# Algorithm for finding effective impedance from e-cloud buildup simulations

The comprehensive simulations of the e-cloud driven instabilities using e.g. POSINST+WARP package [1] require significant computational resources while providing only limited insight into the underlying physics. To better understand the mechanics of the instability it can be useful to study its various aspects separately.

We may start with the e-cloud generation code like POSINST considering e-cloud created by a train of periodically displaced bunches, compute and Fourier analyze the e-cloud electric field acting back on the beam. A convenient quantity which would allow to use the standard theory of coherent instabilities is coupling impedance. For transverse dipole oscillations of the beam (horizontal for definiteness) the impedance introduced by an object of length *L* is defined as (confer Ref. [2] Eq.(2.70))

$$\frac{Z_\perp(\omega)}{L} = -i\,\frac{E_x(\omega)}{I_1(\omega)} \tag{1}$$

where $E_x(\omega)$ and $I_1(\omega)$ are the Fourier transforms of the electric field (contribution from the e-cloud magnetic field is negligible) and the beam current dipole moment:

$$I_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} I_1(t)dt \tag{2}$$

All values are assumed to be taken at a fixed longitudinal position, say *z*=0. The dipole moment is defined as:

$$I_1 = \int x j_z dxdy \tag{3}$$

with $j_z$ being the beam current density, *x* and *y* being the transverse coordinates.

Let us consider a train of bunches with a Gaussian longitudinal profile,

$$\begin{aligned} I_0^{(n)}(t) &= \int j_z dxdy = I_{peak}\exp[-\frac{(t-nT_0-t_0)^2}{2\sigma_t^2}], \\ I_{peak} &= \frac{eN_p}{\sqrt{2\pi}\sigma_t}. \end{aligned} \tag{4}$$

where $T_0$ is the time interval between bunches, *n*=1,2,… Assuming that the bunches are horizontally displaced as

$$x_n = x_0\cos\varphi_n, \quad \varphi_n = n\cdot\Delta\varphi + \varphi_0 \tag{5}$$

we have for the beam current dipole moment

$$I_1 = x_0\sum_n I_0^{(n)}\cos\varphi_n \tag{6}$$

If the number

$$N_m = 2\pi / \Delta\varphi \tag{7}$$

is rational, then $I_1$ is exactly periodic. We will limit ourselves to even simpler case of integer $N_m$ so that the full period is

$$T = N_m T_0 \tag{8}$$

The Fourier transform of a periodic function can be reduced to an integral over one period

$$\begin{aligned} A(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} A(t)dt = \widetilde{A}(\omega)\sum_k \delta(\omega T - 2\pi k), \\ \widetilde{A}(\omega) &= \int_{\tau}^{\tau+T} e^{i\omega t} A(t)dt, \end{aligned} \tag{9}$$

where $\delta(x)$ is Dirac's delta function, $\tau$ is an arbitrary moment of time. It should be noted that $\widetilde{A}(\omega)$ has physical significance only for frequencies

$$\omega = \omega_k = 2\pi k / T \tag{10}$$

The e-cloud electric field in the steady state is also periodic. However, it may require too long computational time to reach it so we will ignore the lack of periodicity and modify the definition of the impedance as

$$\frac{Z_\perp(\omega)}{L} = -i\frac{\widetilde{E}_x(\omega)}{\widetilde{I}_1(\omega)} \tag{11}$$

By changing $\tau$ we can slide the window across the train and observe the development of e-cloud effective impedance in time.

The reduced Fourier transform of the beam dipole moment required in eq. (11) can be obtained analytically. The following result was obtained in the case of short bunches, $\sigma_t << T_0$, but can be generally true:

$$\widetilde{I}_1(\omega_k) = \frac{1}{2} e N_p N_m x_0 e^{i\omega_k t_0 - \omega_k^2 \sigma_t^2 / 2} \sum_j (e^{i\varphi_0}\delta_{k, jN_m - 1} + e^{-i\varphi_0}\delta_{k, jN_m + 1})\,. \tag{12}$$

where $\delta_{i,k}$ is the Kronecker delta: $\delta_{i,i} = 1$, $\delta_{i,k} = 0$, $i \neq k$.

Let us note that Eq. (12) was obtained for the case when each bunch is shifted as a whole by the amount given by Eq. (5). If the displacement along the beam varies continuously according to the sinusoidal law

$$x(t)\big|_{z=0} = x_0 \cos(\frac{2\pi}{T} t + \varphi_0) \tag{13}$$

then for the beam dipole moment Fourier transform we would have a slightly different result:

$$\tilde{I}_1(\omega_k)=\frac{1}{2}eN_pN_m x_0 e^{i\omega_k t_0}\sum_j e^{-\omega_{jN_m}^2\sigma_t^2/2}\left(e^{i\varphi_0}\delta_{k,jN_m-1}+e^{-i\varphi_0}\delta_{k,jN_m+1}\right). \qquad (14)$$

For short bunches, $\sigma_t << T_0$, the difference between Eqs. (12) and (14) is negligible.

These equations show that only first sidebands of the multiples of bunch sequence frequency $2\pi j/T_0$ are excited.

Let us consider now the effect of the e-cloud electric field on the beam dynamics. For that we must know how the phase of beam oscillations (and therefore the phase of the e-cloud electric field) varies with *z*. The initial offset given by Eq. (5) or Eq. (13) must be complemented by the initial values of transverse momentum (or the slope of the trajectory $x'=dx/dz$ that we will refer to as the momentum). We will assume – for simplicity – that at the observation point the beta-function does not have a slope:

$$\alpha_x=-\frac{1}{2}\beta_x'=0 \qquad (15)$$

and that all bunches have the same value of the Courant-Snyder invariant. Then there are two possibilities for bunch-by-bunch transverse momentum (assuming that position is given by Eq. (5)):

$$x_n'=\mp\frac{x_0}{\beta_x}\sin\varphi_n \qquad (16)$$

which are illustrated by Fig. 1 for the case $N_m$=6

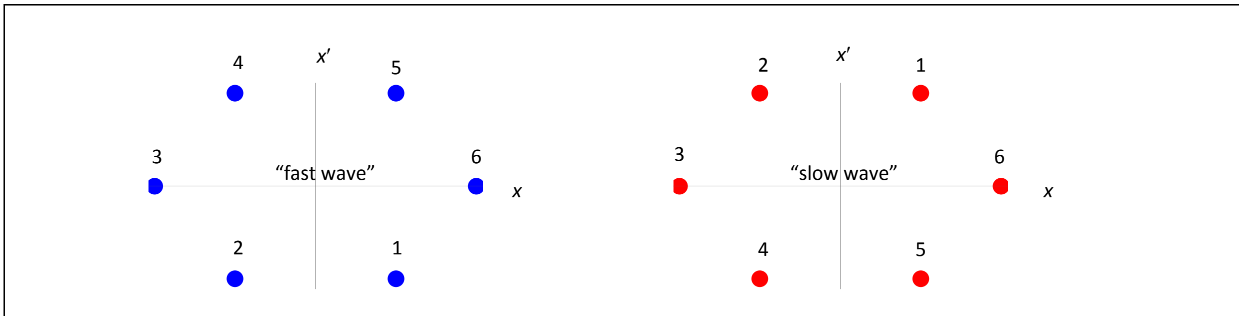


Figure 1. Two possible distributions in phase space corresponding to the same bunch displacements while the momenta are given by Eq. (16) with upper sign (left) and lower sign (right).

When a bunch moves from the initial position *z*=0 its offset and momentum vary according to the betatron phase advance

$$\phi_x=\int_0^z\frac{dz}{\beta_x}\approx\frac{z}{\beta_x} \quad \text{for} \quad z<<\beta_x \qquad (17)$$

so that in the vicinity of the starting point ($\beta_x\approx$ const by assumption (15))

$$x_n \approx x_0 \cos(\frac{z}{\beta_x} \pm \varphi_n), \qquad x'_n \approx -\frac{x_0}{\beta_x} \sin(\frac{z}{\beta_x} \pm \varphi_n) \tag{18}$$

To proceed further let us employ the continuous wave representation (13). The initial phase $\varphi_n$ is specific for a given bunch *n* and for z≠0 "moves" with it so the transition to the continuous wave representation is accomplished by substitution

$$\varphi_n \to \frac{2\pi}{T}(t - \frac{z}{v_0}) + \varphi_0 \tag{19}$$

where $v_0$ is the longitudinal velocity of the beam.

Upon this substitution the beam offset becomes a continuous function (as for a coasting beam):

$$x(z,t) \approx x_0 \cos[\frac{z}{\beta_x} \pm \frac{2\pi}{T}(t - \frac{z}{v_0}) \pm \varphi_0] \tag{20}$$

whereas derivative w.r.t. *z* should be considered as a full derivative

$$x' = \frac{d}{dz}x = (\frac{\partial}{\partial z} + \frac{\partial}{v_0 \partial t})x \tag{21}$$

Let us note in passing that from Eq. (20) we can find the phase velocity of the beam oscillations

$$v_{phase} = -\frac{\partial x / \partial t}{\partial x / \partial z} \approx v_0 /(1 \mp \frac{v_0 T}{2\pi\beta_x}) \tag{22}$$

and verify that the upper sign corresponds to the so-called fast wave ($v_{\text{phase}} > v_0$) while the lower sign corresponds to a slow wave, $v_{\text{phase}} < v_0$.

Using *z* as the independent variable we can write for the equation of motion and its immediate consequence:

$$\frac{d}{dz}x' \equiv (\frac{\partial}{\partial z} + \frac{1}{v_0}\frac{\partial}{\partial t})x' = \frac{e}{m_p \gamma v_0^2}E_x \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dz}x'^2 = \frac{e}{m_p \gamma v_0^2}x'E_x \tag{23}$$

Since the instantaneous current $I_0$ in our case is an integral of motion ($I_0'=0$), we may use it as a weight and obtain

$$\frac{1}{2}\frac{d}{dz}x'^2 I_0 = \frac{e}{m\gamma v_0^2}I_x E_x \tag{24}$$

where the transverse current was introduced: $I_x = x' I_0$.

Noticing that for periodic functions of time the full derivative and integral over the period are commuting operations we can average Eq. (24) over $T = N_m T_0$ and obtain

$$\frac{d}{dz}\left\langle x'^2\right\rangle \approx \frac{2}{N_m N_p m_p \gamma v_0^2}\int_0^T I_x E_x dt \tag{25}$$

Under the same assumptions as were made to derive Eq. (14) we can find for the Fourier transform of the transverse current

$$\widetilde{I}_x(\omega_k) = \pm\frac{i}{2\beta_x} eN_p N_m x_0 e^{i\omega_k t_0}\sum_j e^{-\omega_{jN_m}^2\sigma_t^2/2}(e^{i\varphi_0}\delta_{k,jN_m-1} - e^{-i\varphi_0}\delta_{k,jN_m+1}). \tag{26}$$

and obtain for the work of the electric field over one period

$$\begin{aligned}\int_0^T E_x I_x dt &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\widetilde{E}_x(\omega_k)\widetilde{I}_x(-\omega_k) = \frac{1}{T}[\underbrace{\widetilde{E}_x(0)\widetilde{I}_x(0)}_{=0} + 2\,\mathrm{Re}\sum_{k=1}^{\infty}\widetilde{E}_x(\omega_k)\widetilde{I}_x^*(\omega_k)] = \\ &= \mp\frac{x_0^2}{2\beta_x T}(eN_p N_m)^2\frac{\mathrm{Re}\,Z_{eff}}{L}\end{aligned} \tag{27}$$

where the effective impedance was introduced:

$$Z_{eff} = Z_\perp(\omega_1) + \sum_{j=1}^{\infty} e^{-\omega_{jNm}^2\sigma_t^2/2}[Z_\perp(\omega_{jN_m+1}) - Z_\perp(\omega_{jN_m-1})] \tag{28}$$

Its real part is a single important value which determines the instability growth rate (in absence of Landau damping and decoherence) and which can be used in study of parametric dependencies. In practice it can be found from Eq. (27) read from right to left.

Known the effective impedance we can find the instability growth rate. Since

$$\left\langle x'^2\right\rangle = \frac{1+\alpha_x^2}{2\beta_x^2}x_0^2 \approx \frac{x_0^2}{2\beta_x^2} \tag{29}$$

(assumption (15) applied) we have from Eqs. (25) and (27)

$$\frac{1}{x_0^2}\frac{d}{dz}x_0^2 \approx \mp\frac{2e^2 N_p}{m_p\gamma v_0^2 T_0}\beta_x\frac{\mathrm{Re}\,Z_{eff}}{L} \tag{30}$$

This result illustrates a well-known fact from the theory of coasting beam instabilities: it is the slow wave that may go unstable in a passive medium (Re$Z_{\text{eff}}$ >0) while the fast wave is damped.

Let us take for example parameters of the Recycler: $N_p$=5·10$^{10}$, $T_0$=18.8ns, length of focusing magnets $L$=683.9m and the average beta-function at their locations $<\beta_x>$=44m. Then for specific impedance Re$Z_{\text{eff}}/L$=0.1MΩ/m$^2$ (or the total impedance of e-cloud in focusing magnets Re$Z_{\text{eff}}$ = 68.4 MΩ/m) we have from Eq. (30)

$$\frac{1}{x_0^2}\frac{d}{dn}x_0^2 \approx 0.29/\text{turn} \tag{31}$$

The amplitude of oscillations growth rate is half this value or ~7turns.

**Important remark**

The primary goal of this note was to show how the simulations can be quantified in terms of effective impedance. Its real part is given by Eq. (27) read from right to left. For the slow mode (lower sign) we have:

$$\frac{\mathrm{Re}\, Z_{eff}}{L} = \frac{2\beta_x}{x_0^2 I_{average}^2 T}\int_0^T E_x I_x dt, \quad I_{average} = eN_p N_m / T \tag{32}$$

As for Eq. (30), it serves exclusively the didactic purpose: it implies the translational symmetry in *z* with period $T/v_0$, i.e. – in fact – an absolute instability. To obtain practical results one should solve a more difficult case with boundary condition in *z* at the injection point.