PBAR NOTE 585
MEASUREMENT OF RELATIVE PERMITTIVITY AND PERMEABILITY USING TWO PORT S-PARAMETER TECHNIQUE

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INTRODUCTION

The design of microwave absorbers is crucial to the success of the 4-8 GHz Debuncher Upgrade project. This notes deals with the measurement of the relative permittivity and permeability of materials using a two port S parameter technique.

To measure these parameters, a tight-fitting sample of material of thickness δ is placed in a waveguide perpendicular to the direction of the propagating electromagnetic waves as shown in Figure 1. It is assumed that only the fundamental waveguide mode TE^{z}_{10} can propagate. A calibrated network analyzer will measure the S parameters at the reference planes indicated by dashed lines in Figure 1.

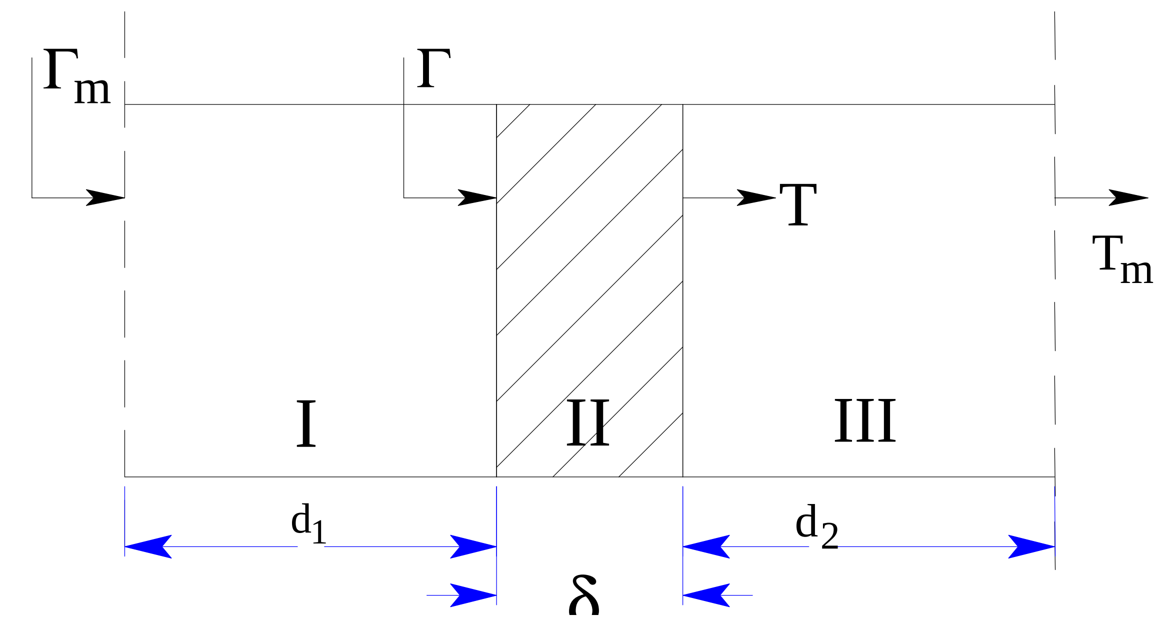

Figure 1. Schematic of waveguide S-parameter measurement.

The reflection coefficient Γ_m (S_{11}) is transformed to the reference plane that is located on the front face (z=0) of the sample by the following expression:

$$\Gamma = \Gamma_m e^{j2\beta_o d_1} \tag{1}$$

For the fundamental mode:

$$\beta^2 = \omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2 \tag{2}$$

where **a** is the width of the waveguide. The transmission coefficient T_m (S_{21}) can also be transformed to the back face of the sample:

$$T = T_m e^{j\beta_o (d_1 + d_2)} \tag{3}$$

DERIVATION OF FORMULAS

The transverse electric field in Region 1 is:

$$E_t = E_1 e^{-j\beta_o z} + E_1 \Gamma e^{+j\beta_o z} \tag{4}$$

The transverse magnetic field in Region 1 is

$$H_t = \frac{E_1}{Z_o} e^{-j\beta_o z} - \frac{E_1}{Z_o} \Gamma e^{+j\beta_o z} \tag{5}$$

where Z is the wave impedance. For the fundamental mode, this impedance is:

$$Z = \frac{\omega\mu}{\beta} \tag{6}$$

The transverse electric field in Region 2 is:

$$E_t = E_2^+ e^{-j\beta_2 z} + E_2^- e^{j\beta_2 z} \tag{7}$$

The transverse magnetic field in Region 2 is:

$$H_t = \frac{E_2^+}{Z_2} e^{-j\beta_2 z} - \frac{E_2^-}{Z_2} e^{j\beta_2 z} \tag{8}$$

In region 3 there is only a forward going wave so the transverse electric field is:

$$E_t = E_3 e^{-j\beta_o (z-\delta)} \tag{9}$$

The transverse magnetic field in Region 3 is:

$$H_t = \frac{E_3}{Z_o} e^{-j\beta_o (z-\delta)} \tag{10}$$

At the front face boundary between Regions 1 and 2 (z=0), the transverse electric and magnetic fields must be continuous:

$$E_1 (1 + \Gamma) = E_2^+ + E_2^- \tag{11}$$

$$E_1 (1 - \Gamma) = \frac{Z_o}{Z_2} \left(E_2^+ - E_2^- \right) \tag{12}$$

At the back face boundary between Regions 2 and 3 (z=δ), the transverse electric and magnetic fields must be continuous:

$$E_2^+ e^{-j\beta_2\delta} + E_2^- e^{j\beta_2\delta} = E_3 \tag{13}$$

$$E_2^+ e^{-j\beta_2\delta} - E_2^- e^{j\beta_2\delta} = \frac{Z_2}{Z_o} E_3 \tag{14}$$

Adding Equations 11 and 12 results in:

$$E_2^+ = \frac{E_1}{2}\left(1 + \frac{Z_2}{Z_o}\right) + \Gamma \frac{E_1}{2}\left(1 - \frac{Z_2}{Z_o}\right) \tag{15}$$

Subtracting Equations 11 and 12 results in:

$$E_2^- = \frac{E_1}{2}\left(1 - \frac{Z_2}{Z_o}\right) + \Gamma \frac{E_1}{2}\left(1 + \frac{Z_2}{Z_o}\right) \tag{16}$$

Adding Equations 13 and 14 results in:

$$E_2^+ = \frac{E_3}{2} e^{j\beta_2\delta}\left(1 + \frac{Z_2}{Z_o}\right) \tag{17}$$

Subtracting Equations 13 and 14 results in:

$$E_2^- = \frac{E_3}{2} e^{-j\beta_2\delta}\left(1 - \frac{Z_2}{Z_o}\right) \tag{18}$$

The transmission amplitude through the sample is:

$$T = \frac{E_3}{E_1} \tag{19}$$

Combining Equations 15 and 17:

$$Te^{j\beta_2\delta}(Z_o + Z_2) = (Z_o + Z_2) + \Gamma(Z_o - Z_2) \tag{20}$$

Combining Equation 16 with 18:

$$Te^{-j\beta_2\delta}(Z_o - Z_2) = (Z_o - Z_2) + \Gamma(Z_o + Z_2) \tag{21}$$

Define an intermediate reflection coefficient:

$$\Gamma_2 = \frac{(Z_2 - Z_o)}{(Z_o + Z_2)} \tag{22}$$

Multiplying Equation 20 by Equation 21:

$$T^2 = 1 - \Gamma\Gamma_2 - \frac{\Gamma}{\Gamma_2} + \Gamma^2 \tag{23}$$

Define the following coefficient:

$$b = \frac{\Gamma^2 - T^2 + 1}{2\Gamma} \tag{24}$$

Equation 23 can be cast into a quadratic equation:

$$\Gamma_2{}^2 + 2b\Gamma_2 + 1 = 0 \tag{25}$$

The roots of Equation 25 are:

$$\Gamma_2 = b \pm \sqrt{b^2 - 1} \tag{26}$$

Equation 22 can be inverted into:

$$\frac{Z_2}{Z_o} = \frac{(1+\Gamma_2)}{(1-\Gamma_2)} \tag{27}$$

Adding Equations 20 and 21 results in:

$$\cos(\beta_2\delta) = \frac{1}{2T}\left(2 - \Gamma\Gamma_2 - \frac{\Gamma}{\Gamma_2}\right) \tag{28}$$

The relative permeability of the sample is found from combining Equations 6, 27 and 28:

$$\mu_r = \frac{\beta_2}{\beta_o}\frac{Z_2}{Z_o} \tag{29}$$

Using Equation 2 and Equation 29, the relative permittivity of the sample is:

$$\varepsilon_r = \frac{1}{\mu_r}\left(\frac{c}{\omega}\right)^2\left(\beta_2{}^2 + \left(\frac{\pi}{a}\right)^2\right) \tag{30}$$

SUMMARY OF EQUATIONS

The following is a summary of the equations needed to find μ_r and ε_r given the S parameters S_{11} (Γ_m) and S_{21} (T_m).

$$\varepsilon_r = \frac{1}{\mu_r}\left(\frac{c}{\omega}\right)^2\left(\beta_2{}^2 + \left(\frac{\pi}{a}\right)^2\right) \tag{31}$$

$$\mu_r = \frac{\beta_2}{\beta_o}\frac{Z_2}{Z_o} \tag{32}$$

$$\beta_o{}^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 \tag{33}$$

$$\cos(\beta_2\delta) = \frac{1}{2T}\left(2 - \Gamma\Gamma_2 - \frac{\Gamma}{\Gamma_2}\right) \tag{34}$$

$$\frac{Z_2}{Z_o} = \frac{(1+\Gamma_2)}{(1-\Gamma_2)} \tag{35}$$

$$\Gamma_2 = b \pm \sqrt{b^2 - 1} \tag{36}$$

$$b = \frac{\Gamma^2 - T^2 + 1}{2\Gamma} \tag{37}$$

$$\Gamma = \Gamma_m e^{j2\beta_o d_1} \tag{38}$$

$$T = T_m e^{j\beta_o(d_1+d_2)} \tag{39}$$

The following is a summary of the equations needed to find the S parameters S_{11} (Γ_m) and S_{21} (T_m) given μ_r and ε_r .

$$\Gamma_m = \Gamma e^{-j2\beta_o d_1} \tag{40}$$

$$T_m = Te^{-j\beta_o(d_1+d_2)} \tag{41}$$

$$\beta_o{}^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 \tag{42}$$

$$\Gamma = \Gamma_2 \frac{1 - e^{j2\beta_2\delta}}{\Gamma_2{}^2 - e^{j2\beta_2\delta}} \tag{43}$$

$$T = e^{-j\beta_2\delta}(1 - \Gamma\Gamma_2) \tag{44}$$

$$\beta_2{}^2 = \mu_r\varepsilon_r\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 \tag{45}$$

$$\Gamma_2 = \frac{\mu_r\beta_o - \beta_2}{\mu_r\beta_o + \beta_2} \tag{46}$$

TEST OF FORMULAS

A 1.55 mm thick sample of Emerson Cuming MF190 absorber was placed in a WR159 waveguide. The magnetic and electric properties are shown in Figures 2-4. These numbers agree well with the manufacture specifications. To further test the above derivations, the waveguide test fixture was modeled using Hewlett Packards's High Frequency Structure Simulator (HFSS). The electromagnetic properties for absorber used in the HFSS model were the measured values at 6 GHz. The S parameters were then calculated using the above derivation and HFSS. This data is shown in Figures 5 and 6.

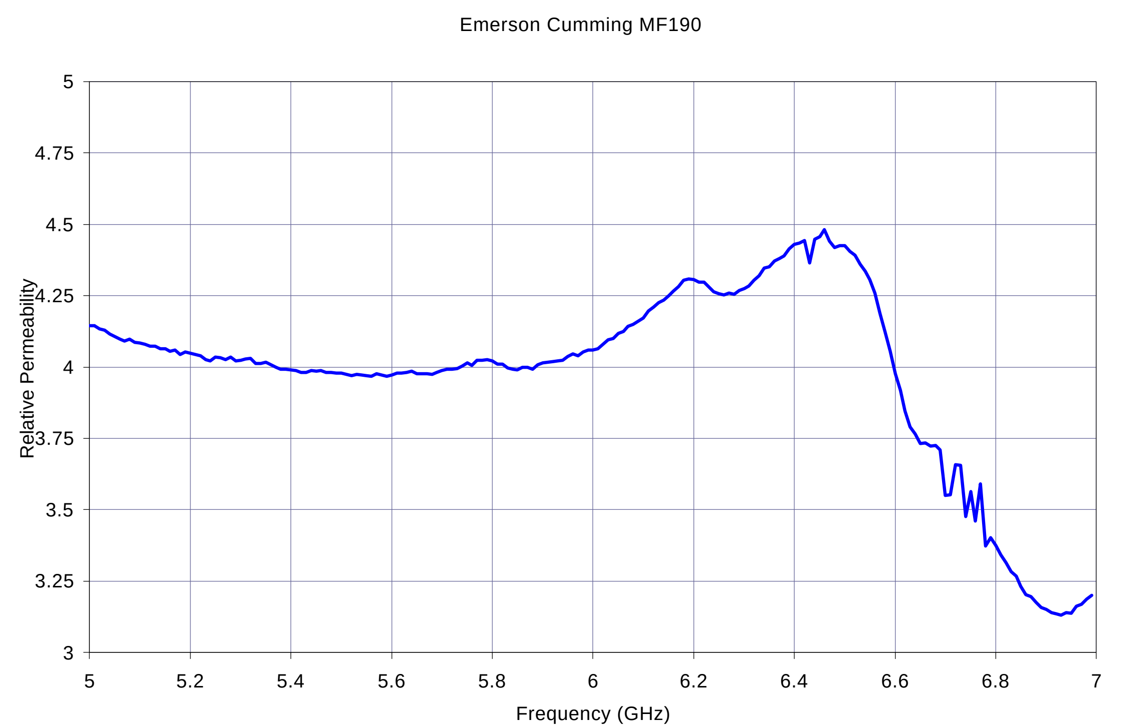

Figure 2. The measured magnitude of the relative permeability (μ_r) for MF190

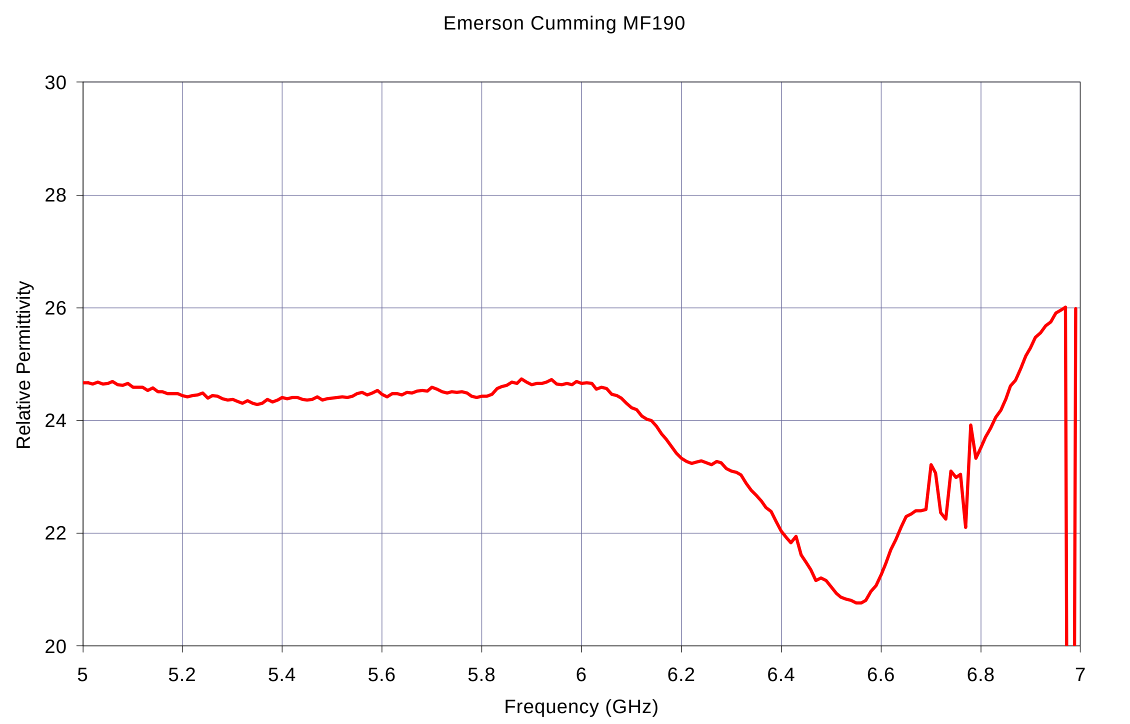

Figure 3. The measured magnitude of the relative permittivity (e_r) for MF190

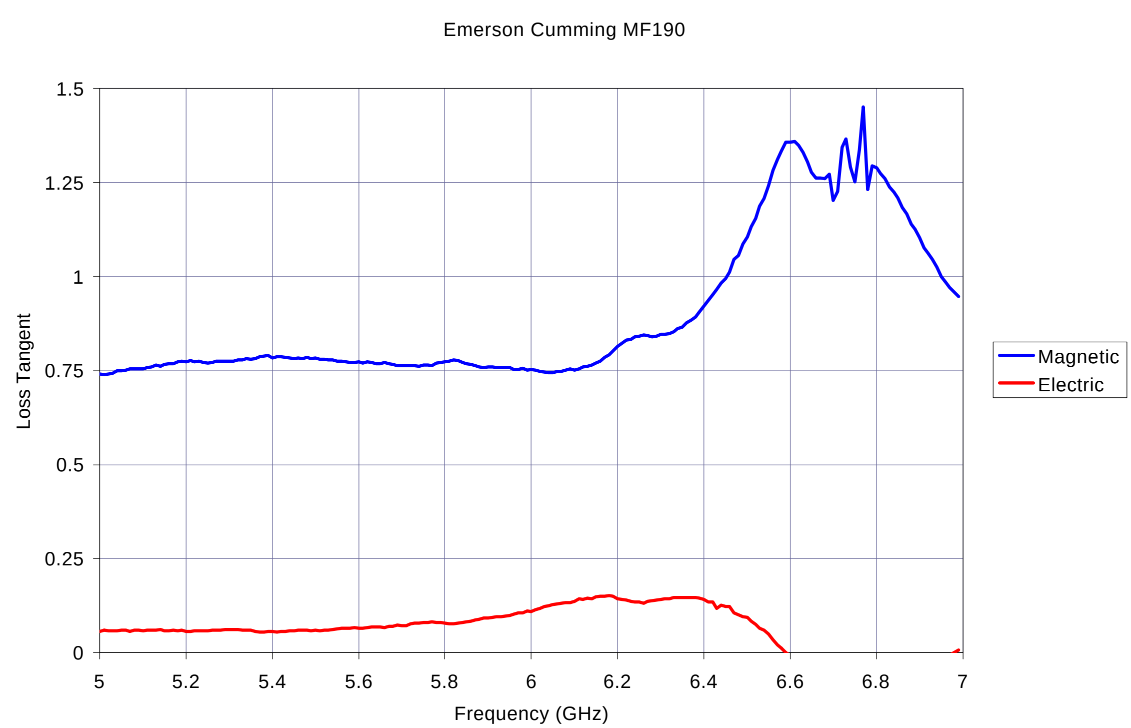

Figure 4. The measured loss tangents for MF190

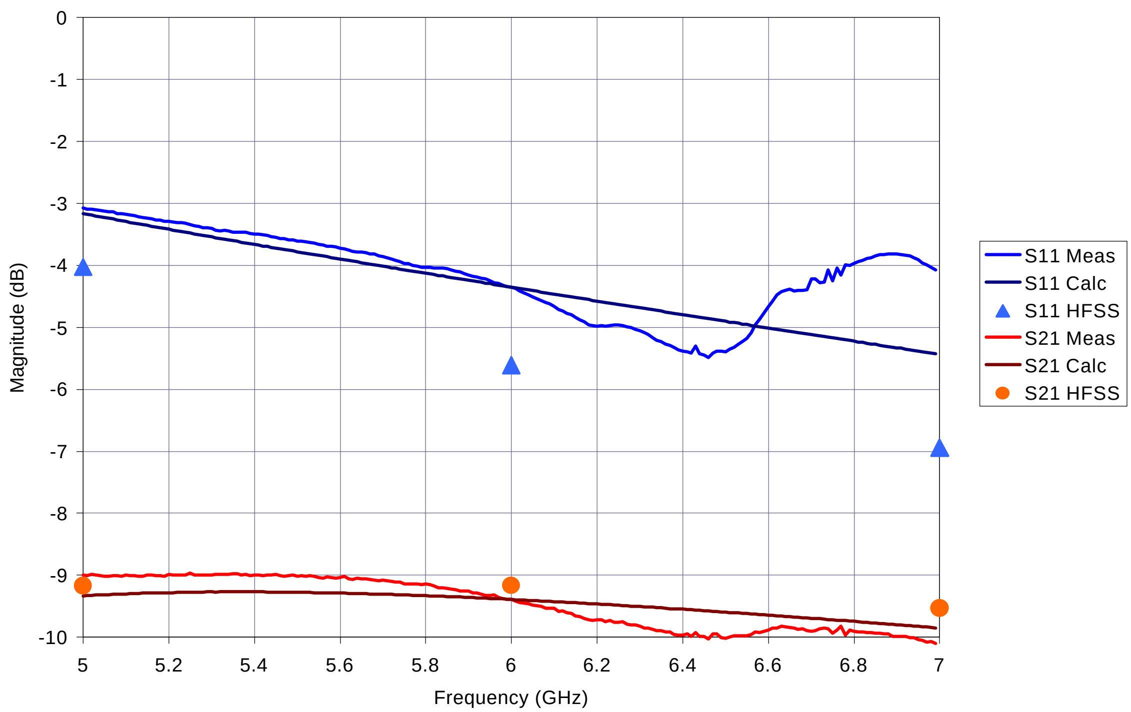

Figure 5. Magnitude of S-parameters for a 1.5 mm thick sample MF190 in a WR159 waveguide.

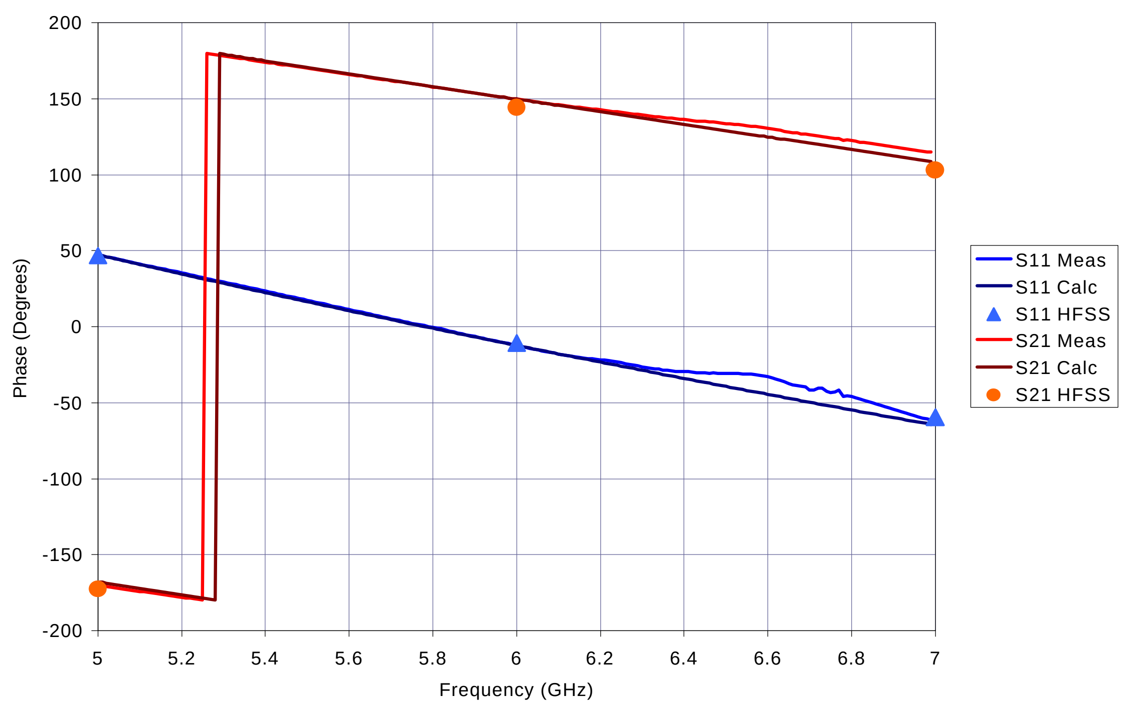

Figure 6. Phase of S-parameters for a 1.5 mm thick sample MF190 in a WR159 waveguide.