ON SECTOR MAGNETS or transverse electromagnetic fields in cylindrical coordinates

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TABLE OF CONTENTS

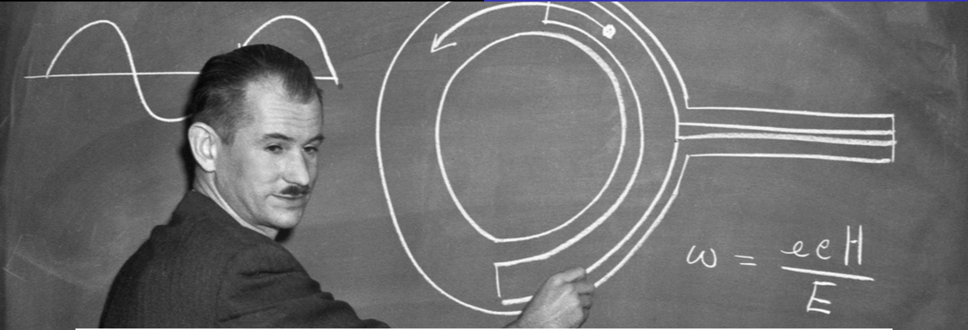
Abstract

The Laplace's equations for the scalar and vector potentials describing electric or magnetic fields in cylindrical coordinates with translational invariance along azimuthal coordinate are considered. The series of special functions which, when expanded in power series in radial and vertical coordinates, in lowest order replicate the harmonic homogeneous polynomials of two variables are found. These functions are based on radial harmonics found by Edwin M. McMillan in his more-than-40-years "forgotten" article, which will be discussed. In addition to McMillan's harmonics, second family of adjoint radial harmonics is introduced, in order to provide symmetric description between electric and magnetic fields and to describe fields and potentials in terms of same special functions. Formulas to relate any transverse fields specified by the coefficients in the power series expansion in radial or vertical planes in cylindrical coordinates with the set of new functions are provided.
This result is no doubt is important for potential theory while also critical for theoretical studies, design and proper modeling of sector dipoles, combined function dipoles and any general sector element for accelerator physics. All results are presented in connection with these problems.

1. Introduction

What do we know about sector magnets?

[1] K. L. Brown, Adv. Part. Phys. **1**, 71 (1968).
[2] E. Forest, *Beam dynamics*, Vol. 8 (CRC Press, 1998).
[3] H. Wiedemann, *Particle accelerator physics* (Springer, 2015).



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LETTER TO THE EDITOR

MULTIPOLES IN CYLINDRICAL COORDINATES*

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Solutions of Laplace's equation for azimuthally symmetrical potentials in cylindrical coordinates are found which can be correlated with two-dimensional multipoles in planes $\theta = \text{const}$. Formulas are presented by which the coefficients for linear combinations of these solutions can be calculated to describe fields whose values are known along given axial or radial lines.

2.1 General equations of motion. Lab frame

Lagrangian of relativistic particle of mass m with electric charge e

In 3-D right-handed Cartesian coordinates, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$,

$$\mathcal{L}[\mathbf{R}, \dot{\mathbf{R}}; t] = -\frac{m\,c^2}{\gamma(\mathbf{V})} - e\,\Phi(\mathbf{R}) + e\,[\mathbf{V}\cdot\mathbf{A}(\mathbf{R})]$$

$\mathbf{R} = (Q_1, Q_2, Q_3)$ position vector in the configuration space,
$\mathbf{V} = (\dot{Q}_1, \dot{Q}_2, \dot{Q}_3)$ vector of matching generalized velocities, $\dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}t}$,
$\Phi(\mathbf{R})$ and $\mathbf{A}(\mathbf{R})$ are scalar electric and magnetic vector potentials,
$\gamma\,(\mathbf{V})$ and $\beta(\mathbf{V})$ are Lorentz factor and ratio of V to speed of light

$$\gamma(\mathbf{V}) = [1 - \beta(\mathbf{V})^2]^{-1/2} \qquad\qquad \beta(\mathbf{V}) = |\mathbf{V}|/c$$

Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{R}}}-\frac{\partial\mathcal{L}}{\partial\mathbf{R}}=0 \qquad \text{with} \qquad \frac{\partial}{\partial\mathbf{a}}=\left(\frac{\partial}{\partial a_1},\frac{\partial}{\partial a_2},\frac{\partial}{\partial a_3}\right)$$

gives the relativistic form of the Lorentz force $\mathbf{F}=e\,[\mathbf{E}+(\mathbf{V}\times\mathbf{B})]$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\, m\,\dot{Q}_i)=e\,(E_i+\epsilon_{ijk}\,\dot{Q}_j B_k)}$$

where the scalar electric and vector magnetic potentials are expressed through electric and magnetic fields respectively

$$\mathbf{E}=(E_1,E_2\,,E_3)\equiv\,-\nabla\,\Phi,$$
$$\mathbf{B}=(B_1,B_2,B_3)\equiv\nabla\times\mathbf{A}.$$

Hamiltonian is defined as the Legendre transformation of $\mathcal{L}$

$$\mathcal{H}[\mathbf{P},\mathbf{Q};t] = \dot{\mathbf{Q}}\,\mathbf{P} - \mathcal{L} = c\sqrt{m^2c^2 + (\mathbf{P} - e\,\mathbf{A})^2} + e\,\Phi$$

where $\mathbf{P}$ and $\mathbf{\Pi}$ are the canonical and kinetic particle's momentum

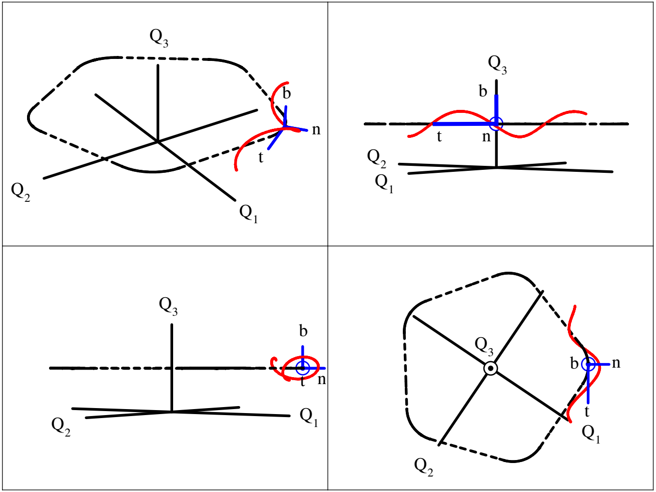
$$\mathbf{P} \equiv \frac{\partial\,\mathcal{L}}{\partial\dot{\mathbf{R}}} = \mathbf{\Pi} + e\,\mathbf{A}, \qquad \mathbf{\Pi} = \gamma\,m\,\mathbf{V}.$$

Hamilton's equations give time evolution of the system

$$\frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}t} = \frac{\partial\,\mathcal{H}}{\partial\mathbf{P}}, \qquad \dot{\mathbf{Q}} = c\,\frac{\mathbf{P} - e\,\mathbf{A}}{\sqrt{m^2c^2 + (\mathbf{P} - e\,\mathbf{A})^2}},$$

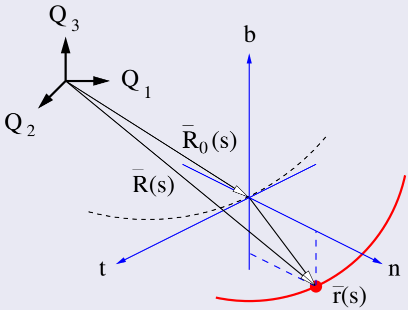
$$\frac{\mathrm{d}\,\mathbf{P}}{\mathrm{d}t} = -\frac{\partial\,\mathcal{H}}{\partial\mathbf{Q}}, \qquad \dot{\mathbf{P}} = e\,(\nabla\mathbf{A})\cdot\dot{\mathbf{Q}} - e\,\nabla\Phi.$$

2.2 Global coordinates associated with Frenet-Serret frame



Local Frenet-Serret frame $\{\hat{\mathbf{n}}, \hat{\mathbf{b}}, \hat{\mathbf{t}}\}$ (sometimes called TNB frame)

- Unit tangent vector
$$\hat{\mathbf{t}} = \frac{\mathrm{d}\,\mathbf{R}_0(s)}{\mathrm{d}s}$$
- Outward-pointing normal
$$\hat{\mathbf{n}} = -\frac{1}{\kappa(s)}\frac{\mathrm{d}\,\hat{\mathbf{t}}}{\mathrm{d}s}$$
- Unit binormal vector
$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$$



$s(t) = \int_0^t |\dot{\mathbf{R}}_0(t)|\,\mathrm{d}\,t$ is natural parametrization of eq. orbit, and,
$\kappa(s) = \left|\mathrm{d}\,\hat{\mathbf{t}}/\mathrm{d}s\right|$ is local curvature.

Frenet-Serret formulas

$$\mathrm{d}\begin{bmatrix}\hat{\mathbf{t}}\\ \hat{\mathbf{n}}\\ \hat{\mathbf{b}}\end{bmatrix} = \begin{bmatrix}0 & -\kappa & 0\\ \kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{t}}\\ \hat{\mathbf{n}}\\ \hat{\mathbf{b}}\end{bmatrix}\mathrm{d}s, \qquad \text{where } \tau(s) \text{ is torsion.}$$

F.-S. formulas allows to express position vector of a test particle

$$\begin{aligned}\mathbf{R}(s) &= \mathbf{R}_0(s) + \mathbf{r}(s) = \mathbf{R}_0(s) + q_1\hat{\mathbf{n}} + q_2\hat{\mathbf{b}},\\ \mathrm{d}\mathbf{R} &= \hat{\mathbf{n}}\,\mathrm{d}q_1 + \hat{\mathbf{b}}\,\mathrm{d}q_2 + (1+\kappa\, q_1)\hat{\mathbf{t}}\,\mathrm{d}q_3 + \tau(q_1\hat{\mathbf{b}} - q_2\hat{\mathbf{n}})\mathrm{d}q_3.\end{aligned}$$

For $\tau = 0$, the local Frenet-Serret frame can be associated with global orthogonal coordinate system with a line element in a form

$$\mathrm{d}\mathbf{l} = \sum_{i=1}^{3} h_i\hat{\mathbf{e}}_i\mathrm{d}q_i, \quad \text{where} \quad h_1 = h_2 = 1 \text{ and } h \equiv h_3 = 1 + \kappa\, q_1.$$

Lagrangian in curvilinear coordinates

$$\mathcal{L}[\mathbf{r},\dot{\mathbf{r}};t] = -m\,c^2\sqrt{1-\frac{v^2}{c^2}} - e\,\Phi + e\,\mathbf{v}\cdot\mathbf{A}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\,m\,\mathbf{v}) = e\,(\mathbf{E} + \epsilon_{ijk}\,\hat{\mathbf{e}}_i\,v_j\,B_k) + \gamma\,m\,\dot{q}_3^2\,\mathbf{K}$$

where $\mathbf{v} = (\dot{q}_1, \dot{q}_2, h\,\dot{q}_3)$ is velocity vector in new coordinates, and, the vector in the RHS defined as

$$\mathbf{K} = (\kappa\,h, 0, \kappa'\,q_1) \qquad \text{with } (\ldots)' \equiv \frac{\mathrm{d}}{\mathrm{d}q_3}$$

Hamiltonian in curvilinear coordinates

$$\mathcal{H}[\mathbf{p},\mathbf{q};t] = c\,\sqrt{m^2c^2 + \sum_{i=1}^{3}\left(\frac{p_i - e\,h_i A_i}{h_i}\right)^2} + e\,\Phi$$

$$\dot{q}_i \times h_i = \frac{c^2}{\mathcal{H} - e\,\Phi}\,\frac{p_i - e\,h_i A_i}{h_i}$$

$$\dot{p}_i \;/\; h_i = \frac{c^2}{\mathcal{H} - e\,\Phi}\left[e\,\epsilon_{ijk}\frac{p_j}{h_j}\,B_k + \frac{K_i}{h^2}\left(\frac{p_3 - e\,h\,A_3}{h}\right)^2\right] + e\,E_i$$

where components of the new canonical momenta are given by

$$\frac{p_i}{h_i} \equiv \frac{1}{h_i}\frac{\partial\,\mathcal{L}}{\partial\dot{q}_i} = \gamma\,m\,v_i + e\,\mathbf{A}_i(\mathbf{r})$$

3. Transverse electromagnetic fields $\Phi = \Phi(q_1, q_2)$, $\mathbf{A} = A_3(q_1, q_2)\hat{\mathbf{e}}_3$.

Laplace's equations in curvilinear coordinates

$$\triangle\Phi = \frac{1}{h}\left[\frac{\partial}{\partial q_1}\left(h\frac{\partial\Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(h\frac{\partial\Phi}{\partial q_2}\right)\right] = 0$$

$$(\text{✡}\mathbf{A})_3 = \frac{\partial}{\partial q_1}\left[\frac{1}{h}\frac{\partial(h A_3)}{\partial q_1}\right] + \frac{\partial}{\partial q_2}\left[\frac{1}{h}\frac{\partial(h A_3)}{\partial q_2}\right] = 0$$

$\mathbf{E} = -\nabla\Phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$ in curvilinear coordinates

$$E_1 = -\frac{\partial\Phi}{\partial q_1} \qquad B_1 = \frac{1}{h}\frac{\partial(h A_3)}{\partial q_2}$$

$$E_2 = -\frac{\partial\Phi}{\partial q_2} \qquad B_2 = -\frac{1}{h}\frac{\partial(h A_3)}{\partial q_1}$$

Differential operators in curvilinear orthogonal coordinates

Gradient	$\nabla\phi$	$\sum_{k=1}^{3}\frac{1}{h_k}\frac{\partial\,\phi}{\partial q^k}\hat{\mathbf{e}}_k$
Divergence	$\nabla\cdot\mathbf{F}$	$\sum_{k=1}^{3}\frac{1}{H}\frac{\partial}{\partial q^k}\left(\frac{H}{h_k}F_k\right)$
Curl	$\nabla\times\mathbf{F}$	$\sum_{k=1}^{3}\frac{h_k\,\hat{\mathbf{e}}_k}{H}\epsilon_{ijk}\frac{\partial}{\partial q^i}\left(h_jF_j\right)$
Laplacian	$\triangle\phi$	$\sum_{k=1}^{3}\frac{1}{H}\frac{\partial}{\partial q^k}\left(\frac{H}{h_k^2}\frac{\partial\,\phi}{\partial q^k}\right)$
	✡$\mathbf{F}$	$\sum_{k=1}^{3}\left\{\frac{1}{h_k}\frac{\partial}{\partial q^k}\left[\frac{1}{H}\frac{\partial}{\partial q^i}\left(\frac{H}{h_i}F_i\right)\right]-\right.$ $\left.-\frac{h_k}{H}\epsilon_{ijk}\frac{\partial}{\partial q^i}\left[\frac{h_j^2\hat{\mathbf{e}}_j}{H}\epsilon_{lmj}\frac{\partial}{\partial q^l}(h_mF_m)\right]\right\}\hat{\mathbf{e}}_k$

Pure electric field. $\kappa = \text{const}$

t-representation. $\mathrm{H}, p_3 = \text{inv}$

Measuring the time in units of $c\,t$ and normalizing the transverse momentums over the longitudinal component, $\tilde{p}_{1,2} = p_{1,2}/p_3$:

$$\boxed{\frac{\mathcal{H}}{p_3} \equiv \mathrm{H}[\tilde{p}_{1,2}, q_{1,2}; c\,t] = \frac{1}{h}\sqrt{1 + h^2(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{m}^2 c^2)} + \frac{e}{p_3\,c}\,\Phi}$$

In paraxial approximation $\tilde{p}_{1,2} \ll 1$, and for $\tilde{p}_{1,2} \gg \tilde{m}\,c$, $\tilde{m} = m/p_3$

$$\mathrm{H}[\tilde{p}_{1,2}, q_{1,2}; c\,t] \approx h\left(\frac{\tilde{p}_1^2}{2} + \frac{\tilde{p}_2^2}{2}\right) + \frac{1}{h} + \frac{e}{p_3\,c}\,\Phi$$

Pure magnetic field. $\kappa = \text{const}$

1. Extended phase space ($\mathcal{H} = \mathrm{inv}$ and $t = \tau + C_0$)

$$0 \equiv \mathcal{O}[\mathbf{p}, -\mathcal{H}; \mathbf{q}, t; \tau] = c\,\sqrt{m^2 c^2 + p_1^2 + p_2^2 + \left(\frac{p_3 - e\,h\,A_3}{h}\right)^2} - \mathcal{H}$$

2. The use of $-p_3$ as a new Hamiltonian

$$\mathcal{K}[p_{1,2}, -\mathcal{H}; q_{1,2}, t; q_3] = -h\,\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m^2 c^2 - p_1^2 - p_2^2} - e\,h\,A_3$$

3. Generating function $G_2(t, -\Pi) = -t\sqrt{\Pi^2 c^2 + (m\,c^2)^2}$

$$\mathcal{K}[p_1, p_2, -\Pi; q_1, q_2, l; q_3] = -h\,\sqrt{\Pi^2 - p_1^2 - p_2^2} - e\,h\,A_3,$$

where $l = -\partial\, G_2/\partial\Pi = \beta c\, t$ is a particle's traversed path.

z-representation. $\mathrm{K}, \Pi = \text{inv}$

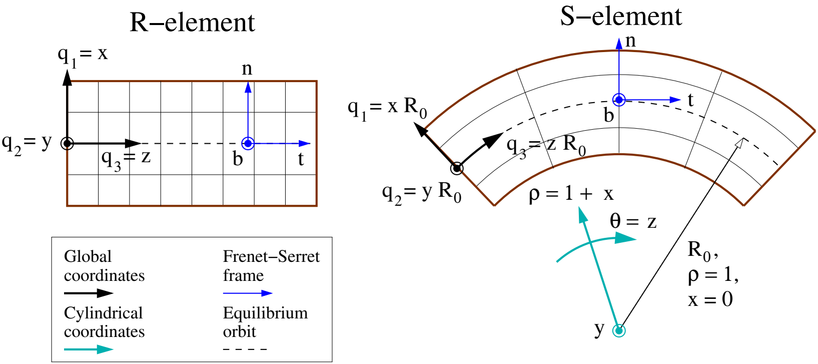
Renormalizng Hamiltonian and momenta by full kinetic monentum, $p_{1,2} \to \tilde{p}_{1,2} = p_{1,2}/\Pi$:

$$\boxed{\frac{\mathcal{K}}{\Pi} \equiv \mathrm{K}[\tilde{p}_{1,2}, q_{1,2}; q_3] = -h\sqrt{1 - \tilde{p}_1^2 - \tilde{p}_2^2} - \frac{e}{\Pi}\, h\, A_3}$$

In paraxial approximation $\tilde{p}_{1,2} \ll 1$

$$\mathrm{K}[\tilde{p}_{1,2}, q_{1,2}; q_3] \approx h\left(\frac{\tilde{p}_1^2}{2} + \frac{\tilde{p}_2^2}{2}\right) - h - \frac{e}{\Pi}\, h\, A_3$$

R- and S- elements



3.2 Multipole expansion in Cartesian coordinates

$$\triangle_\perp \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad ✡_\perp \mathbf{A} = \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} \right) \hat{\mathbf{e}}_z = 0$$

Complex scalar potential and Wirtinger derivatives

$$\Omega(\mathcal{Z}) = A_z(x,y) + i\,\Phi(x,y) \qquad \frac{\partial}{\partial(\mathcal{Z},\overline{\mathcal{Z}})} = \frac{1}{2}\left(\frac{\partial}{\partial x} \mp i\,\frac{\partial}{\partial y}\right)$$

$$\frac{\partial\,\Omega}{\partial\,\mathcal{Z}} = F(\mathcal{Z}): \quad F_x = -\Im\,F(\mathcal{Z}), \quad F_y = -\Re\,F(\mathcal{Z})$$

$$\frac{\partial\,\Omega}{\partial\,\overline{\mathcal{Z}}} = 0: \quad F_x = -\frac{\partial\,\Phi}{\partial x} = \frac{\partial\,A_z}{\partial y}, \quad F_y = -\frac{\partial\,\Phi}{\partial y} = -\frac{\partial\,A_z}{\partial x}$$

$$\frac{\partial\,F}{\partial\,\overline{\mathcal{Z}}} = 0: \quad \nabla \cdot \mathbf{F} = 0, \quad \nabla \times \mathbf{F} = 0$$

Harmonic homogeneous polynomials

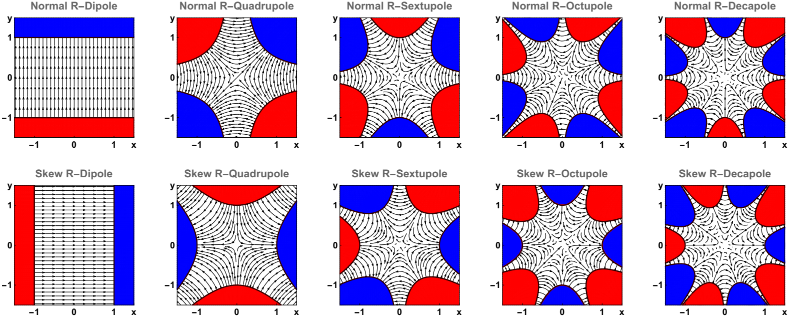
$$n\,\{\mathcal{A},\mathcal{B}\}_{n-1} = \frac{\partial}{\partial x}\{\mathcal{A},\mathcal{B}\}_n = \pm\frac{\partial}{\partial y}\{\mathcal{B},\mathcal{A}\}_n$$

n	$\Re\,\mathcal{Z}^n$ $\mathcal{A}_n = \sum_{k=0}^{n}\binom{n}{k} x^{n-k}y^k \cos\frac{k\,\pi}{2}$	$\Im\,\mathcal{Z}^n$ $\mathcal{B}_n = \sum_{k=0}^{n}\binom{n}{k} x^{n-k}y^k \sin\frac{k\,\pi}{2}$
0	1	0
1	x	y
2	$x^2 - y^2$	$2\,x\,y$
3	$x^3 - 3\,x\,y^2$	$3\,x^2y - y^3$
4	$x^4 - 6\,x^2y^2 + y^4$	$4\,x^3y - 4\,x\,y^3$
5	$x^5 - 10\,x^3y^2 + 5\,x\,y^4$	$5\,x^4y - 10\,x^2y^3 + y^5$

Normal and skew R-multipoles

One can define two independent sets of solutions

Normal, $\overline{\Omega}^{(n)} = -\,\overline{C}_n \dfrac{\mathcal{Z}^n}{n!}$	Skew, $\underline{\Omega}^{(n)} = -i\,\underline{C}_n \dfrac{\mathcal{Z}^n}{n!}$
$\overline{\Phi}^{(n)} = -\overline{C}_n \dfrac{\mathcal{B}_n}{n!}$ $\overline{A}_z^{(n)} = -\overline{C}_n \dfrac{\mathcal{A}_n}{n!}$	$\underline{\Phi}^{(n)} = -\underline{C}_n \dfrac{\mathcal{A}_n}{n!}$ $\underline{A}_z^{(n)} = \underline{C}_n \dfrac{\mathcal{B}_n}{n!}$
$\overline{F}_{\mathrm{x}}^{(n)} = \overline{C}_n \dfrac{\mathcal{B}_{n-1}}{(n-1)!}$ $\overline{F}_{\mathrm{y}}^{(n)} = \overline{C}_n \dfrac{\mathcal{A}_{n-1}}{(n-1)!}$	$\underline{F}_{\mathrm{x}}^{(n)} = \underline{C}_n \dfrac{\mathcal{A}_{n-1}}{(n-1)!}$ $\underline{F}_{\mathrm{y}}^{(n)} = -\underline{C}_n \dfrac{\mathcal{B}_{n-1}}{(n-1)!}$



n	$\overline{C}_n$	$\underline{C}_n$	$\overline{C}_n$	$\underline{C}_n$
1	F_y	F_x	F_y	F_x
2	$\partial_y F_x$	$-\partial_y F_y$	$\partial_x F_y$	$\partial_x F_x$
3	$-\partial_y^2 F_y$	$-\partial_y^2 F_x$	$\partial_x^2 F_y$	$\partial_x^2 F_x$
4	$-\partial_y^3 F_x$	$\partial_y^3 F_y$	$\partial_x^3 F_y$	$\partial_x^3 F_x$

3.3 Multipole expansion in cylindrical coordinates

$$\triangle_{\curvearrowright}\Phi \quad = 0 = \triangle_{\perp}\Phi + \frac{1}{\rho}\frac{\partial\,\Phi}{\partial\rho} = \quad \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial y^2}\right]\Phi$$

$$(\text{✡}_{\curvearrowright}\mathbf{A})_\theta = 0 = \triangle_{\curvearrowright}A_\theta - \frac{A_\theta}{\rho^2} = \frac{\partial^2 A_\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\,A_\theta}{\partial\rho} + \frac{\partial^2 A_\theta}{\partial y^2} - \frac{A_\theta}{\rho^2}$$

3.3 Multipole expansion in cylindrical coordinates

$$\triangle_{\curvearrowright}\Phi \quad = 0 = \triangle_{\perp}\Phi + \frac{1}{\rho}\frac{\partial\,\Phi}{\partial\rho} = \quad \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial y^2}\right]\Phi$$

$$(\text{✡}_{\curvearrowright}\mathbf{A})_\theta = 0 = \triangle_{\curvearrowright}A_\theta - \frac{A_\theta}{\rho^2} \quad = \frac{1}{\rho}\left[\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial y^2}\right](\rho\,A_\theta)$$

We will look for solutions in a form

$$\Phi \quad = \quad -\sum_{k=0}^{n} \quad \frac{\mathcal{F}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\overline{C}_n \sin\frac{k\,\pi}{2} + \underline{C}_n \cos\frac{k\,\pi}{2}\right),$$

$$A_\theta \quad = \quad -\sum_{k=0}^{n} \frac{1}{\rho}\frac{\mathcal{G}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\overline{C}_n \cos\frac{k\,\pi}{2} - \underline{C}_n \sin\frac{k\,\pi}{2}\right),$$

where $\mathcal{F}_n(\rho)$ and $\mathcal{G}_n(\rho)$ are the functions to be determined.

$$\left(\frac{\partial^2}{\partial\rho^2} \pm \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\{\mathcal{F},\mathcal{G}\}_n = n\,(n-1)\,\{\mathcal{F},\mathcal{G}\}_{n-2}$$

Lowering operators

$$\mathcal{F}_n = \frac{1}{(n+1)(n+2)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial}{\partial\rho}\right)\right]\mathcal{F}_{n+2}, \quad \mathcal{F}_{n-1} = \frac{1}{n}\frac{1}{\rho}\frac{\partial\,\mathcal{G}_n}{\partial\rho},$$

$$\mathcal{G}_n = \frac{1}{(n+1)(n+2)}\left[\rho\,\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\right]\mathcal{G}_{n+2}, \quad \mathcal{G}_{n-1} = \frac{1}{n}\,\rho\,\frac{\partial\,\mathcal{F}_n}{\partial\rho}.$$

Raising operators

$$\mathcal{F}_n = n\,(n-1)\int_1^\rho \frac{1}{\rho}\int_1^\rho \rho\,\mathcal{F}_{n-2}\,\mathrm{d}\,\rho\,\mathrm{d}\,\rho,$$

$$\mathcal{G}_n = n\,(n-1)\int_1^\rho \rho\int_1^\rho \frac{1}{\rho}\,\mathcal{G}_{n-2}\,\mathrm{d}\,\rho\,\mathrm{d}\,\rho.$$

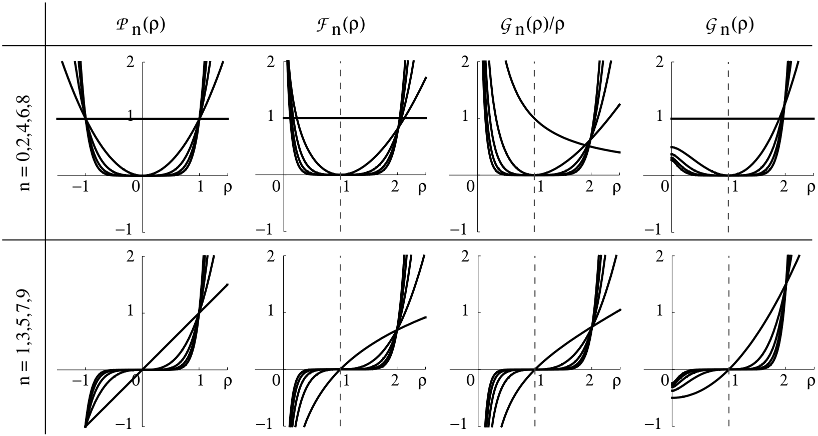
McMillan radial harmonics, $\mathcal{F}_n$

0	1
1	$\ln\rho$
2	$\frac{1}{2}(\rho^2-1)-\ln\rho$
3	$\frac{3}{2}\left[-(\rho^2-1)+(\rho^2+1)\ln\rho\right]$
4	$3\left[\frac{1}{8}(\rho^4-1)+\frac{1}{2}(\rho^2-1)-\left(\rho^2+\frac{1}{2}\right)\ln\rho\right]$
5	$\frac{15}{2}\left[-\frac{3}{8}(\rho^4-1)+\left(\frac{1}{4}\rho^4+\rho^2+\frac{1}{4}\right)\ln\rho\right]$

Adjoint radial harmonics, $\mathcal{G}_n$

0	1
1	$\frac{1}{2}\left(\rho^2-1\right)$
2	$1\left[-\frac{1}{2}\left(\rho^2-1\right)+\rho^2\ln\rho\right]$
3	$\frac{3}{2}\left[\frac{1}{4}\left(\rho^4-1\right)-\rho^2\ln\rho\right]$
4	$3\left[-\frac{5}{8}\left(\rho^4-1\right)+\frac{1}{2}\left(\rho^2-1\right)+\rho^2\left(\frac{\rho^2}{2}+1\right)\ln\rho\right]$
5	$\frac{15}{4}\left[\frac{1}{12}\left(\rho^6-1\right)+\frac{3}{4}\rho^2\left(\rho^2-1\right)-\rho^2\left(\rho^2+1\right)\ln\rho\right]$

$\mathcal{F}_n$, $\mathcal{G}_n$ plot



Sector harmonics

Finally, we will define 4 sets of **sector harmonics**:

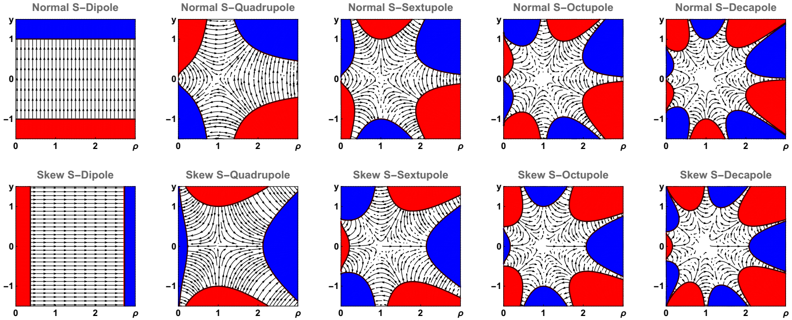
$$
\begin{aligned}
\mathcal{A}_n^{(\mathrm{e})}(\rho, y) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}_{n-k}(\rho)\, y^k \cos\frac{k\,\pi}{2}, \\
\mathcal{A}_n^{(\mathrm{m})}(\rho, y) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{G}_{n-k}(\rho)}{\rho}\, y^k \cos\frac{k\,\pi}{2}, \\
\mathcal{B}_n^{(\mathrm{e})}(\rho, y) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}_{n-k}(\rho)\, y^k \sin\frac{k\,\pi}{2}, \\
\mathcal{B}_n^{(\mathrm{m})}(\rho, y) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{G}_{n-k}(\rho)}{\rho}\, y^k \sin\frac{k\,\pi}{2}.
\end{aligned}
$$

$$n\,\{\mathcal{A},\mathcal{B}\}^{(\mathrm{e})}_{n-1} = \pm\frac{\partial\,\{\mathcal{B},\mathcal{A}\}^{(\mathrm{e})}_{n}}{\partial y} = \frac{1}{\rho}\frac{\partial\left(\rho\,\{\mathcal{A},\mathcal{B}\}^{(\mathrm{m})}_{n}\right)}{\partial\rho}$$

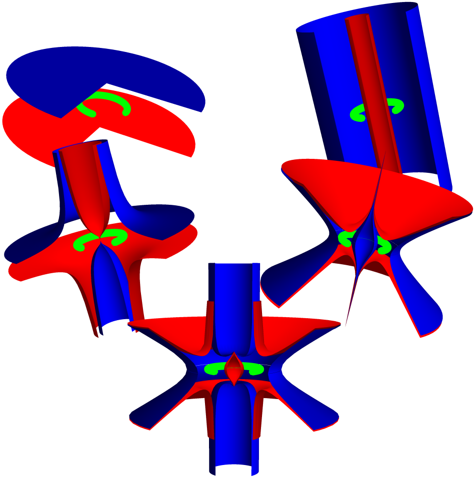
$$n\,\{\mathcal{A},\mathcal{B}\}^{(\mathrm{m})}_{n-1} = \pm\frac{1}{\cancel{\rho}}\frac{\partial\left(\cancel{\rho}\,\{\mathcal{B},\mathcal{A}\}^{(\mathrm{m})}_{n}\right)}{\partial y} = \frac{\partial\,\{\mathcal{A},\mathcal{B}\}^{(\mathrm{e})}_{n}}{\partial\rho}$$

$\overline{\Phi}^{(n)} = -\overline{C}_n\dfrac{\mathcal{B}^{(\mathrm{e})}_n}{n!}$	$\underline{\Phi}^{(n)} = -\underline{C}_n\dfrac{\mathcal{A}^{(\mathrm{e})}_n}{n!}$
$\overline{A}^{(n)}_{\theta} = -\overline{C}_n\dfrac{\mathcal{A}^{(\mathrm{m})}_n}{n!}$	$\underline{A}^{(n)}_{\theta} = \underline{C}_n\dfrac{\mathcal{B}^{(\mathrm{m})}_n}{n!}$
$\overline{F}^{(n)}_{\rho} = \overline{C}_n\dfrac{\mathcal{B}^{(\mathrm{m})}_{n-1}}{(n-1)!}$	$\underline{F}^{(n)}_{\rho} = \underline{C}_n\dfrac{\mathcal{A}^{(\mathrm{m})}_{n-1}}{(n-1)!}$
$\overline{F}^{(n)}_{\mathrm{y}} = \overline{C}_n\dfrac{\mathcal{A}^{(\mathrm{e})}_{n-1}}{(n-1)!}$	$\underline{F}^{(n)}_{\mathrm{y}} = -\underline{C}_n\dfrac{\mathcal{B}^{(\mathrm{e})}_{n-1}}{(n-1)!}$

Pure normal and skew S-multipoles



Pure normal and skew S-multipoles. 3D view



	n	$x=0$	$y=0$
$\overline{C}_n$	1	F_y	F_y
	2	$\partial_y F_x$	$\partial_x F_y$
	3	$-\partial_y^2 F_y$	$\partial_x^2 F_y + \partial_x F_y$
	4	$-\partial_y^3 F_x$	$\partial_x^3 F_y + \partial_x^2 F_y - \partial_x F_y$
	5	$\partial_y^4 F_y$	$\partial_x^4 F_y + 2\,\partial_x^3 F_y - \partial_x^2 F_y + \partial_x F_y$
$\underline{C}_n$	1	F_x	F_x
	2	$-\partial_y F_y$	$\partial_x F_x + F_x$
	3	$-\partial_y^2 F_x$	$\partial_x^2 F_x + \partial_x F_x - F_x$
	4	$\partial_y^3 F_y$	$\partial_x^3 F_x + 2\,\partial_x^2 F_x - \partial_x F_x + F_x$
	5	$\partial_y^4 F_x$	$\partial_x^4 F_x + 2\,\partial_x^3 F_x - 3\,\partial_x^2 F_x + 3\,\partial_x F_x - 3\,F_x$

3.4 Recurrence equations for sector coordinates

Power series ansatz

$$\Phi = -\sum_{m,n\geq 0}^{\infty} V_{m,n} \frac{x^m}{m!} \frac{y^n}{n!} \qquad A_\theta = -\sum_{m,n\geq 0}^{\infty} \frac{1}{1+x} V_{m,n} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$\boxed{V_{m+2,n} + V_{m,n+2} = -(m \pm 1)\, V_{m+1.n} - m\, V_{m-1,n+2}}$$

In order to solve these recurrences, one can look for a solution

$$V_{i,j} = V_{i,j}^{*} + V_{i,j}^{(i+j-1)} + V_{i,j}^{(i+j-2)} + \ldots, \qquad V_{m+2,n}^{*} + V_{m,n+2}^{*} \equiv 0,$$

where starred variables are the "design" terms given by pure multipole fields, and $V_{i,j}^{(n)}$ for $i+j>n$ are terms induced by lower orders and are subject to be determined.

4. Summary

- The scalar and vector Laplace's equations for static transverse electromagnetic fields in curvilinear orthogonal coordinates with zero and constant curvatures are solved.
- Described a family of solutions to the Laplace's equations in cylindrical coordinates which we call sector harmonics. The radial part is given by the set of newly introduced McMillan and adjoint radial harmonics.
- Sector harmonics, when expanded around equilibrium orbit, in its lowest order replicate the solution in Cartesian geometry .
- This set of solutions does not require any truncation and exactly satisfies Laplace equation, and, provides a well defined full basis of functions which can be related to any field by its expansion in radial or vertical planes.
- Including the model Hamiltonians for t- and z-representations, where no assumptions but the field symmetry has been used, one can construct numerical scheme integrating equations of motion.

4. Summary

- **Thus, I would like to suggest the set of sector harmonics as a new basis for description and design of any sector magnets with translational symmetry along azimuthal coordinate.**

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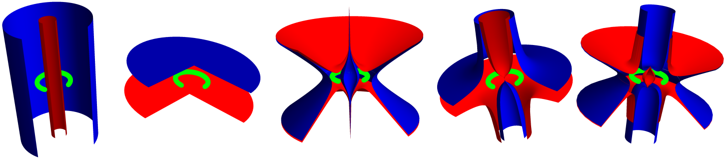
LAST SLIDE

Thank you for your attention!



Questions?