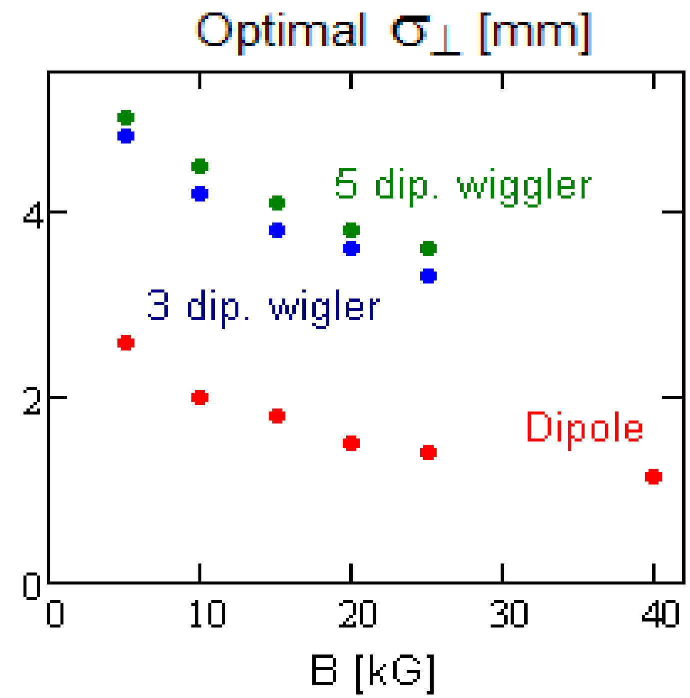
and $\frac{1}{2}\int_0^z \left(\frac{dx}{dz'}\right)^2 dz'$ represents the path length difference between light and beam introduced by wiggler (relative to wiggler center)

Energy Kick in Dipole Wiggler

- Wiggler consists of positive and negative dipoles which are immediately followed by dipole of the same field for further separation of beams
  - Dipole length, $\sigma_{\perp}$ and the beam centroid offset are adjusted to maximize the kick
  - $\sigma_{\perp}$ is much larger than the beam transverse size
- Because of tighter light focusing the kick in a dipole is only marginally lower than in the 3 dipole wiggler

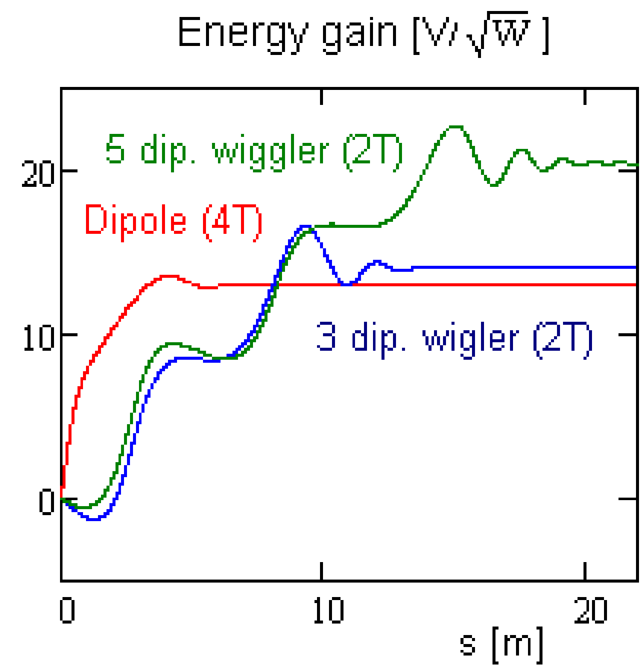






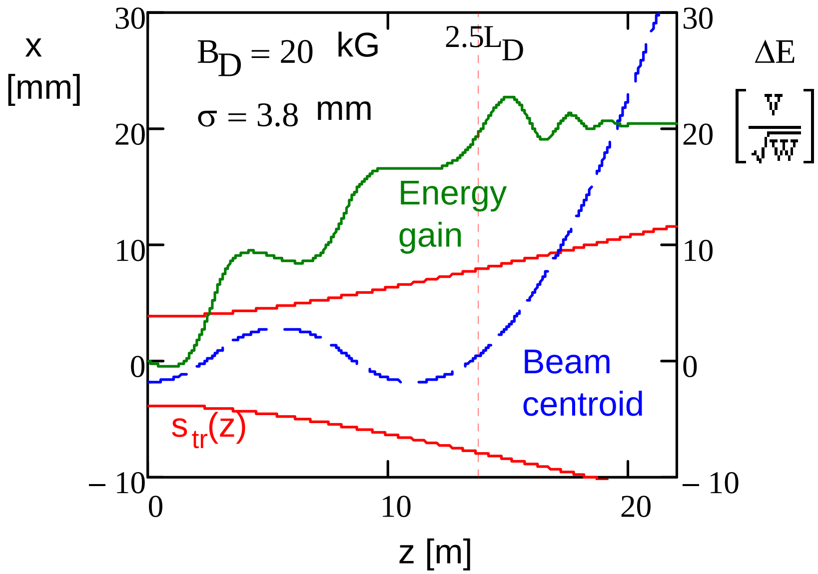
Energy Kick in Dipole Wiggler

- Both E_x and E_z fields contribute to the kick
  - That allows one to get additional kick in the case of single dipole
- Kick in 4 T dipole is 64% of the 5 dipole 2T wiggler
  - Length of 5 dipoles is 27.5 m
  - The total length of 5 dipole system determined by beam separation is ~40 m
- Taking into account available space and comparatively high kick efficiency in a dipole as well as other limitations it looks possible to use a standard Tevatron dipole instead of wiggler



Beam acceleration, $e\int(\mathbf{E}\cdot d\mathbf{s})$, starting from wiggler center

5 dipole wiggler



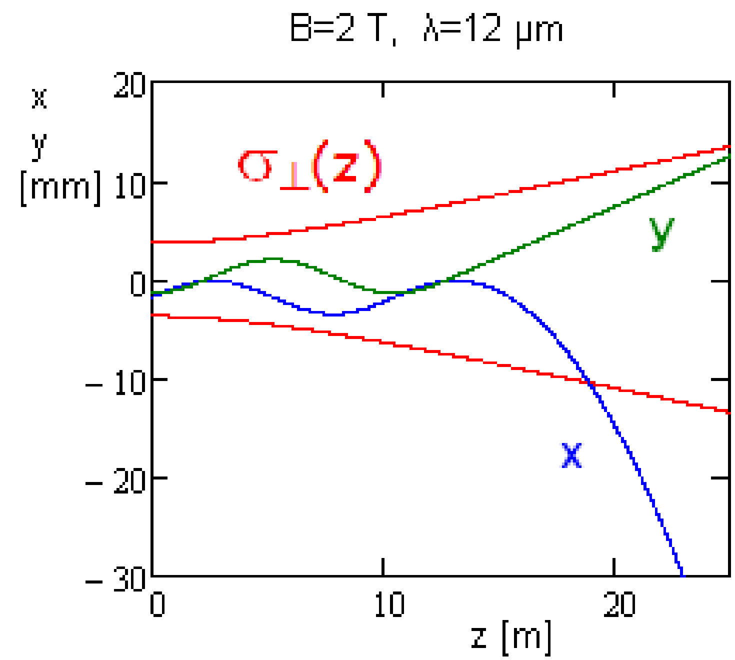
Energy Kick in Helical Wiggler

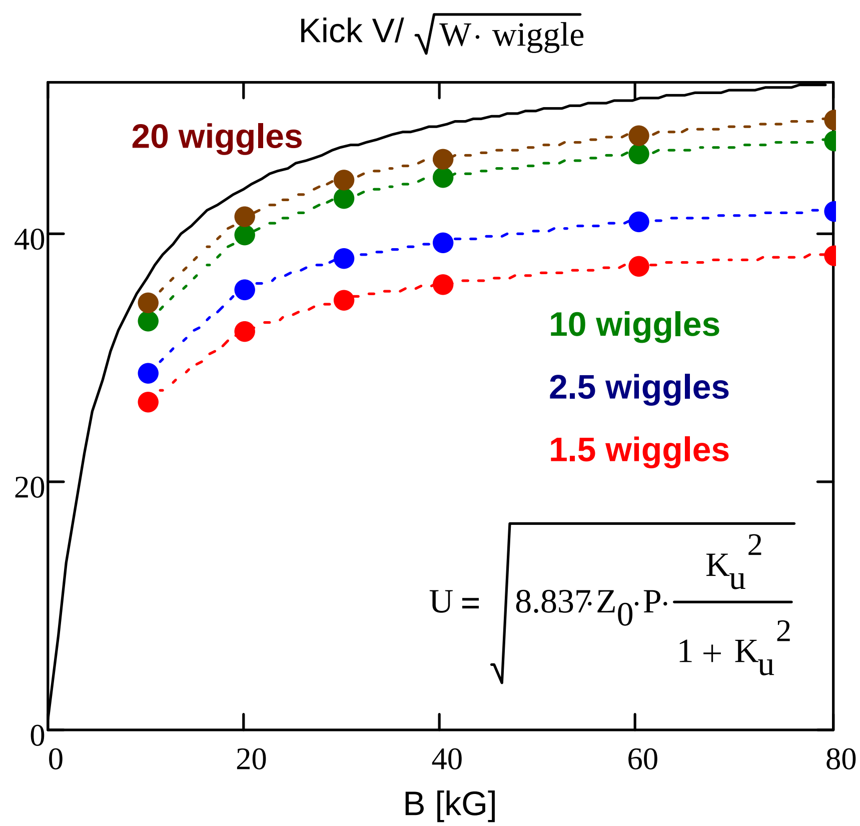
- Helical dipole suggest √2 times better kicker efficiency
  - Circular polarized light
- For large number of periods (n_{wgl} >> 1) the kicker strength is[♦]

$$\frac{\Delta E_{max}}{e} \approx \sqrt{8.837 n_{wgl} P Z_0 \frac{K_u^{\,2}}{1+K_u^{\,2}}}$$

where $K_u = \frac{2\pi}{\lambda_{wgl}} \frac{eB}{mc^2}$, Z_0=377 Ω

- The waist size is growing with kicker length - $\sigma_\perp \approx \sqrt{0.946 L \lambda_w}$
- The kicker is less effective than formula prediction for small n_{wgl}
  - $\rho_{wgl} \sim \sigma_\perp$
  - Negative contribution of E_z

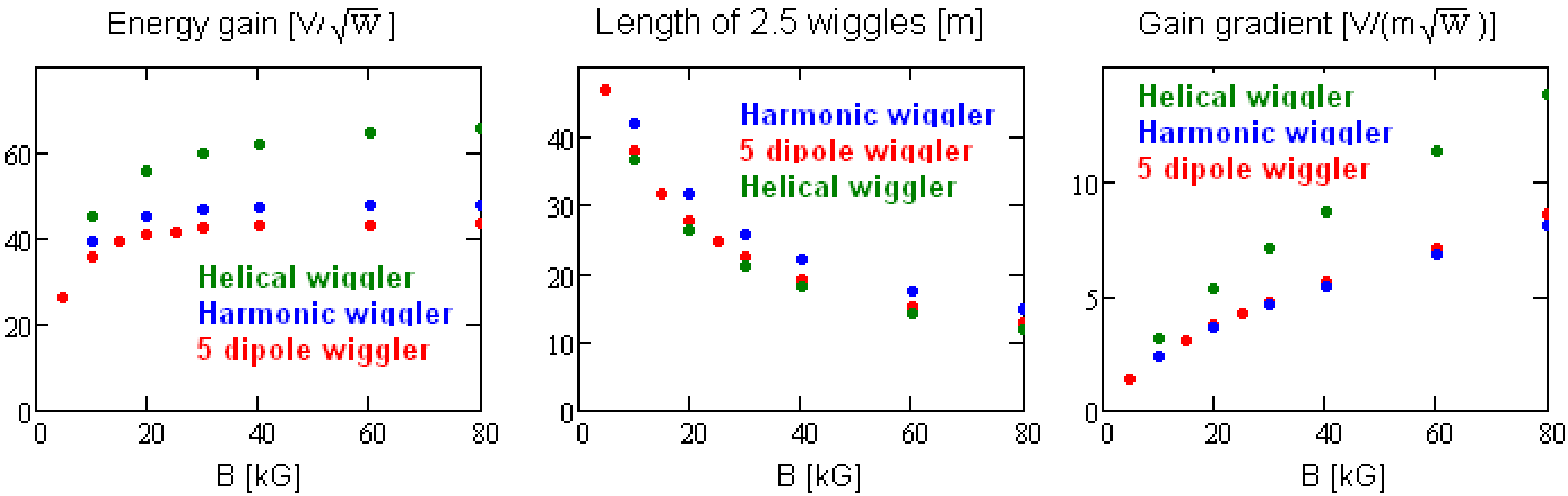




♦M. Zolotorev

Comparison of Different Wiggler Types

- For large wiggler period the wiggler consisting of dipoles is easier to make than a usual harmonic wiggler
  - Little loss in efficiency is compensated by shorter length
- Helical dipole wiggler is ~√2 time more efficient

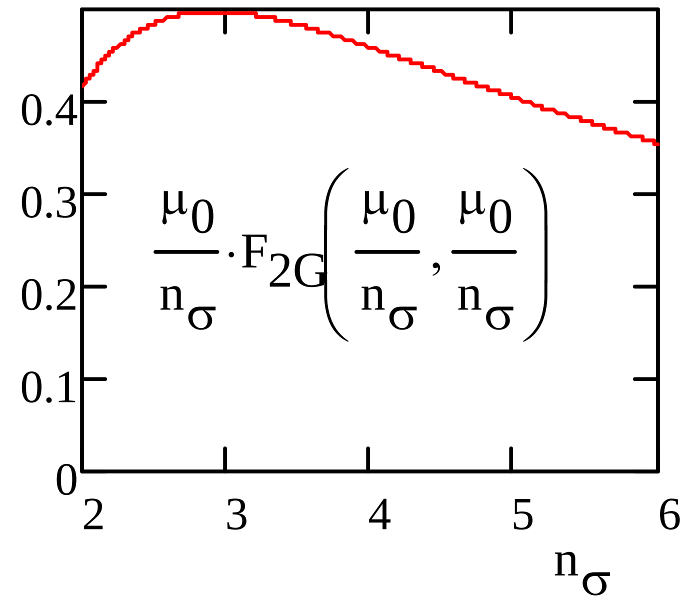


Comparison of wiggler parameters for λ_w = 12 μm and different wigglers (2.5 wiggles each)

Longitudinal Damping Rate

- Long. cooling decrement is proportional to the kick amplitude (ΔE_{max}) excited by a single particle
  - Requirement to have the cooling range of σn_σ times yields

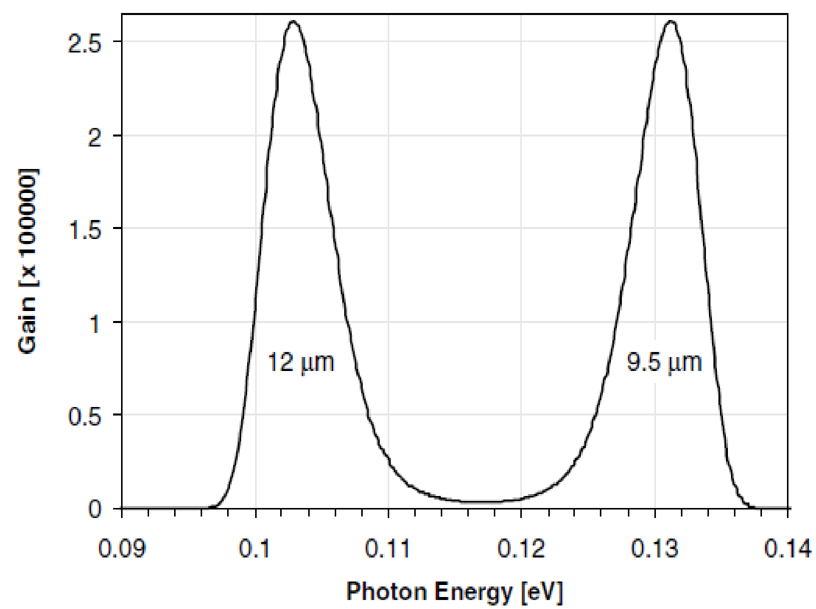
$$\lambda_2 = \frac{1}{2} f_0 \frac{\Delta E_{max}}{cp\sigma_p} \frac{\mu_0}{n_\sigma} F_{2G}\left(\frac{\mu_0}{n_\sigma}, \frac{\mu_0}{n_\sigma}\right)$$



  - In optimum the long. damping rate does not depend on details of beam optics

- For Gaussian dependence of laser gain on f the energy in a single particle pulse is related to the peak power and the FWHM bandwidth (power) as:

$$\int P(t)dt = \sqrt{\frac{\ln(2)}{\pi}} \frac{P_{peak}}{\Delta f_{FWHM}}$$



  - RHIC proposal (2004), λ_w=12 µm, $(\Delta f/f)_{FWHM}$=6%

Longitudinal Damping Rate (2)

- For beam with n_b bunches and N_p particles/bunch the average laser power is

$$P_{laser} = n_b N_p f_0 \sqrt{\frac{\ln(2)}{\pi}} \frac{P_{peak}}{\Delta f_{FWHM}} = \frac{n_b N_p f_0}{\Delta f_{FWHM}} \sqrt{\frac{\ln(2)}{\pi}} \left(\frac{\Delta E_{max}}{G_{kick}} \right)^2$$

where G_{kick} is the kicker efficiency determined by the equation for monochromatic wave $\Delta E_{max} = G_{kick} \sqrt{P}$

⇨ For helical dipole with large number of wiggles

$$P_{laser} = 1.26 \left(\frac{1}{n_{wgl} \left(\Delta f / f \right)_{FWHM}} \frac{1 + K_u^2}{K_u^2} \right) \frac{n_b N_p \lambda_2^2 \lambda_w}{c f_0} \frac{\left(c p \sigma_p / e \right)^2}{Z_0}$$

$$\xrightarrow{K_u >> 1,\ n_{wgl} \left(\Delta f / f \right)_{FWHM} = 1} \approx \frac{n_b N_p \lambda_2^2 \lambda_w}{c f_0} \frac{\left(c p \sigma_p / e \right)^2}{Z_0}$$

- ♦ Number of wiggles is limited by bandwidth: $n_{wgl} \le 1/(\Delta f / f)$
- ♦ For efficient kick the undulator parameter $K_u \ge 2$
  - For larger magnetic field the kicker is shorter for same n_{wgl}
- ♦ In optimal setup ⊥ cooling does not require additional power
  - but requires an optimized optics

Possible Choice of OSC Parameters

Damping time 4.5 hour, N_p=3•10^{11}, n_b=36, σ_p=1.2•10^{-4} , λ_2^{-1} =4.5 hour

⇨ Amplitude of single particle kick, ΔE_{max} = 0.66 eV

Wave length [μm]	Wiggler type/n_{wgl}	B [T]	Total length [m]	G_{kicker} [eV/√W]	$\Delta f/f_{FWHM}$ %	P [W]
12	Tevatron dipole/(N/A)	4	N/A	26	6	125
6				18		133
2				14		71
12	Helical dipole/2.5	2	40	56	6	28
	Helical dipole/8	8	44	132	6	5
6	Helical dipole/7	6	38	110	6	3.5
2	Helical dipole/12	6	36	116	6	1.05

- Peak optical amplifier power is ~100 times larger than the average one
- Bandwidth is limited by optical amplifier

Discussion

- OSC would double the average Tevatron luminosity
- Cooling installation requires a modification of beam optics
  - C0 straight is available
  - New optics implies
    - new quad circuits
    - may be new quads
    - shuffling existing and/or installation of new dipoles
    - Installation of wigglers?
  - Considerable work
    - Fractional tunes should stay the same
    - Helices should not be affected
- Antiproton beam has less particles but requires faster cooling
  - That results in approximately the same power requirements for optics amplifier but its larger gain

- 2 μm wavelength
  - 2 μm parametric optical amplifier is feasible (MIT-Bates)
    - 20-100 W (pumped by Nd:YAG laser)
  - Can be used with Tevatron dipoles being pickups and kickers (no wigglers), 70 W amplifier per beam
    - 2T helical wiggler (~20 m) requires ~12 W amplifier per beam
  - Optics stability and path length control are questionable
    - We will continue to look into optics issues
- 12 μm wavelength
  - Looks good for control of optics and the path length
  - Parametric optical amplifier pumped by 2-nd harmonic of CO_2 laser
    - Was not demonstrated yet
      - Attempt for RHIC was not quite successful
    - 5-10 W looks reasonable request
      - But R&D is required to prove feasibility
  - Requires ~6-8 T helical wiggler (≥4 years)
- There is no fast way (2-3 years) to introduce OSC in Tevatron
  - looks possible for 5-6 years

This Work Results and Plans for Further Studies

- Done
  - Better understanding of beam optics issues for OSC
    - Formulation of requirements for optimal beam optics
    - Understanding of cooling range
  - Better understanding of kicker efficiency
    - Helical undulator allows to reduce its length and/or laser power
- Future work
  - Look into realistic Tevatron optics
  - Study its sensitivity
    - Is the 2 μm wavelength possible?

      ⇒ If yes then the fast scenario can work with 60 W amplifier (No wigglers, pickup and kicker are in dipoles)
- Making experiment in Bates would be extremely helpful but ?

Backup Viewgraphs

Damping Rates of Optical Stochastic Cooling

Transfer Matrix Parameterization

- Vertical degree of freedom is uncoupled and we will omit it in further consideration

$$\mathbf{M}=\begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{X}=\begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p/p \end{bmatrix}$$

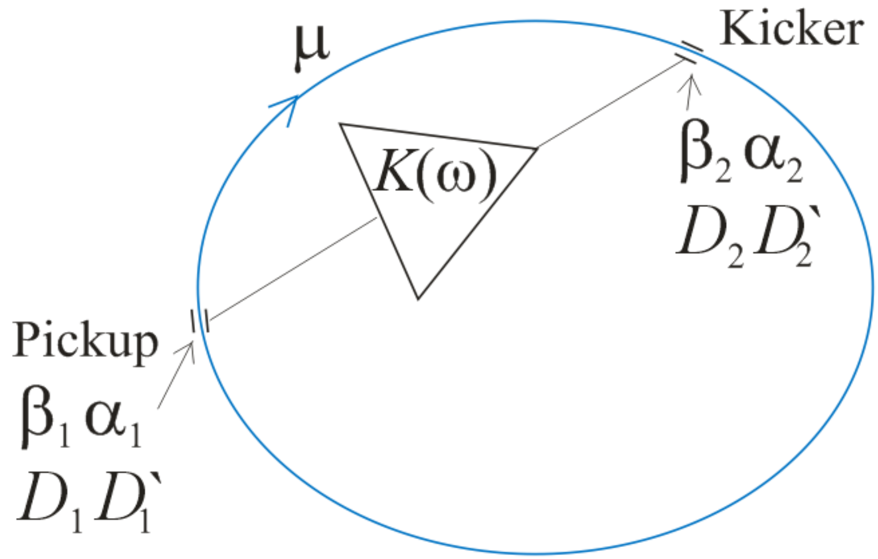
- M_{16} & M_{26} can be expressed through dispersion

$$\begin{bmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_1' \\ 1 \end{bmatrix}=\begin{bmatrix} D_2 \\ D_2' \\ 1 \end{bmatrix}$$

That yields

$$M_{16}=D_2-M_{11}D_1-M_{12}D_1'$$
$$M_{26}=D_2'-M_{21}D_1-M_{22}D_1'$$

$$M_{11}=\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu+\alpha_1\sin\mu\right)$$
$$M_{22}=\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu-\alpha_2\sin\mu\right)$$
$$M_{12}=\sqrt{\beta_1\beta_2}\,\sin\mu$$
$$M_{21}=\frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu-\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu$$



Transfer Matrix Parameterization (continue)

- Symplecticity ($\mathbf{M}^T\mathbf{U}\,\mathbf{M} = \mathbf{U}$) binds up M_{51}, M_{52} and M_{16}, M_{26}
- That yields

$$M_{51} = M_{21}M_{16} - M_{11}M_{26}$$

$$M_{52} = M_{22}M_{16} - M_{12}M_{26}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Finally one can write

$$M_{16} = D_2 - D_1\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu + \alpha_1 \sin\mu\right) - D_1'\sqrt{\beta_1\beta_2}\sin\mu$$

$$M_{26} = D_1\left(\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu + \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\mu\right) + D_1' - D_1'\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu - \alpha_2\sin\mu\right)$$

$$M_{51} = -D_2\left(\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu + \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\mu\right) + D_2' - D_2'\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu + \alpha_1\sin\mu\right)$$

$$M_{52} = -D_1 + D_2\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu - \alpha_2\sin\mu\right) - D_2'\sqrt{\beta_1\beta_2}\sin\mu$$

- In the first order the orbit lengthening due to betatron motion is equal to zero if $D_1 = D`_1 = D_2 = D`_2 = 0$

Transfer Matrix Parameterization (continue)

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to M_{56}

$$\Delta s_{1\to 2} \equiv 2\pi R\eta_1 \frac{\Delta p}{p} = M_{51}D_1\frac{\Delta p}{p} + M_{52}D_1'\frac{\Delta p}{p} + M_{56}\frac{\Delta p}{p} + \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

  - Further we assume that $\mathrm{v} = c$, v=c, i.e. $1/\gamma^2 = 0$ and $\eta_1 = \alpha_{1\to 2}$.

- That results in $\eta_1 = \dfrac{M_{51}D_1 + M_{52}D_1' + M_{56}}{2\pi R}$ or

$$M_{56} = 2\pi R\eta_1 + D_1D_2\left(\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu + \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\mu\right) + D_1D_2'\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu + \alpha_1\sin\mu\right)$$
$$- D_1'D_2\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu - \alpha_2\sin\mu\right) + D_1'D_2'\sqrt{\beta_1\beta_2}\sin\mu$$

- Thus, the entire transfer matrix from a point 1 to a point 2 can be expressed through the β-functions, dispersions and their derivatives at these points and the partial slip factor

Parameterization of the Entire Ring Transfer Matrix

- Formulas for the entire ring look more compact

$$M_{11} = \cos\mu + \alpha\sin\mu \qquad M_{12} = \beta\sin\mu$$

$$M_{21} = -\frac{1+\alpha^2}{\beta}\sin\mu \qquad M_{22} = \cos\mu - \alpha\sin\mu$$

$$M_{16} = D(1-\cos\mu - \alpha\sin\mu) - D'\beta\sin\mu$$

$$M_{26} = D\frac{1+\alpha^2}{\beta}\sin\mu + D'(1-\cos\mu + \alpha\sin\mu)$$

$$M_{51} = -D\frac{1+\alpha^2}{\beta}\sin\mu + D'(1-\cos\mu - \alpha\sin\mu)$$

$$M_{52} = -D(1-\cos\mu + \alpha\sin\mu) - D'\beta\sin\mu$$

$$M_{56} = 2\pi R\alpha_{1\to 2} + D^2\frac{1+\alpha^2}{\beta}\sin\mu + 2DD'\alpha\sin\mu + D'^2\beta\sin\mu$$

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p / p \end{bmatrix}$$

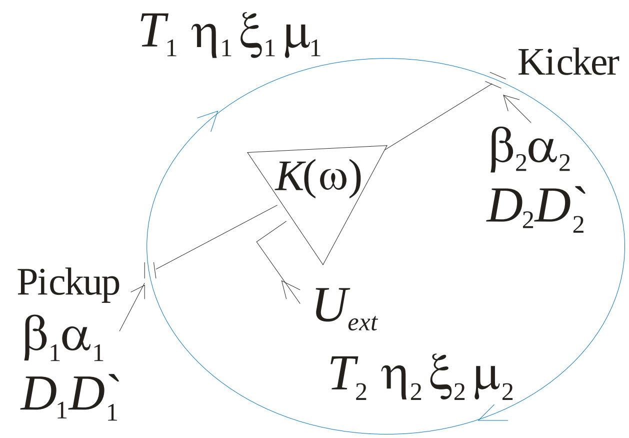
Damping Rates of Optical Stochastic Cooling

Longitudinal kick

$$\frac{\delta p}{p} = \kappa\,\Delta L = \kappa\left(M_{151} x_1 + M_{152}\theta_{x1} + M_{156}\frac{\Delta p}{p}\right)$$

Or in the matrix form: $\boldsymbol{\delta}\mathbf{X} = \mathbf{M}_c \mathbf{X}_1$

$$\mathbf{M}_c = \kappa \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{151} & M_{152} & 0 & M_{156} \end{bmatrix}$$



M_1 - pickup-to-kicker matrix

M_2 - kicker-to-pickup matrix

$M = M_1 M_2$ – ring matrix

$\mu = \mu_1 + \mu_2$

$$\mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

Total ring matrix related to kicker (Ring&RF&damper)

$$\mathbf{M}_{tot}\mathbf{X}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{X}_2 + \boldsymbol{\delta}\mathbf{X}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{X}_2 + \mathbf{M}_c\mathbf{X}_1 = (\mathbf{M}_1\mathbf{M}_2 + \mathbf{M}_c\mathbf{M}_2)\mathbf{X}_2$$

⇨ $\mathbf{M}_{tot} = \mathbf{M} + \Delta\mathbf{M}_c$ where $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\ ,\quad \Delta\mathbf{M} = \mathbf{M}_c\mathbf{M}_2$

Damping Rates of Optical Stochastic Cooling (continue)

Perturbation theory yields that the eigen-value correction is [HB2008]:

$$\delta\lambda_k = \frac{i}{2}\mathbf{v}_k^{+}\,\mathbf{U}\,\Delta\mathbf{M}\,\mathbf{v}_k = \frac{i}{2}\mathbf{v}_k^{+}\mathbf{U}\,\mathbf{M}_c\,\mathbf{M}_1^{-1}(\mathbf{M}_1\mathbf{M}_2)\mathbf{v}_k = \frac{i}{2}\lambda_k\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\,\mathbf{M}_1^{-1}\,\mathbf{v}_k$$

Corresponding tune shift is:

$$\delta Q_k = \frac{i}{2\pi}\frac{\delta\lambda_k}{\lambda_k} = -\frac{1}{4\pi}\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\,\mathbf{M}_1^{-1}\,\mathbf{v}_k$$

Symplecticity relates the transfer matrix and its inverse:

$$\mathbf{M}_1^{-1} = -\mathbf{U}\,\mathbf{M}_1^{T}\,\mathbf{U}$$

$$\Rightarrow \qquad \delta Q_k = \frac{1}{4\pi}\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\,\mathbf{U}\,\mathbf{M}_1^{T}\mathbf{U}\,\mathbf{v}_k$$

Performing matrix multiplication and taking into account that symplecticity binds up M_{51},M_{52} and M_{16},M_{26} one finally obtains:

$$\delta Q_k = \frac{\kappa}{4\pi}\mathbf{v}_k^{+}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{v}_k$$

Eigen-vectors and Damping Decrements (Mode 1)

- There are two eigen-vectors
  - One related to the betatron motion $\mathbf{v}_1$
  - And one related to the synchrotron motion $\mathbf{v}_2$
- They are normalized as: $\mathbf{v}_k^{+}\,\mathbf{U}\,\mathbf{v}_k = -2i$
- If the synchrotron tune and dispersion in RF cavities are small the effect of RF can be neglected in the computation of $\mathbf{v}_1$
  - In this case $\lambda_1 = e^{-i\mu}$ and the eigen-vector related to the kicker position is

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ \mathrm{v}_{13} \\ 0 \end{bmatrix}, \quad \mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k \,, \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first 2 components are the same as for uncoupled case.

The third component has to be found from the third equation

=> $$\mathrm{v}_{13} = -\frac{iD_2(1-i\alpha_2)+D_2'\beta_2}{\sqrt{\beta_2}}$$

- Corresponding damping rate is

$$\lambda_1 = -2\pi \operatorname{Im} \delta Q_1$$

$$= -\frac{\kappa}{2} \operatorname{Im}\left(\begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ v_{1_3} \\ 0 \end{bmatrix}^+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ v_{1_3} \\ 0 \end{bmatrix} \right)$$

$$= -\frac{\kappa}{2}\left(D_2 M_{1_{2,6}} - D_2' M_{1_{1,6}} \right)$$

That yields

$$\lambda_1 = -\frac{\kappa}{2}\left[D_1 D_2 \frac{(1+\alpha_1\alpha_2)\sin\mu_1 + (\alpha_2-\alpha_1)\cos\mu_1}{\sqrt{\beta_1\beta_2}} - D_1' D_2 \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu_1 - \alpha_2\cos\mu_1) \right.$$
$$\left. + D_1 D_2' \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_1 + \alpha_1 \sin\mu_1) + D_1' D_2' \sqrt{\beta_1\beta_2}\sin\mu_1 \right]$$

Expressing it through the partial slip factor one gets

$$\lambda_1 = -\frac{\kappa}{2}\left(M_{56} - 2\pi R \eta_1 \right)$$

Eigen-vectors and Damping Decrements (Mode 2)

- To find the second eigen-vector we will ignore the second order effects of betatron motion on the longitudinal dynamics
  - The linerazed RF kick is

$$\frac{\delta p}{p} = -\Phi_s s$$

  - Simple calculations yield for the eigen value $\lambda_1 = e^{-i\mu_s}$ where the synchrotron tune $\mu_s = \sqrt{2\pi R \eta \Phi_s}$
  - Corresponding eigen-vector related to the kicker position is

$$\mathbf{v}_1 = \begin{bmatrix} -iD_2 / \sqrt{\beta_s} \\ -iD_2' / \sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i / \sqrt{\beta_s} \end{bmatrix}$$

where the longitudinal beta-function $\beta_s = 2\pi R \eta / \mu_s$

- Corresponding damping rate is

$$\lambda_2 = -2\pi \operatorname{Im} \delta Q_2$$

$$= -\frac{\kappa}{2} \operatorname{Im}\left(\begin{bmatrix} -iD_2/\sqrt{\beta_s} \\ -iD_2'/\sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i/\sqrt{\beta_s} \end{bmatrix}^{+} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -iD_2/\sqrt{\beta_s} \\ -iD_2'/\sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i/\sqrt{\beta_s} \end{bmatrix} \right)$$

$$= -\frac{\kappa}{2}\left(M_{1_{56}} - D_2 M_{1_{26}} + D_2' M_{1_{16}} \right)$$

Expressing the matrix elements through Twiss parameters one obtains

$$\lambda_2 = -\frac{\kappa}{2} M_{1_{56}} - \lambda_1 = -\pi \kappa R \eta_1$$

The last expression can be directly obtained from the definition of the partial slip factor

- The above equation yields the sum of the decrements is

$$\lambda_1 + \lambda_2 = -\frac{\kappa}{2} M_{1_{56}}$$

Damping Rates for Smooth Lattice Approximation

- For zero derivatives of beta-function and dispersion at pickup and kicker one obtains

$$\lambda_1 = -\frac{\kappa}{2}\frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1$$

$$\lambda_2 = -\frac{\kappa}{2}\left[M_{156} - \frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1\right]$$

- Smooth lattice approximation additionally yields

$$\beta = \frac{R}{\nu}, \quad D = \frac{R}{\nu^2}, \quad \mu_1 = \nu\frac{L_{pk}}{R} \quad \eta_1 = -\frac{L_{pk}}{2\pi\nu^2 R}, \quad M_{156} = -\frac{L_{pk}}{\nu^2} + \frac{R}{\nu^3}\sin\left(\nu\frac{L_{pk}}{R}\right),$$

where L_{pk} is the pickup-to-kicker path length, and ν is the betatron tune

$$\Rightarrow \quad \lambda_1 = -\frac{\kappa}{2}\frac{R}{\nu^3}\sin\left(\nu\frac{L_{pk}}{R}\right)$$

$$\lambda_2 = \frac{\kappa}{2}\frac{L_{pk}}{\nu^2}$$

Comparison to Zholents-Zolotorev result

PRST-AB, v.7,p.12801 (2004)

Eqs. (A9) and (A11) in the paper Appendix can be rewritten in the following simplified form

$$\lambda_1 = \frac{\kappa}{2}\left(D_2 M_{1_{51}}^{-1} + D_2' M_{1_{52}}^{-1}\right)$$

$$\lambda_2 = -\frac{\kappa}{2}\left(D_2 M_{1_{51}}^{-1} + D_2' M_{1_{52}}^{-1} + M_{1_{56}}^{-1}\right)$$

The inverse of the matrix is

$$\mathbf{M}_1^{-1} = -\mathbf{U}\,\mathbf{M}_1^{T}\,\mathbf{U} = \begin{bmatrix} M_{1_{22}} & -M_{1_{12}} & 0 & M_{1_{52}} \\ -M_{1_{21}} & M_{1_{11}} & 0 & M_{1_{51}} \\ M_{1_{26}} & M_{1_{16}} & 1 & -M_{1_{56}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting expressions for matrix elements into above Eqs. for decrements one arrives to the same results

Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

$$\sigma_{\Delta L}^{2} = \int \left(M_{51} x + M_{52} \theta_x + M_{56} \tilde{p} \right)^2 f\left(x, \theta_x, \tilde{p}\right) dx d\theta_x d\tilde{p} \,, \quad \tilde{p} = \frac{\Delta p}{p}$$

where for Gaussian distribution

$$f\left(x, \theta_x, \tilde{p}\right) = \frac{\exp\left(-\frac{\gamma_p \left(x - D_p \tilde{p}\right)^2 + 2\alpha_p \left(\theta_x - D_p' \tilde{p}\right)\left(x - D_p \tilde{p}\right) + \beta_p \left(x - D_p \tilde{p}\right)}{2\varepsilon} - \frac{\tilde{p}^2}{2\sigma_p^2} \right)}{\sqrt{2\pi}\, 2\pi \sigma_p \varepsilon} \,, \quad \gamma_p = \frac{1 + \alpha_p^2}{\beta_p}$$

- Performing integration one obtains

$$\sigma_{\Delta L}^{2} = \varepsilon \left(\beta_p M_{51}^2 - 2\alpha_p M_{51} M_{52} + \gamma_p M_{52}^2 \right) + \sigma_p^2 \left(M_{51} D_p + M_{52} D_p' + M_{56} \right)^2$$

- Expressing matrix elements through Twiss parameters yields

$$\sigma_{\Delta L}^{2} = \varepsilon F_\varepsilon + \sigma_p^2 \left(2\pi R \alpha_{1\to 2} \right)^2$$

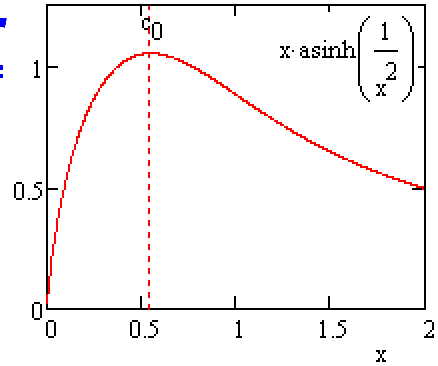
$$F_\varepsilon = D_p^2 \gamma_p + D_k^2 \gamma_k - \frac{2 D_p D_k}{\sqrt{\beta_p \beta_k}} \left(\left(1 + \alpha_p \alpha_k\right) \cos\mu_1 + \left(\alpha_p - \alpha_k\right) \sin\mu_1 \right) + D_p'^2 \beta_p + D_k'^2 \beta_k + 2 D_p D_p' \alpha_p +$$

$$2 D_p D_p' \alpha_p + 2 D_p D_k' \sqrt{\frac{\beta_k}{\beta_p}} \left(\sin\mu_1 - \alpha_p \cos\mu_1 \right) - 2 D_k D_p' \sqrt{\frac{\beta_p}{\beta_k}} \left(\sin\mu_1 + \alpha_k \cos\mu_1 \right) - 2 D_k' D_p' \sqrt{\beta_p \beta_k} \cos\mu_1$$

- For zero derivatives it yields

$$\sigma_{\Delta L}{}^2 = \varepsilon\left(\frac{D_k{}^2}{\beta_k} + \frac{D_p{}^2}{\beta_p} - \frac{2D_k D_p}{\sqrt{\beta_k \beta_p}}\cos\mu_1\right) + \sigma_p{}^2\left(M_{1_{56}} - \frac{D_k D_p}{\sqrt{\beta_k \beta_p}}\sin\mu_1\right)$$

Estimate of Energy Kick in Helical Wiggler



- Assuming that $\rho_\perp \ll \sigma_\perp$ the kick amplitude is

$$\frac{\Delta E}{e} = \sqrt{\frac{4P}{c\sigma_\perp^{\ 2}}}\theta_0 2\int_0^{L/2} \frac{\sigma_\perp^{\ 2} dz}{\left|\sigma_\perp^{\ 2} - iz/k\right|} = 4\sqrt{\frac{P}{c}}\theta_0 k\sigma_\perp a\sinh\left(\frac{L}{2k\sigma_\perp^{\ 2}}\right)$$

- The function $x\sinh\left(1/x^2\right)$ achieves its maximum at $x = c_0 \approx 0.54884$

⇨ Maximum kick of $\left.\frac{\Delta E}{e}\right|_{opt} = \frac{4c_0}{\sqrt{2}}\sinh\left(\frac{1}{c_0^{\ 2}}\right)\sqrt{\frac{P}{c}}\theta_0\sqrt{kL}$ is achieved at $\sigma_\perp = \sqrt{\frac{c_0^{\ 2}}{2}\frac{L}{k}}$

- Taking into account that $4\pi/c = Z_0$ and $kL = 2\pi n_{wgl}$ we obtain

$$\left.\frac{\Delta E}{e}\right|_{opt} = 2c_0\sinh\left(\frac{1}{c_0^{\ 2}}\right)\theta_0\sqrt{PZ_0 n_{wgl}}$$

- The condition of resonance is: $k\left(1/(2\gamma^2) + \theta_0^{\ 2}/2\right) = k_{wgl}$, where the particle angle (relative to wave direction) is $\theta_0 = \frac{1}{k_{wgl}R_L}$, $R_L = \frac{pc}{eB_0}$

- That yields

$$\left.\frac{\Delta E}{e}\right|_{opt} = c_0\sinh\left(\frac{1}{c_0^{\ 2}}\right)\sqrt{\frac{8PZ_0 n_{wgl}K_u^{\ 2}}{1+K_u^{\ 2}}} \approx \sqrt{\frac{8.837PZ_0 n_{wgl}K_u^{\ 2}}{1+K_u^{\ 2}}}\ , \quad K_u = \frac{eB_0}{pck_{wgl}}$$

References

HB2008 - V. Lebedev, A. Burov, "Coupling and its Effects on Beam Dynamics", HB-2008