Analysis of Tipping in 3-Point Component Stands

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Motivation

Using a 3 point stand to support a component offers some advantages and disadvantages. The main advantage is that it avoids creating an over-constrained system, making alignment of the component easy and preventing "hidden" large loads due to binding or preloading that can occur with a statically indeterminate 4 point stand. The main disadvantage is that it is easier to tip the component over than with an equivalent 4 point stand. One way to solve this issue is to provide positive retention so that the component cannot tip relative to the stand base. If this is not an option, however, then an analysis should be performed to understand the minimum force, and its point of application, that will cause the component to separate from the stand and begin tipping.

Geometry

Analysis of the 3 point stand system requires 5 defined points. Of them, 3 are the 3 pivot points between the component and the stand, 1 is the center of gravity (CG) of the component being tipped, and 1 is the point of application of the tipping force. In addition to the 5 points, there is 1 vector required. This unit vector will be the direction vector for the tipping force acting at the tipping point.

Finding the Static Loads on the Stand

Before we can compute a tipping load, we need to solve the static loads on the 3 point stand. This is as simple as summing the total forces and moments to zero to solve for the unknowns. We assume that the forces at the 3 points of contact act straight up.

$$\Sigma\vec{F} = 0$$

$$\boxed{\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + m\vec{g} = 0}$$

$$\Sigma\vec{M} = 0$$

$$\left(\overrightarrow{P_1} \times \overrightarrow{F_1}\right) + \left(\overrightarrow{P_2} \times \overrightarrow{F_2}\right) + \left(\overrightarrow{P_3} \times \overrightarrow{F_3}\right) + \left(\overrightarrow{P_4} \times m\vec{g}\right) = 0$$

The previous equation can easily be turned into 2 scalar equations for this problem.

$$\Sigma M_x = 0$$

$$\boxed{\left(P_{1y}F_1\right) + \left(P_{2y}F_2\right) + \left(P_{3y}F_3\right) + \left(P_{4y}mg\right) = 0}$$

$$\Sigma M_y = 0$$

$$\boxed{\left(P_{1x}F_1\right) + \left(P_{2x}F_2\right) + \left(P_{3x}F_3\right) + \left(P_{4x}mg\right) = 0}$$

Solving the 3 boxed equations will give us the values of F_1, F_2, and F_3.

Finding the Tipping Force

Because most components are longer in the x direction than in the y direction, the support locations will reflect this and the tipping direction will generally be more closely about the x axis (roll) than the y axis (pitch). We will assume that to be the case here. While it is possible to tip the component with only a single point of stand contact remaining, it is unlikely. Most of the time, tipping will occur with 2 of the original 3 stand pivot points in contact. The line through these remaining 2 pivot points forms the axis about which the component can tip. This is the also the axis that the moments need to be summed about to calculate the tipping force. For a 3 point stand, 3 independent axes of tipping exist. They are formed from the 3 possible 2 pivot-point combinations. We will show how to solve the tipping force for only one axis, but the other 2 are solved in the same way. To start, let's solve the tipping force about the axis through P_1 and P_3. In this scenario, tipping is assumed to occur when F_2 becomes 0, meaning that contact at that point has been lost. A simple moment summation about the tipping axis will solve for the tipping force required.

$$\Sigma M_{tip\_axis} = 0$$

In this case, the tip axis goes though P_1 and P_3, so the only forces that will generate a moment about that axis are the weight of the object acting at P_4 and the tipping force acting at P_5 (P_2 is zero at the initiation of tipping so it can't contribute any moment). We can now formulate this into an equation, which can then be solved for F_5.

$$\left(\left(\left(\overrightarrow{P_4}-\overrightarrow{P_1}\right)\times m\vec{g}\right)\cdot\left[\frac{\overrightarrow{P_3}-\overrightarrow{P_1}}{\left|\overrightarrow{P_3}-\overrightarrow{P_1}\right|}\right]\right)+\left(\left(\left(\overrightarrow{P_5}-\overrightarrow{P_1}\right)\times\overrightarrow{F_5}\right)\cdot\left[\frac{\overrightarrow{P_3}-\overrightarrow{P_1}}{\left|\overrightarrow{P_3}-\overrightarrow{P_1}\right|}\right]\right)=0$$

We can expand each term of this equation out one at a time.

$$\begin{aligned}\left(\overrightarrow{P_4}-\overrightarrow{P_1}\right)\times m\vec{g}&\\ &=\left(\left((P_{4y}-P_{1y})mg_z\right)-\left((P_{4z}-P_{1z})mg_y\right)\right)i\\ &+\left(\left((P_{4z}-P_{1z})mg_x\right)-(P_{4x}-P_{1x})mg_z\right)j\\ &+\left(\left((P_{4x}-P_{1x})mg_y\right)-\left(P_{4y}-P_{1y}\right)mg_x\right)k\end{aligned}$$

Because g_x and g_y are 0, this expression can be simplified.

$$\left(\overrightarrow{P_4}-\overrightarrow{P_1}\right)\times m\vec{g}=\left((P_{4y}-P_{1y})mg_z\right)i+\left((P_{1x}-P_{4x})mg_z\right)j$$

The other cross product can also be expanded.

$$(\overrightarrow{P_5} - \overrightarrow{P_1}) \times \overrightarrow{F_5} \\ = \Big(\big((P_{5y} - P_{1y})F_{5z}\big) - \big((P_{5z} - P_{1z})F_{5y}\big)\Big) i \\ + \Big(\big((P_{5z} - P_{1z})F_{5x}\big) - \big((P_{5x} - P_{1x})F_{5z}\big)\Big) j \\ + \Big(\big((P_{5x} - P_{1x})F_{5y}\big) - \big((P_{5y} - P_{1y})F_{5x}\big)\Big) k$$

We can also expand out the pivot axis unit vector calculation.

$$\frac{\overrightarrow{P_3} - \overrightarrow{P_1}}{|\overrightarrow{P_3} - \overrightarrow{P_1}|} = \frac{(P_{3x} - P_{1x})i + (P_{3y} - P_{1y})j + (P_{3z} - P_{1z})k}{\sqrt{(P_{3x} - P_{1x})^2 + (P_{3y} - P_{1y})^2 + (P_{3z} - P_{1z})^2}}$$

Now we can recombine the parts to get the original equation.

$$\frac{(P_{4y}P_{3x} - P_{1y}P_{3x} - P_{4y}P_{1x} + P_{1y}P_{1x} + P_{1x}P_{3y} - P_{4x}P_{3y} - P_{1x}P_{1y} + P_{4x}P_{1y})mg_z}{\sqrt{(P_{3x} - P_{1x})^2 + (P_{3y} - P_{1y})^2 + (P_{3z} - P_{1z})^2}} + \\ \frac{(P_{5y}F_{5z}P_{3x} - P_{1y}F_{5z}P_{3x} - P_{5y}F_{5z}P_{1x} + P_{1y}F_{5z}P_{1x} - P_{5z}F_{5y}P_{3x} + P_{1z}F_{5y}P_{3x} + P_{5z}F_{5y}P_{1x} - P_{1z}F_{5y}P_{1x})}{\sqrt{(P_{3x} - P_{1x})^2 + (P_{3y} - P_{1y})^2 + (P_{3z} - P_{1z})^2}} + \\ \frac{(P_{5z}F_{5x}P_{3y} - P_{1z}F_{5x}P_{1y} - P_{5z}F_{5x}P_{1y} + P_{1z}F_{5x}P_{1y} - P_{5x}F_{5z}P_{3y} + P_{1x}F_{5z}P_{3y} + P_{5x}F_{5z}P_{1y} - P_{1x}F_{5z}P_{1y})}{\sqrt{(P_{3x} - P_{1x})^2 + (P_{3y} - P_{1y})^2 + (P_{3z} - P_{1z})^2}} + \\ \frac{(P_{5x}F_{5y}P_{3z} - P_{1x}F_{5y}P_{3z} - P_{5x}F_{5y}P_{1z} + P_{1x}F_{5y}P_{1z} - P_{5y}F_{5x}P_{3z} + P_{1y}F_{5x}P_{3z} + P_{5y}F_{5x}P_{1z} - P_{1y}F_{5x}P_{1z})}{\sqrt{(P_{3x} - P_{1x})^2 + (P_{3y} - P_{1y})^2 + (P_{3z} - P_{1z})^2}} = 0$$

This equation can be reduced further.

$$(P_{4y}P_{3x} - P_{1y}P_{3x} - P_{4y}P_{1x} + P_{1y}P_{1x} + P_{1x}P_{3y} - P_{4x}P_{3y} - P_{1x}P_{1y} + P_{4x}P_{1y})mg_z + \\ (P_{5z}P_{3y} - P_{1z}P_{1y} - P_{5z}P_{1y} + P_{1z}P_{1y} - P_{5y}P_{3z} + P_{1y}P_{3z} + P_{5y}P_{1z} - P_{1y}P_{1z})F_{5x} + \\ (P_{5x}P_{3z} - P_{1x}P_{3z} - P_{5x}P_{1z} + P_{1x}P_{1z} - P_{5z}P_{3x} + P_{1z}P_{3x} + P_{5z}P_{1x} - P_{1z}P_{1x})F_{5y} + \\ (P_{5y}P_{3x} - P_{1y}P_{3x} - P_{5y}P_{1x} + P_{1y}P_{1x} - P_{5x}P_{3y} + P_{1x}P_{3y} + P_{5x}P_{1y} - P_{1x}P_{1y})F_{5z} = 0$$

To solve for F_5 we need to assume in which direction it will act. That direction assumption will provide 2 more equations tying F_{5x}, F_{5y}, and F_{5z} together, and create a system of 3 equations with 3 unknowns, which can then be solved. If we want to find the minimum force F_{5min} that will cause tipping, the direction will have to be orthogonal to the line between P_1 and P_3 (the tipping axis) and the shortest line from P_5 to that axis. Simply finding F_{5min} is not particularly difficult. The solution stems from the problem of finding the minimum distance from the origin to a plane defined by its x, y, and z intercepts. In this case, those intercepts represent the solutions for F_{5x}, F_{5y}, and F_{5z} in the absence of any other force. All we do is solve the previous equation three times, each time setting two of those three forces equal to zero and solving for the remaining one. These solutions give the x, y, and z intercepts we seek.

$$F_{5x} = \frac{-\left(P_{4y}P_{3x} - P_{1y}P_{3x} - P_{4y}P_{1x} + P_{1y}P_{1x} + P_{1x}P_{3y} - P_{4x}P_{3y} - P_{1x}P_{1y} + P_{4x}P_{1y}\right)mg_z}{P_{5z}P_{3y} - P_{1z}P_{1y} - P_{5z}P_{1y} + P_{1z}P_{1y} - P_{5y}P_{3z} + P_{1y}P_{3z} + P_{5y}P_{1z} - P_{1y}P_{1z}}$$

$$F_{5y} = \frac{-\left(P_{4y}P_{3x} - P_{1y}P_{3x} - P_{4y}P_{1x} + P_{1y}P_{1x} + P_{1x}P_{3y} - P_{4x}P_{3y} - P_{1x}P_{1y} + P_{4x}P_{1y}\right)mg_z}{P_{5x}P_{3z} - P_{1x}P_{3z} - P_{5x}P_{1z} + P_{1x}P_{1z} - P_{5z}P_{3x} + P_{1z}P_{3x} + P_{5z}P_{1x} - P_{1z}P_{1x}}$$

$$F_{5z} = \frac{-\left(P_{4y}P_{3x} - P_{1y}P_{3x} - P_{4y}P_{1x} + P_{1y}P_{1x} + P_{1x}P_{3y} - P_{4x}P_{3y} - P_{1x}P_{1y} + P_{4x}P_{1y}\right)mg_z}{P_{5y}P_{3x} - P_{1y}P_{3x} - P_{5y}P_{1x} + P_{1y}P_{1x} - P_{5x}P_{3y} + P_{1x}P_{3y} + P_{5x}P_{1y} - P_{1x}P_{1y}}$$

Now that we have the intercepts, we can solve for the minimum tipping force.

$$F_{5min} = \hat{n} \cdot (-F_{5x}, 0{,}0)$$

$$\hat{n} = \frac{\left(F_{5y} - F_{5x}\right) \times \left(F_{5z} - F_{5x}\right)}{\left|\left(F_{5y} - F_{5x}\right) \times \left(F_{5z} - F_{5x}\right)\right|} = \frac{\left(-F_{5x}, F_{5y}, 0\right) \times \left(-F_{5x}, 0, F_{5z}\right)}{\left|\left(-F_{5x}, F_{5y}, 0\right) \times \left(-F_{5x}, 0, F_{5z}\right)\right|}$$

$$\hat{n} = \frac{\left(F_{5y}F_{5z}\right)i + \left(F_{5x}F_{5z}\right)j + \left(F_{5x}F_{5y}\right)k}{\left|\left(F_{5y}F_{5z}\right)i + \left(F_{5x}F_{5z}\right)j + \left(F_{5x}F_{5y}\right)k\right|}$$

$$F_{5min} = \frac{-F_{5x}F_{5y}F_{5z}}{\sqrt{\left(F_{5y}F_{5z}\right)^2 + \left(F_{5x}F_{5z}\right)^2 + \left(F_{5x}F_{5y}\right)^2}}$$

<u>The Tipping Factor</u>

When designing loaded structures the concept of a safety factor is used. It describes the multiple of rated load a structure can handle before physically failing in some way. I propose a similar concept for perched loads that can tip over. There is more than one way to construct such a metric, the simplest being the ratio of structure weight to minimum tipping force. Such a metric, however, doesn't account for things like the increased risk associated with heavy components mounted high up as opposed to mounted near the floor. Instead, a metric comparing stored potential energy in the system to minimum tipping force seems more appropriate. To avoid over-complication in calculating the stored energy, I propose the following.

$$Tipping\ Factor = \frac{Weight \cdot Component\ Height\ from\ Floor}{Minimum\ Tipping\ Force}$$

In the case of the previous calculations, the minimum tipping force would be F_{5min}. In any scenario, a low Tipping Factor (TF) is desired. That being said, rather than give a target value for this expression from the outset, it is probably more practical to evaluate several existing designs to get an idea of what TF values are typical. If a new design has a TF value that is significantly higher than the norm, it is probably a good idea to evaluate it closely for design improvement before fabrication. This alleviates the risky problem of "testing the limits" of an allowable TF value.