Einzel Lens Chopper – Transmission Line Mismatches

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ABSTRACT: A quick calculation to see the effect of a transmission line that connects the load resistor to the Einzel lens. In this treatment, a capacitor with a small capacitance is used to model the Einzel lens. If the load resistor is not equal to the impedance of the transmission line, there are two surprising results: (i) the time constant from each reflection is $Z_0 C_E$ rather than RC_E. Here Z_0 is the impedance of the transmission line, R is the load resistor and C_E is the capacitance of the Einzel lens. (ii) for a very long cable, the decay rate of the voltage on the Einzel lens when it is turned off instantaneously depends on both the length of the transmission line and the reflection coefficient and is independent of C_E.

We assume that the Einzel lens can be modelled as a capacitor C_E. We will solve the system shown in Figure 1 for a current source $I_s(t)$ which we assume is representable in Fourier space. If we work in Fourier space, the initial conditions are defined at $t = -\infty$ and in this formalism it is the superposition of steady state solutions for current sources which have sinusoidal outputs. Therefore, only boundary conditions are specified here.

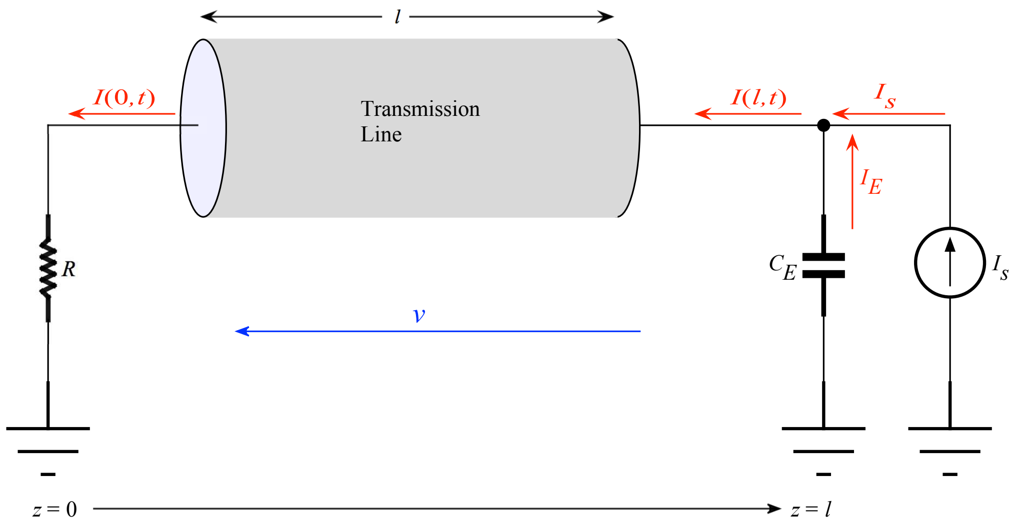

Figure 1 This is the steady state representation of the system. Note that v is pointing from $z = \ell$ to $z = 0$, this means that $v < 0$.

The equations at the transmission line boundaries are

$$\left.\begin{aligned} I(0,t) &= \frac{V(0,t)}{R} \\ I(\ell,t) &= I_s(t) - C_E \dot{V}(\ell,t) \end{aligned}\right\} \tag{1}$$

because $I_E(t) = -C_E \dot{V}(\ell,t)$.

The transmission line model satisfies the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c} I(z,t) \\ V(z,t) \end{array}\right\} = 0 \tag{2}$$

where $v = -1/\sqrt{LC}$. Note: the negative sign comes from the way we have set up the system because the source is pushing current from $z = \ell$ to $z = 0$. It is easy to show that the relationship between I and V is

$$\left.\begin{aligned} \frac{\partial V}{\partial z} &= -L\frac{\partial I}{\partial t} \\ \frac{\partial I}{\partial z} &= -C\frac{\partial V}{\partial t} \end{aligned}\right\} \tag{3}$$

We will define the Fourier transform pair to be

$$\left.\begin{aligned} \tilde{f}(z,\omega) &= \int_{-\infty}^{\infty} dt\ f(z,t)e^{-i\omega t} \\ f(z,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \tilde{f}(z,\omega)e^{i\omega t} \end{aligned}\right\} \tag{4}$$

Therefore, the transmission line differential equation in Fourier space is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\left\{\begin{array}{c} \tilde{I}(z,\omega) \\ \tilde{V}(z,\omega) \end{array}\right\} = 0 \tag{5}$$

where $k^2 = \omega^2/v^2$.

The general solutions to the wave equation (2) are

$$\left.\begin{aligned} \tilde{V}(z,\omega) &= A(\omega)e^{ikz} + B(\omega)e^{-ikz} \\ \tilde{I}(z,\omega) &= \frac{1}{Z_0}\left[B(\omega)e^{-ikz} - A(\omega)e^{ikz}\right] \end{aligned}\right\} \tag{6}$$

where $Z_0 = \sqrt{L/C}$.

The boundary conditions from (1) in Fourier space are

$$\left.\begin{aligned} \tilde{I}(0,\omega) &= \frac{\tilde{V}(0,\omega)}{R} \\ \tilde{I}(\ell,\omega) &= \tilde{I}_s(\omega) - i\omega C_E\tilde{V}(\ell,\omega) \\ &= \tilde{I}_s(\omega) - \frac{\tilde{V}(\ell,\omega)}{Z_E(\omega)} \end{aligned}\right\} \tag{7}$$

where $Z_E(\omega) = 1/i\omega C_E$.

When we substitute (6) into (7), we get

$$\left.\begin{aligned} A\left(\frac{1}{R} + \frac{1}{Z_0}\right) + B\left(\frac{1}{R} - \frac{1}{Z_0}\right) &= 0 \\ A\left(\frac{1}{Z_0} - \frac{1}{Z_E}\right)e^{ik\ell} - B\left(\frac{1}{Z_0} + \frac{1}{Z_E}\right)e^{-ik\ell} &= -\tilde{I}_s \end{aligned}\right\} \tag{8}$$

which can be solved for A and B in terms of determinants

$$\left.\begin{aligned} A &= \frac{\begin{vmatrix} 1/R - 1/Z_0 & 0 \\ -(1/Z_0 + 1/Z_E)e^{-ik\ell} & \tilde{I}_s \end{vmatrix}}{\begin{vmatrix} 1/R + 1/Z_0 & 1/R - 1/Z_0 \\ (1/Z_0 - 1/Z_E)e^{ik\ell} & -(1/Z_0 + 1/Z_E)e^{-ik\ell} \end{vmatrix}} \\ B &= -\frac{\begin{vmatrix} 1/R + 1/Z_0 & 0 \\ (1/Z_0 - 1/Z_E)e^{ik\ell} & \tilde{I}_s \end{vmatrix}}{\begin{vmatrix} 1/R + 1/Z_0 & 1/R - 1/Z_0 \\ (1/Z_0 - 1/Z_E)e^{ik\ell} & -(1/Z_0 + 1/Z_E)e^{-ik\ell} \end{vmatrix}} \end{aligned}\right\} \tag{9}$$

When we substitute this into (6), we have[†]

$$\tilde{V}(z,\omega) = \frac{\tilde{I}_s(\omega)}{\mathcal{D}(z,\omega)}\left[\left(\frac{1}{R} - \frac{1}{Z_0}\right)e^{ikz} - \left(\frac{1}{R} + \frac{1}{Z_0}\right)e^{-ikz}\right] \tag{10}$$

where $\mathcal{D}(\ell,\omega) = \begin{vmatrix} 1/R + 1/Z_0 & 1/R - 1/Z_0 \\ (1/Z_0 - 1/Z_E)e^{ik\ell} & -(1/Z_0 + 1/Z_E)e^{-ik\ell} \end{vmatrix}$.

In particular, we are interested in how the Einzel lens will behave. Therefore, at $z = \ell$, we have

$$\tilde{V}(\ell,\omega) = \frac{\tilde{I}_s(\omega)}{\mathcal{D}(\ell,\omega)}\left[\left(\frac{1}{R} - \frac{1}{Z_0}\right)e^{ik\ell} - \left(\frac{1}{R} + \frac{1}{Z_0}\right)e^{-ik\ell}\right] \tag{11}$$

$$I_s(t) \;=\; I_0(1 - u(t))$$

We can write $I_s(t)$ so that it is "on" from $-\infty < t < 0-$ and "off" for $t > 0+$.

$$I_s(t) = I_0\Big(1 - u(t)\Big) \tag{12}$$

where I_0 is the peak current and $u(t)$ is the Heaviside operator. The system is shown in Figure 2. Its Fourier transform is

$$\tilde{I}_s(\omega) = \frac{i}{\omega} + \pi\delta(\omega) \tag{13}$$

† And similarly for $\tilde{I}$ when the solutions are substituted into (6). But we are more interested in voltage for now.

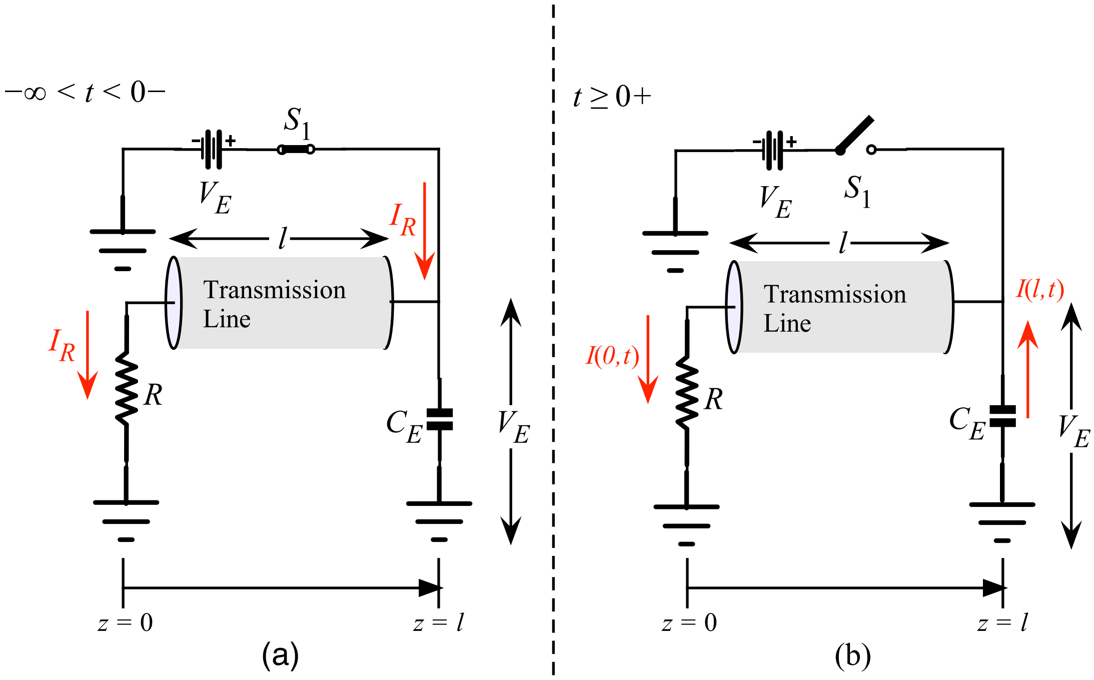

Figure 2 The system is at steady state at $t = 0-$. At $t = 0+$, the switch S_1 is thrown open and the capacitor starts discharging into the load resistor.

where $\delta(x)$ is the Dirac δ-function.

Therefore, the voltage on the Einzel lens is

$$\tilde{V}(\ell,\omega) = \frac{I_0}{\mathcal{D}(\ell,\omega)}\left[\frac{i}{\omega} + \pi\delta(\omega)\right]\left[\left(\frac{1}{R} - \frac{1}{Z_0}\right)e^{ik\ell} - \left(\frac{1}{R} + \frac{1}{Z_0}\right)e^{-ik\ell}\right] \tag{14}$$

$$I_s(t) \;=\; I_0(1 - u(t)) \text{: check}$$

There is an analytic solution when $\ell = 0$, i.e. no transmission line. We expect the voltage across the Einzel lens to be $V(t) = V_E e^{-t/RC_E}$. This is trivial to prove. Using Kirchoff's equations with the sign convention of positive current direction I shown in

Figure 3 and recalling that the voltage drop across a resistor is always negative, we have

$$\left.\begin{aligned} -IR + Q/C_E = 0 \quad &\Rightarrow Q/C_E = IR \\ \therefore \quad &Q/C_E = -\dot{Q}R \end{aligned}\right\} \tag{15}$$

because $I = -\dot{Q}$ where the negative sign comes from the loss of charge over time. Hence,

$$\frac{dQ}{Q} = -\frac{dt}{RC_E} \tag{16}$$

and so

$$Q = Q(0)e^{-t/RC_E} \tag{17}$$

The voltage across the resistor is

$$V(t) = I_R R = -I_E R = -\dot{Q}R = \frac{Q(0)}{C_E}e^{-t/RC_E} = V_E e^{-t/RC_E} \tag{18}$$

and the voltage across the capacitor (or Einzel lens) is

$$V(t) = \frac{Q(0)}{C_E}e^{-t/RC_E} = V_E e^{-t/RC_E} \tag{19}$$

which is the same as across the resistor (17) as expected.

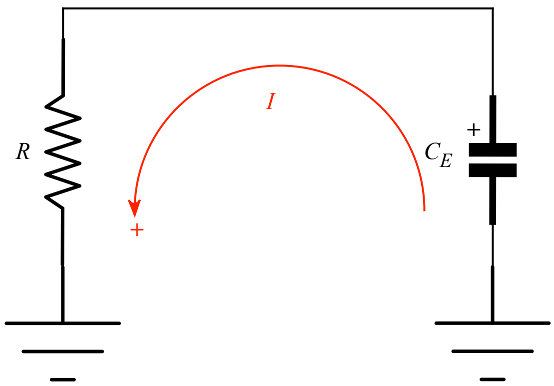

Figure 3 The circuit used for checking (14) for $\ell = 0$. The convention we have chosen is that the current I flowing in the counter-clockwise direction is positive.

Now, we have to check that $\tilde{V}(0,s)$ from (11) gives the same answer as (14). In fact when $\ell = 0$, we get

$$\tilde{V}(0,\omega) = \frac{iI_0R}{\omega\Big(1+i\omega RC_E\Big)} + \frac{\pi I_0 R\delta(\omega)}{1+i\omega RC_E} \tag{20}$$

And when we inverse Fourier transform (20), we get

$$V(0,t) = V_E\left[1+u(t)\left(e^{-t/RC_E}-1\right)\right] \tag{21}$$

where $V_E = I_0R$. Therefore, when $t \geq 0$, we get

$$V(0,t) = V_E e^{-t/RC} \tag{22}$$

which is identical to (19). Note: at $t = 0$, $u(0)$ is undefined, but $(e^{-t/RC_E}-1)|_{t=0} = 0$.

Special Case 1: $R = Z_0$

The first special case which we will consider is when $R = Z_0$. When we substitute this into (14), we have

$$\tilde{V}(\ell,s) = I_0R\left(\frac{Z_E}{R+Z_E}\right)\left[\frac{i}{\omega}+\pi\delta(\omega)\right] \tag{23}$$

which is independent of ℓ.

The inverse Fourier transform of (23) is

$$V_B(\ell,t) = V_E\left[1+u(t)\left(e^{-t/RC_E}-1\right)\right] \tag{24}$$

where we have made $\lim_{\omega\to\infty} Z_E/(R+Z_E) = 1$. This is, again, what we expect: there are no reflections when the load resistor matches the impedance of the cable.

PERFORMING $\mathrm{FT}^{-1}[\tilde{V}(\ell,\omega)]$

We have to apply a few tricks in order to $\mathrm{FT}^{-1}[\tilde{V}(\ell,\omega)]$ because it cannot be done directly. Let us define new functions $\tilde{F}(\ell,\omega)$ and $\tilde{D}(\ell,\omega)$ to be

$$\left.\begin{aligned}\tilde{F}(\ell,\omega) &= I_0\left[\frac{i}{\omega}+\pi\delta(\omega)\right]\left[\left(\frac{1}{R}-\frac{1}{Z_0}\right)e^{ik\ell}-\left(\frac{1}{R}+\frac{1}{Z_0}\right)e^{-ik\ell}\right]\\ \tilde{D}(\ell,\omega) &= \frac{1}{\mathcal{D}(\ell,\omega)}\end{aligned}\right\} \tag{25}$$

(14) becomes

$$\tilde{V}(\ell,\omega)=\tilde{F}(\ell,\omega)\tilde{D}(\ell,\omega) \tag{26}$$

Now $\tilde{F}(\ell,\omega)$ is invertible by using the following relationships

$$\left.\begin{aligned}\mathrm{FT}\Big[1-u(t)\Big] &= \frac{i}{\omega}+\pi\delta(\omega)\\ \mathrm{FT}\Big[\delta(t\pm\ell/v)\Big] &= e^{\pm i\omega\ell/v}\\ \mathrm{FT}\Big[F(t)*G(t)\Big] &= \tilde{F}(\omega)\tilde{G}(\omega)\end{aligned}\right\} \tag{27}$$

Therefore,

$$F(\ell,t)=-I_0\left[\frac{2}{Z_0}+\left(\frac{1}{R}-\frac{1}{Z_0}\right)u(t+\ell/v)-\left(\frac{1}{R}+\frac{1}{Z_0}\right)u(t-\ell/v)\right] \tag{28}$$

We cannot invert $\tilde{D}(\omega)$ directly and so we need to expand it and do it term by term in powers of a dimensionless variable to be defined below. Rather than lugging the matrix entries of $\mathcal{D}$ around, let us define the following new variables a_1, a_2, b_1, b_2 for the entries

$$\mathcal{D}(\ell,\omega)=\begin{vmatrix}1/R+1/Z_0 & 1/R-1/Z_0\\ (1/Z_0-1/Z_E)e^{ik\ell} & -(1/Z_0+1/Z_E)e^{-ik\ell}\end{vmatrix}\equiv\begin{vmatrix}a_1 & a_2\\ b_1e^{ik\ell} & b_2e^{-ik\ell}\end{vmatrix} \tag{29}$$

Therefore,

$$\tilde{D}(\ell,\omega)=\frac{1}{\begin{vmatrix}a_1 & a_2\\ b_1e^{ik\ell} & b_2e^{-ik\ell}\end{vmatrix}}=\frac{e^{ik\ell}}{a_1b_2\left(1-\frac{a_2b_1}{a_1b_2}e^{i2k\ell}\right)} \tag{30}$$

Note that $|a_2 b_1/a_1 b_2| < 1$ because

$$\begin{aligned} \left|\frac{a_2 b_1}{a_1 b_2}\right| &= \left|\frac{(R-Z_0)(i+\omega Z_0 C_E)}{(R+Z_0)(-i+\omega Z_0 C_E)}\right| \\ &= \left|\frac{R-Z_0}{R+Z_0}\right| < 1 \quad \text{because } |-i+\omega Z_0 C_E| = |i+\omega Z_0 C_E| \end{aligned} \tag{31}$$

And we can conveniently identify $\left|\frac{R-Z_0}{R+Z_0}\right|$ as the reflection coefficient.

If we define

$$\rho(\omega) = \frac{a_2 b_1}{a_1 b_2} \tag{32}$$

then (30) can be expanded in terms of $\rho e^{i2k\ell}$

$$\left.\begin{aligned} \tilde{D}(\ell,\omega) &= \frac{e^{ik\ell}}{a_1 b_2\left(1-\rho(\omega)e^{i2k\ell}\right)} \\ &= \frac{e^{ik\ell}}{a_1 b_2}\left[1+\rho(\omega)e^{i2k\ell}+\rho(\omega)^2 e^{i4k\ell}+\dots\right] \\ &\equiv \tilde{D}_0(\ell,\omega)+\tilde{D}_1(\ell,\omega)+\tilde{D}_2(\ell,\omega)+\dots \end{aligned}\right\} \tag{33}$$

because the series is convergent for $|\rho(\omega)| < 1$. It is clear that the rhs has reflections because of the $e^{i(2n+1)k\ell}$ terms where $n \in \mathbb{Z}^+$.

Now, we can inverse Fourier transform $\tilde{D}(\omega)$ term by term on the rhs of (33)

$$\left.\begin{aligned} D_0(\ell,t) &= -\frac{e^{-t/Z_0C_E}e^{-(\ell/v)(1/Z_0C_E)}RZ_0 u(t+\ell/v)}{C_E(R+Z_0)} \\ D_1(\ell,t) &= \frac{R(R-Z_0)Z_0^2}{(R+Z_0)^2}\left\{-\frac{e^{-t/Z_0C_E}e^{-3(\ell/v)(1/Z_0C_E)}(t+3\ell/v)u(t+3\ell/v)}{Z_0^2C_E^2}\right. \\ &\quad +\frac{e^{-t/Z_0C_E}e^{-3(\ell/v)(1/Z_0C_E))}}{vZ_0^2C_E^2}\left[3Z_0C_E\ell\delta\left(t+\frac{3\ell}{v}\right)+Z_0C_Evt\delta\left(t+\frac{3\ell}{v}\right)\right. \\ &\quad \left.\left.+\Big(-3\ell-vt+vZ_0C_E\Big)u\left(t+\frac{3\ell}{v}\right)\right]\right\} \\ D_2(\ell,t) &= \dots \end{aligned}\right\} \tag{34}$$

Note: terms like $u(t+n\ell/v)$ where $n \in \mathbb{Z}^+$ in the above equations seem to give non-causal solutions. However, we recall that $v < 0$ because of the way the system was set up (See Figure 1) and so, in fact, each equation is causal.

Since each $\tilde{D}_n$ can be Fourier inverted, we can easily calculate the $V(\ell, t)$ term by term by using (28) and (34) because

$$\left.\begin{aligned} V(\ell,t) &= F(\ell,t) * D_0(t) + F(\ell,t) * D_1(\ell,t) + \ldots \\ &\equiv V_0(\ell,t) + V_1(\ell,t) + \ldots \end{aligned}\right\} \tag{35}$$

We will use (35) for numerical inversion in the *Numerical Inversion of* $\tilde{V}$ section.

As an example, we can write down V_0

$$\left.\begin{aligned} V_0(\ell,t) = -\frac{I_0 Z_0}{R+Z_0} e^{-t/Z_0 C_E} \Big\{ &\left(e^{t/Z_0 C_E} - 1\right)(R+Z_0)u(t) \\ &+ (Z_0 - R)e^{-2(\ell/v)(1/Z_0 C_E)} u\left(t + \frac{2\ell}{v}\right) + \\ &+ e^{t/C_E Z_0}\left[-2R + (R - Z_0)u\left(t + \frac{2\ell}{v}\right)\right]\Big\} \end{aligned}\right\} \tag{36}$$

The other terms $V_{1\ldots\infty}$ are too onerous to write down and we will let *Mathematica* take care of these terms.

More Observations

The first observation of (36) is that the reflection happens at $t = 2\ell/v$. This is expected because a signal must travel down the transmission line and then reflect back again, which means the first reflection can only occur at $t = 2\ell/v$. We expect that for V_1 will have reflections occur at $t = 4\ell/v$. In fact, the *Mathematica* solutions show that reflections occur at every $t = 2n(\ell/v)$, where $n \in \mathbb{Z}$.

The second observation is that the time constant for each reflection is $Z_0 C_E$ and not RC_E. This is an unexpected result for us because an ideal transmission line does not have any resistance! This result implies that the L's and C's of a transmission line behave like a resistor in this type of system.

The third observation is that it is not obvious that the sum of V_n will for $\ell = 0$, i.e. no reflections, has a time constant of RC_E. We will prove that it does indeed have a time constant RC_E in the section *Special Case A: $\ell = 0$*.

Special Case A: $\ell = 0$

We can check that (33) makes sense by considering the special case $\ell = 0$. When we substitute this value into (28), we have

$$F(0,t) = -\frac{2I_0}{Z_0}\Big[1 - u(t)\Big] \tag{37}$$

and from (30)

$$\begin{aligned}\tilde{D}(\omega) &= -\frac{1}{a_2 b_1} \times \frac{1}{1 - a_1 b_2/a_2 b_1} \\ &= -\frac{1}{a_2 b_1 - a_1 b_2} = \frac{iRZ_0}{2(-i + \omega RC_E)}\end{aligned} \tag{38}$$

which has an inverse Fourier transform

$$D(t) = -\tfrac{1}{2}\left(\frac{Z_0}{C_E}\right) e^{-t/RC_E} u(t) \tag{39}$$

Therefore, the convolution is give $V_A(0,t)$ and is

$$\begin{aligned}V_A(0,t) = F(0,t) * D(t) &= \int_{-\infty}^{\infty} d\tau \, \left(-\frac{2I_0}{Z_0}\Big[1 - u(t)\Big]\right)\left(-\tfrac{1}{2}\left(\frac{Z_0}{C_E}\right) e^{-(t-\tau)/RC_E} u(t-\tau)\right) \\ &= V_E\left[1 + \left(e^{-t/RC_E} - 1\right) u(t)\right]\end{aligned} \tag{40}$$

where $V_E = I_0 R$ and (40) is identical to (19) for $t > 0$, which is what we expect.

Special Case B: Numerical Inversion of $\tilde{V}$: $Z_0 C_E \gg \ell/|v|$

Let us consider the case when the time constant $Z_0 C_E$ is much greater than the temporal length of the transmission line $\ell/|v|$. We expect the temporal behaviour of $V(t)$ to be nearly the same as that for $\ell = 0$. In this example, we have set $\ell = 0.03$ m and so $(Z_0 C_E = 5 \text{ ns}) \gg (\ell/|v| = 0.15 \text{ ns})$. The plots of $V_{0\ldots11}$ are shown in Figure 4. We notice the slow convergence of $\sum V_n(t)$ and even after 12 terms, it is still insufficient to reproduce $V_{\text{ref}}(t) = V_E e^{-t/RC_E}$. We cannot use more terms because *Mathematica* becomes

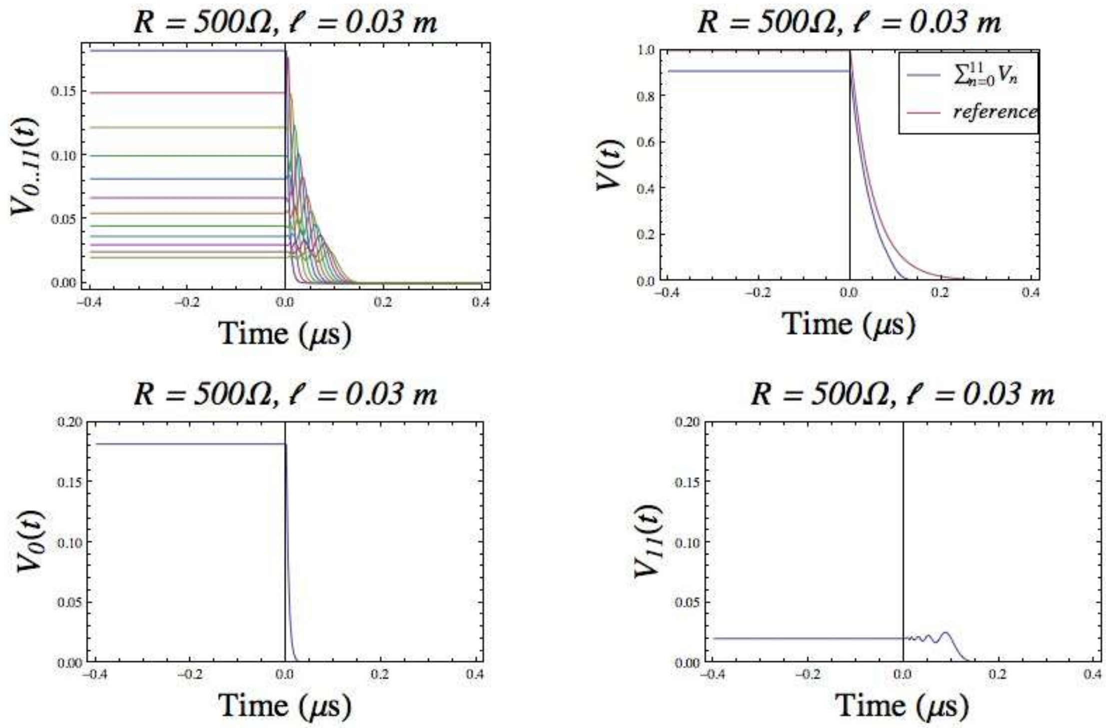

Figure 4 These plots are for $(Z_0C_E = 5 \text{ ns}) \ll (\ell/|v| = 0.15 \text{ ns})$. The top left plot shows $V_0(t)$ to $V_{11}(t)$ which clearly shows reflections. The bottom two graphs show $V_0(t)$ and $V_{11}(t)$ explicitly. The top right graph shows the $\sum_{n=0}^{11} V_n(t)$ compared to the reference $V_E \times e^{-t/RC_E}$.

excruciatingly slow in calculating V_n for $n \geq 12$. Despite this, we will assume and expect that the infinite sum of V_n does indeed converge to V_{ref}.

Special Case C: Numerical Inversion of $\tilde{V}$: $Z_0C_E \ll \ell/|v|$

The special case which we will consider here is when the Z_0C_E time constant is shorter than the length of the transmission line, i.e. $Z_0C_E \ll \ell/|v|$. In this case, the temporal length of the transmission line $\ell/|v|$ is a lot longer than the time constant Z_0C_E. In this example, we have set $\ell = 5$ m and so $(Z_0C_E = 5 \text{ ns}) < (\ell/|v| = 25) \text{ ns})$ and is sufficient to illustrate this case. The plots are shown in Figure 5. Again, we were unable to sum V_n above $n = 7$ because of large numerical errors. However, the reflections are clearly visible in both V_0 and V_5. The slope of the overall curve is clearly much slower than V_{ref} which is not surprising.

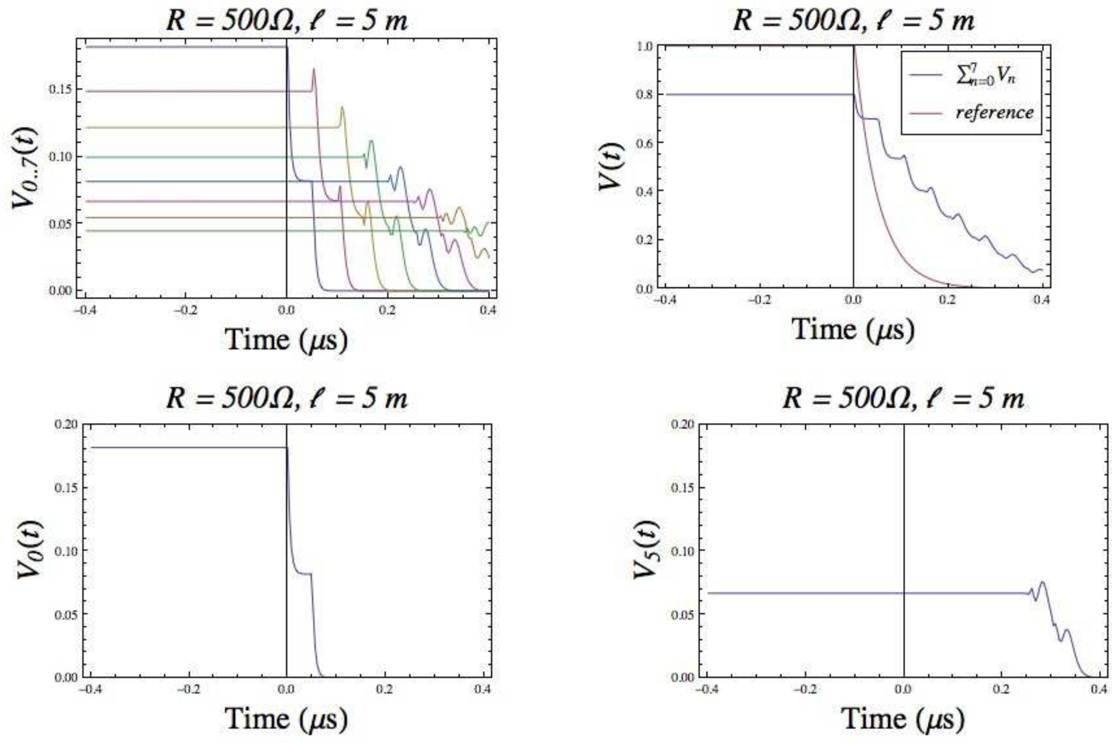

Figure 5 These plots are for $(Z_0 C_E = 5\text{ ns}) < (\ell/|v| = 25\text{ ns})$. The top left plot shows $V_0(t)$ to $V_7(t)$ which clearly shows very strong reflections. The bottom two graphs show $V_0(t)$ and $V_5(t)$ explicitly. The top right graph shows the $\sum_{n=0}^{7} V_n(t)$ compared to the reference $V_E \times e^{-t/RC_E}$.

Table 1 Parameters used for the numerical inversion of $\tilde{V}$

Parameter	Value	Comments
C_E	100 pF	Einzel lens capacitance
Z_0	50Ω	impedance of coaxial transmission line
R	500Ω	load resistor
v	$-0.67c$	group velocity of transmission line
V_E	1 V	initial steady state voltage
I_0	V_E/R	R is defined in each special case

Ansatz for $Z_0 C_E \ll \ell/|v|$

It is clear from Figure 5 that we can come up with an ansatz for the behaviour of $V(t)$ for $\ell/|v| \gg Z_0 C_E$. The spacing of each reflection is $t = 2n\ell/v$ and the relationship between the amplitude of after each reflection is

$$V_n = \Gamma V_{n-1} \tag{41}$$

where $\Gamma = (R - Z_0)/(R + Z_0)$ is the reflection coefficient. In this ansatz we must have $R > Z_0$.

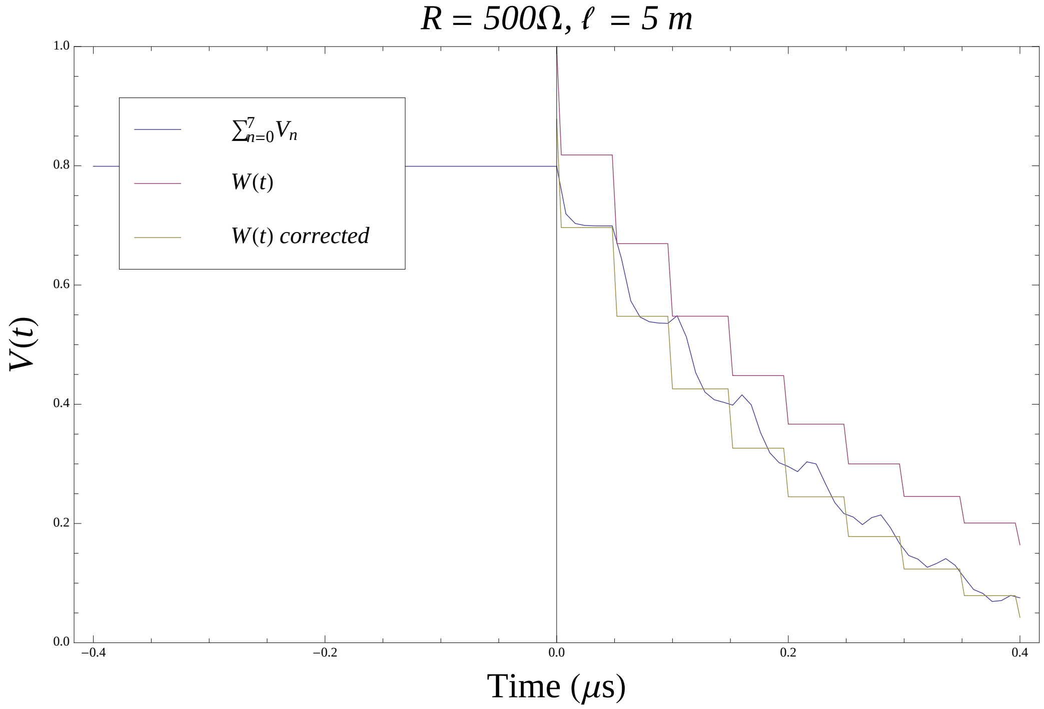

Figure 6 The ansatz solution $W(t)$ compared to $\sum_{n=0}^{7} V_n(t)$ from the section *Special Case C*. It is clear that the ansatz works quite well.

Therefore, the ansatz is

$$W(t) = V_E \sum_{n=0}^{\infty} \left[u\left(t - \frac{2(n-1)\ell}{|v|}\right) - u\left(t - \frac{2n\ell}{|v|}\right) \right] \Gamma^n \quad \text{for } t \geq 0. \tag{42}$$

It is interesting to note that (42) is independent of C_E because in this ansatz it is assumed that the time constant $Z_E C_E = 0$.

In Figure 6, we have plotted the ansatz solution $W(t)$ together with $\sum_{n=0}^{7} V_n(t)$ for $\ell = 5$ m and $R = 500\Omega$ from *Special Case C*. We have to subtract a vertical offset from $W(t)$ to better compare to $\sum_{n=0}^{7} V_n(t)$ because of the problems with summing V_n which was previously discussed.

CONCLUSION

We have shown that there are reflections if $R \neq Z_0$. These reflections are minimised if $\ell/v \ll Z_0 C_E$ and they manifest themselves very clearly if $\ell/v \gg Z_0 C_E$. The two surprising results are:

(i) the correct comparison for the checking whether the length ℓ/v is long or short is to $Z_0 C_E$ and not RC_E.

(ii) And even more surprising is that the decay rate for $\ell/v \gg Z_0 C_E$, is essentially independent of C_E and depends only on the length of the transmission line and the reflection coefficient.

Of course, a similar calculation can be done for the case when the chopper goes from an "off" state to an "on" state. The results should be very similar to the case we have already considered here.