Synergia simulation of space charge modes

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Collaborators:

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Outline

- **Modal analysis. Dynamic Mode Decomposition.**
- **Synergia simulations.**
- **Results**
  - Transverse space charge modes.
  - Parametric Landau damping.

Motivation

- **Tools for modal analysis of beams simulated with particle tracking codes are necessary.**
    - The popular method, Singular Value Decomposition, does not capture modes dynamics.
    - Dynamic Mode Decomposition works (provides damping/growing rates).

- **The space charge modes in bunched beams are not fully understood.**
    - Comparison with existing analytical results.
    - Modes properties in parametric regions inaccessible by analytical approach.
    - Numerical investigation of the Landau damping mechanism.
        - In the proximity of transverse coupling resonance we found a novel damping mechanism.

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Data from simulation

data sequence: $X(q,t_0), X(q,t_1), \ldots, X(q,t_{N-1})$

Examples:

$$X(q,t) \equiv \rho(z,\delta,t), \;\; q \equiv (z,\delta), \;\; \delta = \frac{\Delta p}{p}$$ **longitudinal plane distribution**

$$X(q,t) \equiv X(z,\delta,t) = \frac{\int dx\, dy\, x\, \rho(x,y,z,\delta,t)}{\int dx\, dy\, \rho(x,y,z,\delta,t)}$$ **horizontal dipole density**

Data can be stored as an MxN **matrix:**

$$X(q,t) = \begin{array}{cc} \begin{bmatrix} X_0^0 & X_0^1 & . & . & . & X_0^{N-1} \\ X_1^0 & X_1^1 & . & . & . & X_1^{N-1} \\ . & . & . & . & . & . \\ X_{M-1}^0 & X_{M-1}^1 & . & . & . & X_{M-1}^{N-1} \end{bmatrix} & \begin{matrix} q_0 \\ q_1 \\ \\ q_{M-1} \end{matrix} \\ \begin{matrix} t_0 & \quad t_1 & \quad\quad\quad & t_{N-1} \end{matrix} & \text{time} \end{array}$$

q represent points in the phase space

Singular Value Decomposition

(*aka Principal component analysis, Proper orthogonal decomposition, …)*

$$X(q,t)=\begin{bmatrix} X_0^0 & X_0^1 & . & . & . & X_0^{N-1} \\ X_1^0 & X_1^1 & . & . & . & X_1^{N-1} \\ . & . & . & . & . & . \\ X_{M-1}^0 & X_{M-1}^1 & . & . & . & X_{M-1}^{N-1} \end{bmatrix}$$

$$X=U\Sigma V^{\dagger}=\sum_m \sigma_m u_m(q)\, v_m(t)$$

$$\Sigma=\begin{bmatrix} \sigma_1 & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & \sigma_r & 0 & . & 0 \\ 0 & . & 0 & 0 & . & 0 \end{bmatrix}$$

$$C(q_1,q_2)=\frac{1}{N}\sum_t X(q_1,t)\, X^{\dagger}(t,q_2)$$ covariance matrix

$$C u_m=\sigma_m^2 u_m$$

- **The SVD singular vectors $u_m(q)$ diagonalize the covariance matrix.**
- **For systems with no higher than second moment correlations the singular vectors represent independent modes.**
- **The singular value σ_m is a measure of the mode variance.**
- **It makes sense for stationary systems.**

Dynamic Mode Decomposition

P.J. Schmid, Journal of Fluid Mechanics, 656, 2010.
C.W. Rowley et al., Journal of Fluid Mechanics, 641, 2009.

- **data sequence:** $X(q,t_0), X(q,t_1), \ldots, X(q,t_N)$ $\quad t_k = k\,\Delta t$

- **find a linear operator** A **such as:**

$$X(q,t_{k+1}) = A\,X(q,t_k) \qquad \textbf{for all} \qquad k=\overline{1,N-1}$$

- **the eigenvalues and the eigenvectors of** A **describe the dynamics**

$$A\varphi_j(q) = \mu_j \varphi_j(q)$$

$$X(t_0) = \sum_j \alpha_j \varphi_j = \sum_j \psi_j$$

$$X(t_k) = A^k X(t_0) = \sum_j \mu_j^k \psi_j = \sum_j \mathrm{e}^{-\lambda_j \Delta t\, k} e^{-i\omega_j \Delta t\, k} \psi_j = \sum_j \mathrm{e}^{-\lambda_j t_k} e^{-i\omega_j t_k} \psi_j$$

Dynamic Mode Decomposition

$$X_0 = \begin{bmatrix} X_0^0 & X_0^1 & . & . & . & X_0^{N-1} \\ X_1^0 & X_1^1 & . & . & . & X_1^{N-1} \\ . & . & . & . & . & . \\ X_{M-1}^0 & X_{M-1}^1 & . & . & . & X_{M-1}^{N-1} \end{bmatrix} \qquad X_1 = \begin{bmatrix} X_0^1 & X_0^2 & . & . & . & X_0^N \\ X_1^1 & X_1^2 & . & . & . & X_1^N \\ . & . & . & . & . & . \\ X_{M-1}^1 & X_{M-1}^2 & . & . & . & X_{M-1}^N \end{bmatrix}$$

DMD equation: $A X_0 = X_1$

A size is MxM

X_0 and X_1 sizes are MxN

- **The system might be overdetermined or underdetermined.**
- **A solution might not exist or there might be more than one solutions.**

Dynamic Mode Decomposition

- **DMD equation:** $A X_0 = X_1$ **not a well defined problem**
- **DMD solution for finding A: least square fit**
  the problem is projected onto the subspace spanned by the Single Value Decomposition modes.

$$X_0 = U \Sigma V^\dagger$$

$$\Sigma = \begin{bmatrix} \sigma_1 & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & \sigma_r & 0 & . & 0 \\ 0 & . & 0 & 0 & . & 0 \end{bmatrix}$$

$$\Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & . & 0 & 0 \\ . & . & . & . \\ 0 & . & \sigma_r^{-1} & 0 \\ 0 & . & 0 & 0 \\ . & . & . & . \\ 0 & . & 0 & 0 \end{bmatrix}$$

$$A = U F U^\dagger \quad \Rightarrow \quad X_1 - U F U^\dagger X_0 = 0 \quad \Rightarrow \quad F = U^+ X_1 V \Sigma^+$$

F is rank r

- ***y* eigenvector of *F* in the SVD projected space**

$Fy = \mu y \quad \Rightarrow \quad A\varphi = \mu \varphi$ **where** $\varphi = Uy$ **is a DMD mode**

Dynamic Mode Decomposition

- **It finds the best (least square) approximation of a linear operator which describes the data evolution in time.**

- **The eigenvectors of this operator are the dynamic modes.**

- **The eigenvalues of this operator provide the modes' frequencies and damping/growing rates.**

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Synergia code

https://cdcvs.fnal.gov/redmine/projects/synergia2

- **Single-particle physics**
  - direct symplectic tracking
  - arbitrary-order polynomial maps
- **Collective effects** (single and multiple bunches)
  - space charge (different solvers)
  - wake fields (arbitrary wakes)
- **Apertures** (different shapes)
- **Slip stacking**

Typical number of macroparticles in realistic simulations is of order $\mathbf{10^5}$**-**$\mathbf{10^9}$.
Large parallel computers are required for large problems.

Space charge modes simulations

- **Gaussian beams with equal transverse emittances**
- **10 x OFORODO linear lattice**
- **10^8 -10^9 macroparticles**
- **3D space charge Poisson solver with open boundary conditions (**Hockney**)**
- **Beam initially excited with an approximate (guessed) mode shape f(z,δ)**

$$x_i \rightarrow x_i + a\, f(z_i, \delta_i)$$ a is the excitation amplitude

- **Store** $X(z, \delta, t_n)$ **at every turn** $X(z,\delta,t_0), X(z,\delta,t_1), \ldots\ldots, X(z,\delta,t_N)$

$$X(z,\delta,t) = \frac{\int dx\, dy\, x\, \rho(x,y,z,\delta,t)}{\rho(z,\delta,t)}$$ **dipole density**

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Transverse space charge modes

Equation of motion (smooth approximation)**:**

$$\ddot{x}_i + \omega_0^2 Q_\beta^2 x_i = \frac{1}{m\gamma} F_x\left(x_i - \overline{x}(z_i), y_i - \overline{y}, z_i\right)$$

$$F_x(z) \propto \rho(z) \propto e^{\frac{-z^2}{2\sigma_z^2}}$$

the space charge force is proportional to the charge density

$$z_i(t) = Z_i \cos\left(\omega_0 Q_s t + \psi_i\right)$$

particles execute synchrotron oscillations

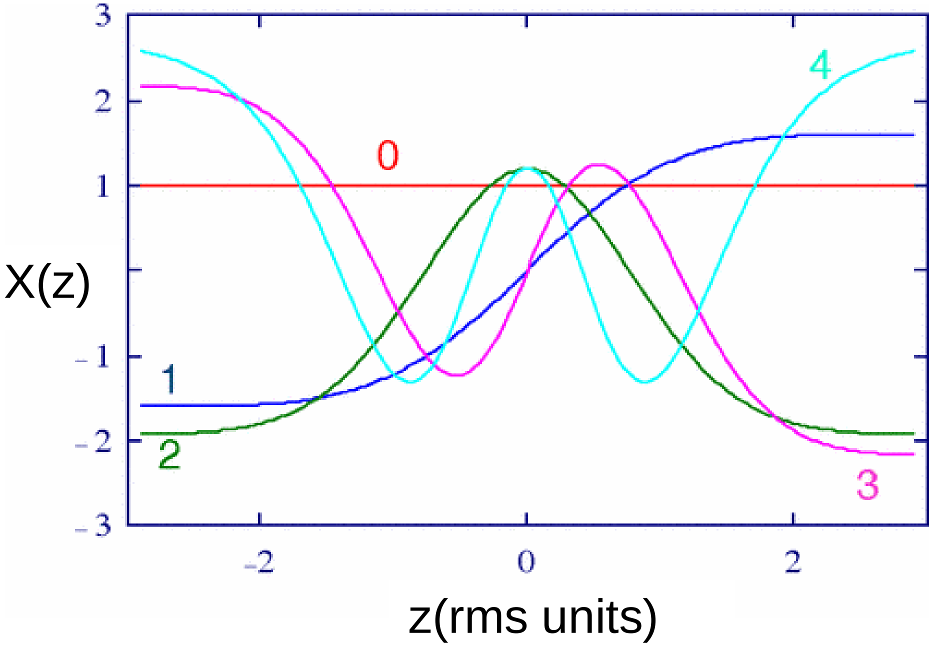
$$\overline{x}(z) = \frac{\int X(z,\delta)\rho(z,\delta)\,d\delta}{\rho(z)}$$

mode dependent

The simulations and the following results are not based on these equations.

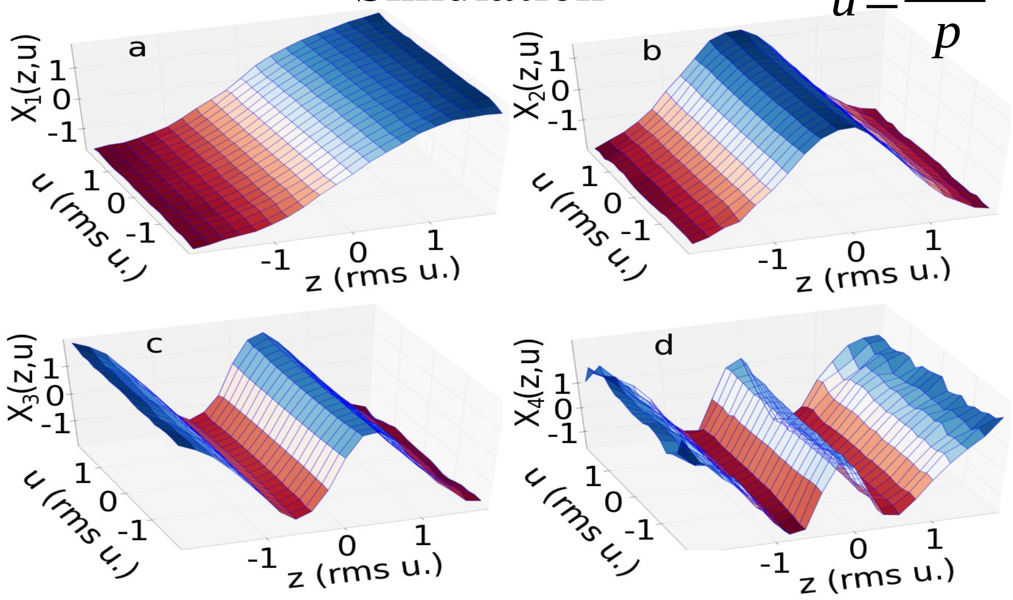
Strong space charge limit

Analytical results, transverse modes

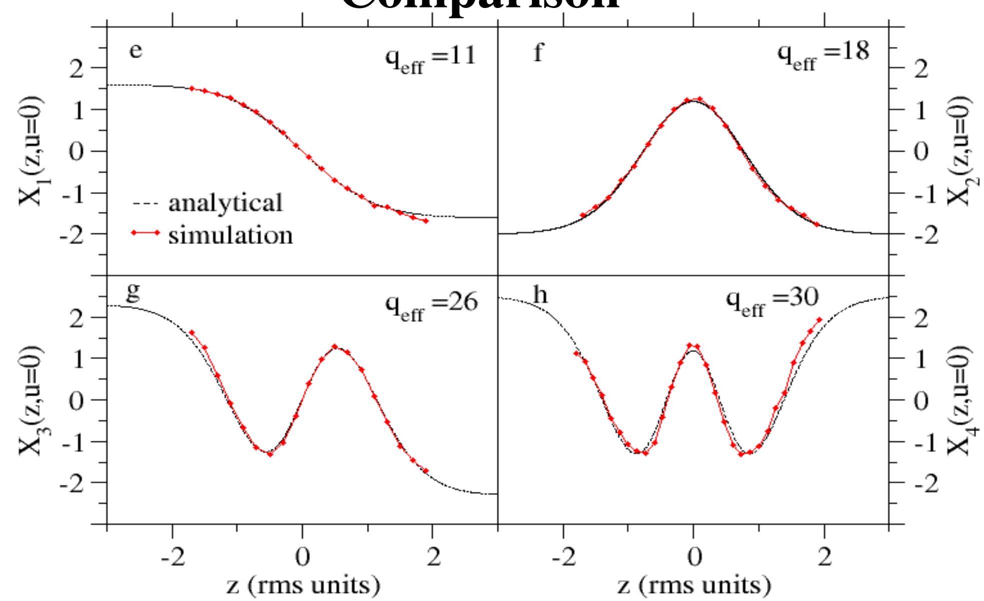


Simulation

$$u = \frac{\Delta p}{p}$$



Comparison



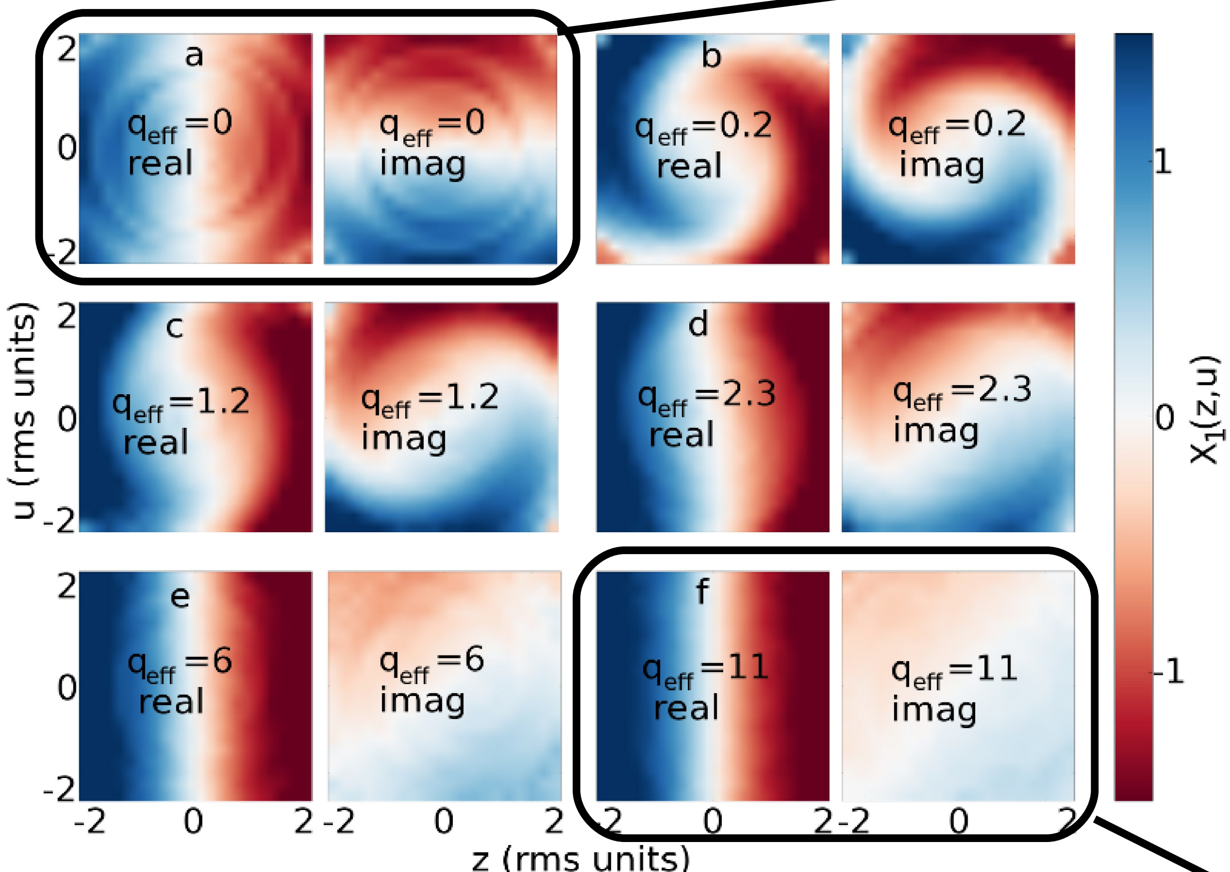
A. Burov, PRSTAB 12, 109901, 2009.
V. Balbekov. PRSTAB12, 124402, 2009.

- **Modes shape depends only on the longitudinal coordinate z**

$$X_k(z, \delta) = X_k(z)$$

Mode 1

$$q_{eff} = 0.5\,\frac{maximum\ \ tune\ \ shift}{synchrotron\ \ tune}$$



$q_{eff} = 0$

rotationally invariant harmonics

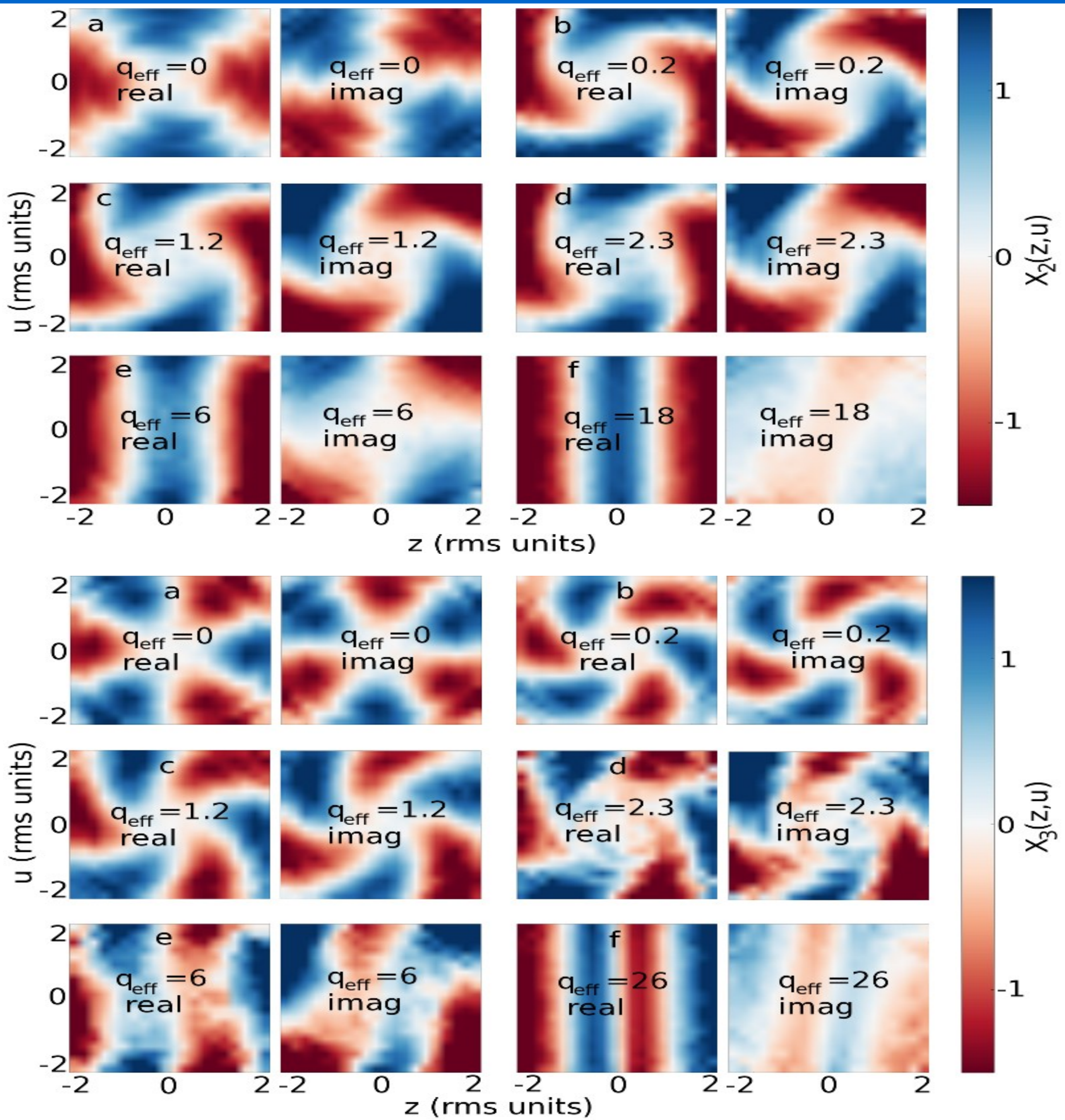
$$X_k(z,\delta) = X_k(r,\theta) = R(r)\,e^{ik\theta}$$

$q_{eff} \rightarrow \infty$

space charge harmonics

$$X_k(z,\delta) = X_k(z)$$

Mode 2 and mode 3



$q_{eff}=0$

rotationally invariant harmonics

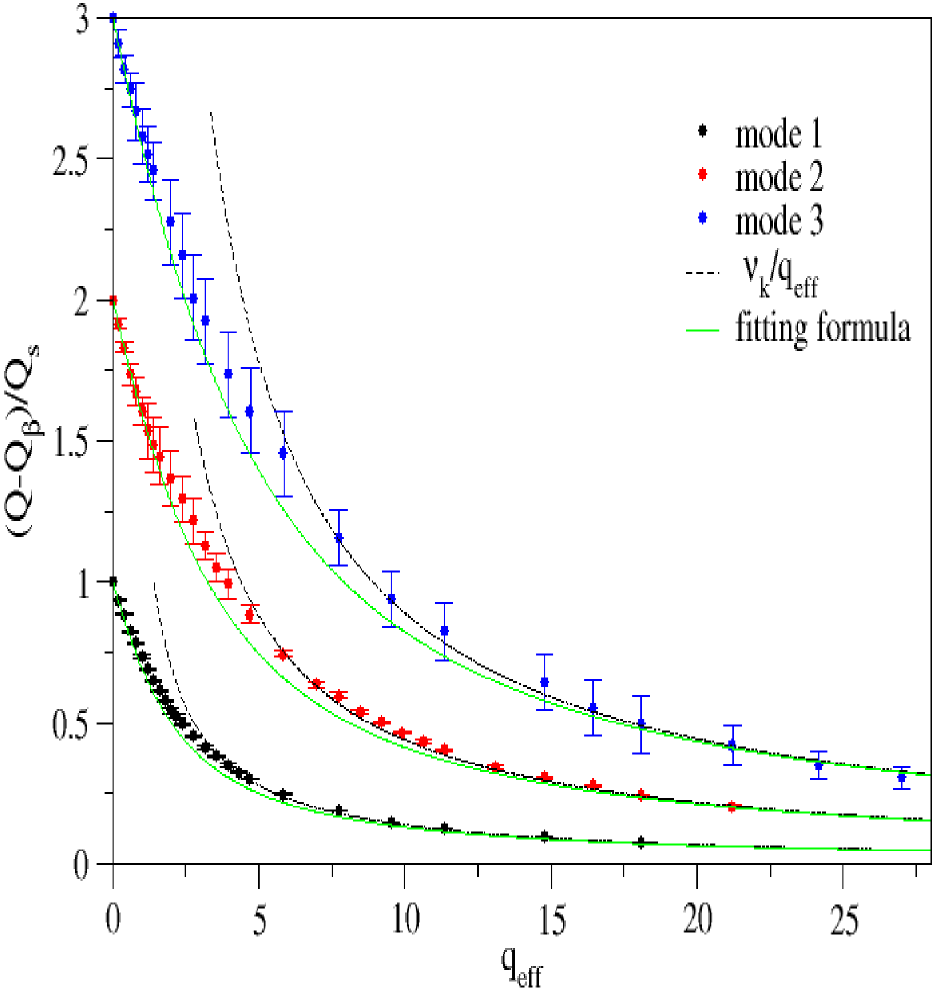
$$X_k(z,\delta)=X_k(r,\theta)=R(r)\,e^{ik\theta}$$

$q_{eff}\rightarrow\infty$

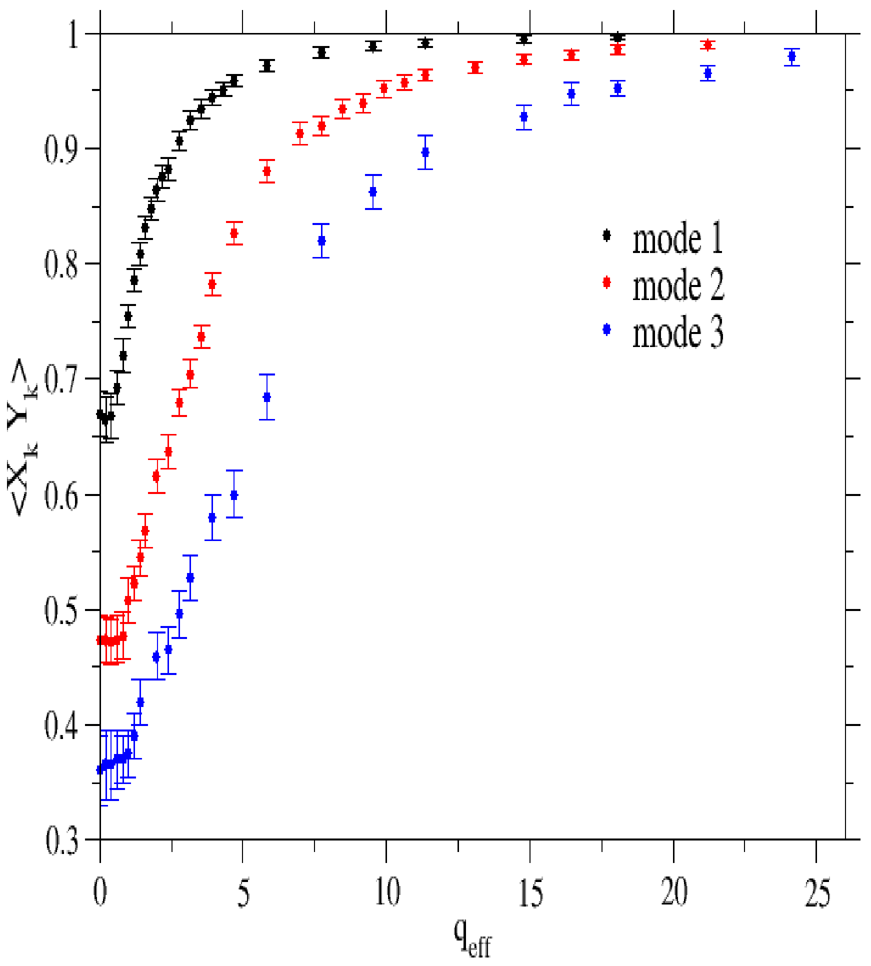
space charge harmonics

$$X_k(z,\delta)=X_k(z)$$

Modes frequencies and shapes



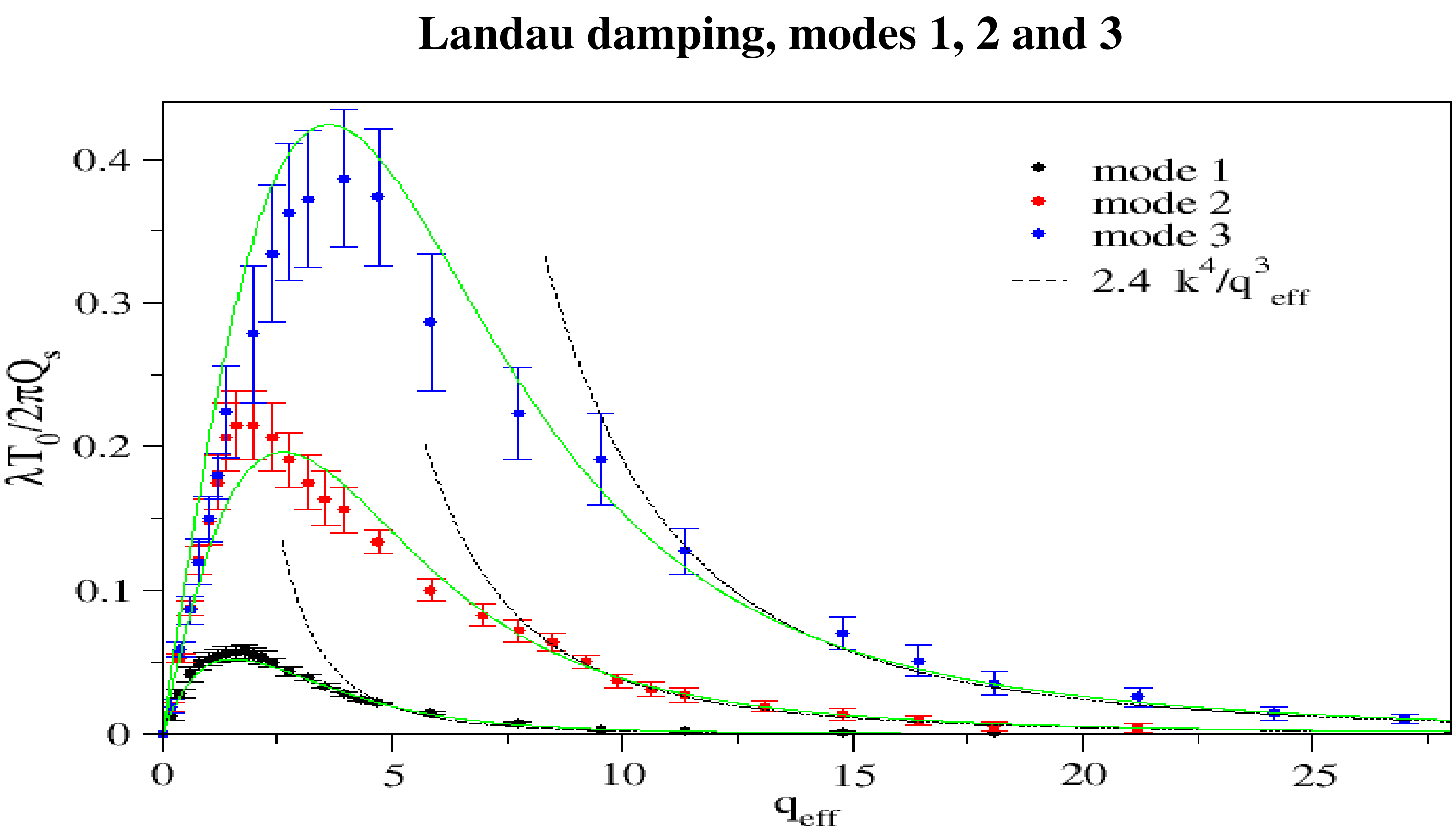
Tune versus the space charge strength q.



Mode k **shape overlap with the** k **space charge harmonic versus the space charge strength** q.

Landau damping

A. Macridin et al., PRSTAB, 074402, 2015



- **In the strong interaction regime the damping rate is proportional to k^4/q^3,** k **is the mode number,** q **is the space charge parameter** (agreement with A. Burov, *PRSTAB 12, 109901, 2009*)

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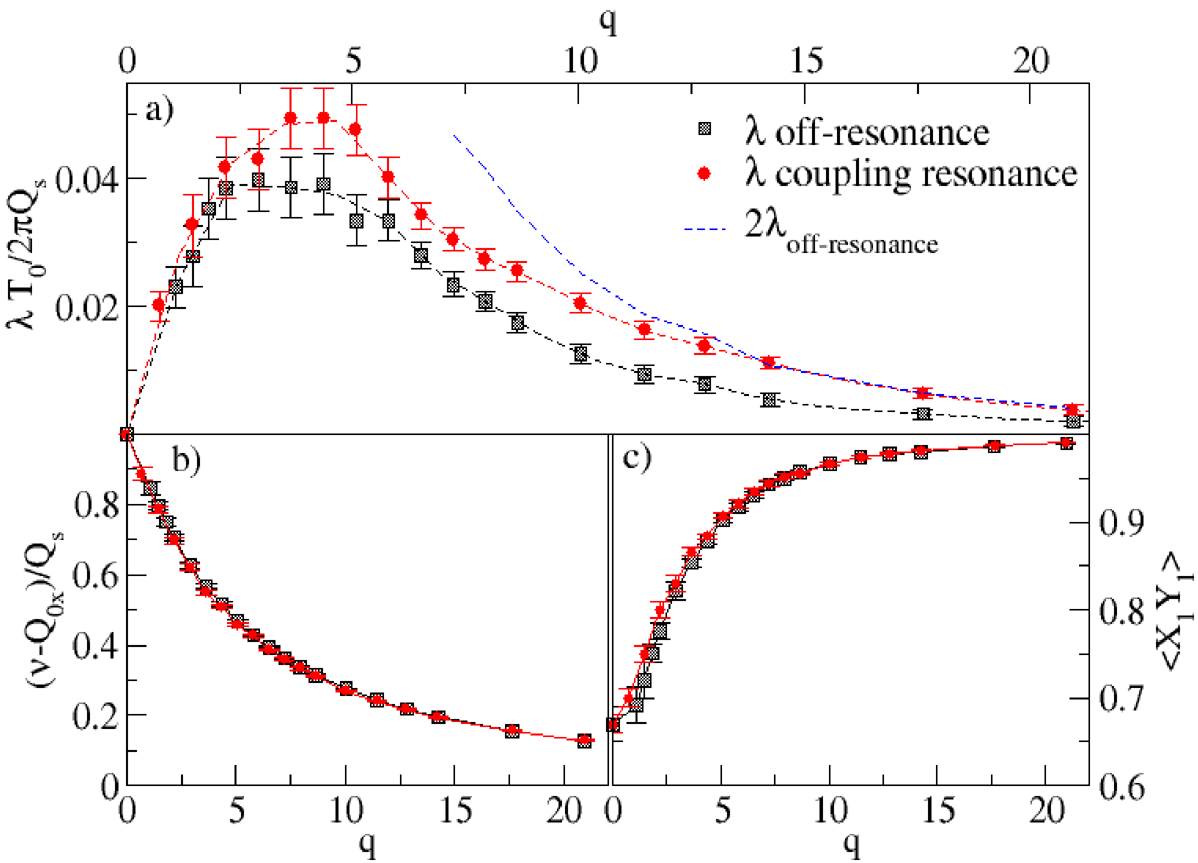
Damping is enhanced in the CR region

mode 1



Coupling resonance, $Q_{x0}=Q_{y0}$

- **Nonlinear coupling resulting from the term proportional to x^2y^2 in space charge potential**
- **Montague's resonance, $2Q_x-2Q_y=0$**

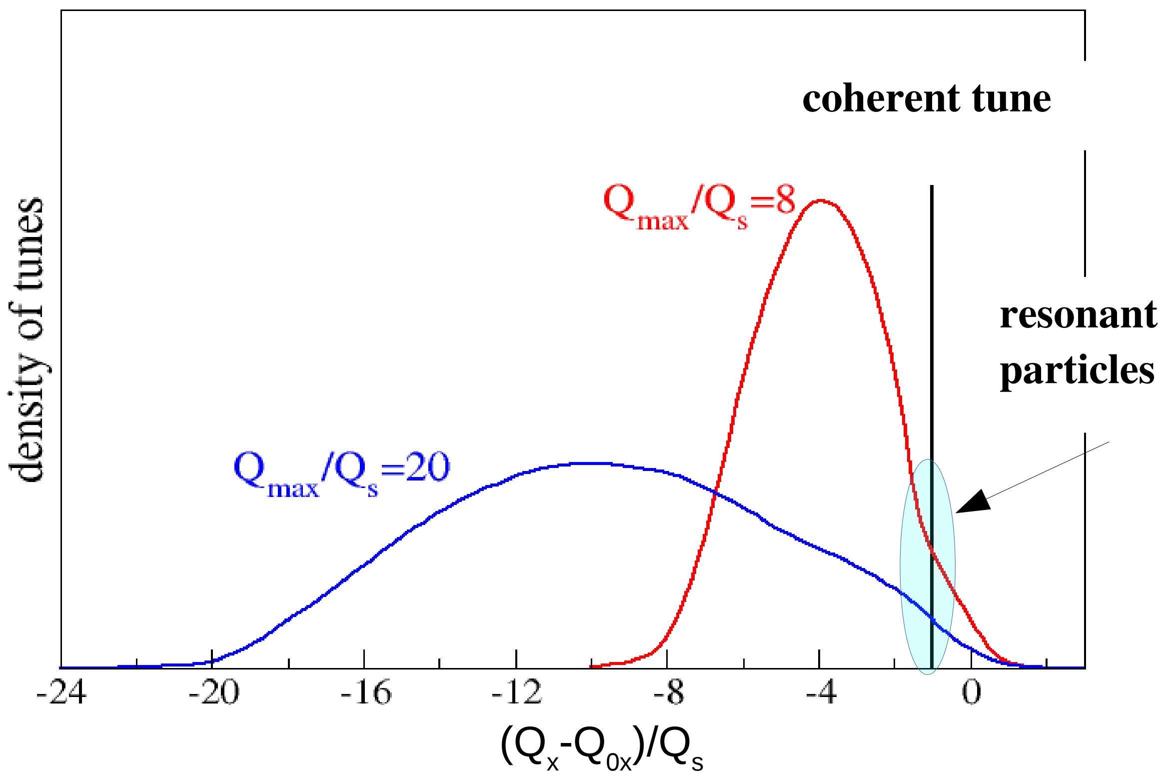
2 times larger damping at CR in strong space charge regime.

Why?

Conventional Landau damping mechanism

$$\ddot{x}_i+\omega_0^2\left(Q_{0x}-\delta Q_i\right)^2 x_i=-2\,\omega_0^2\,Q_{0x}\,\delta Q_i\,\bar{x}$$

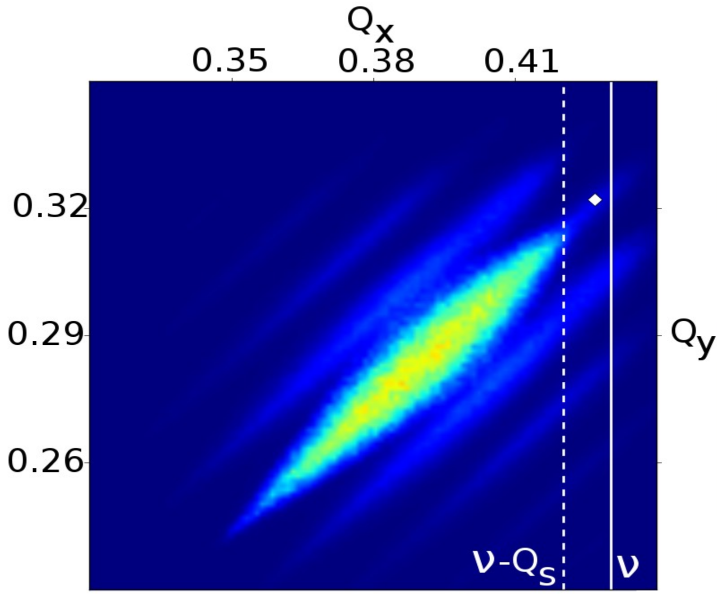
mode-particle coupling



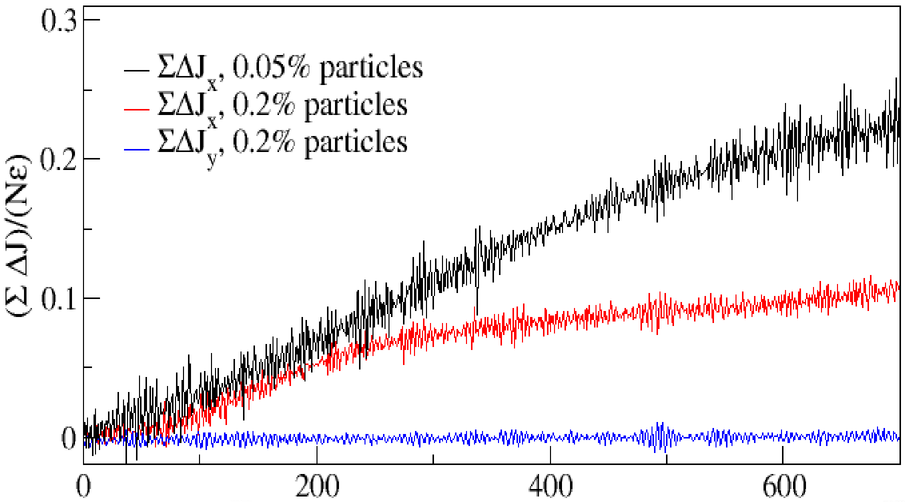
- **The mode energy is transferred to the resonant particles.**
- **The resonant particles tune equal the coherent tune** Q_c**.**

Landau damping off-resonance

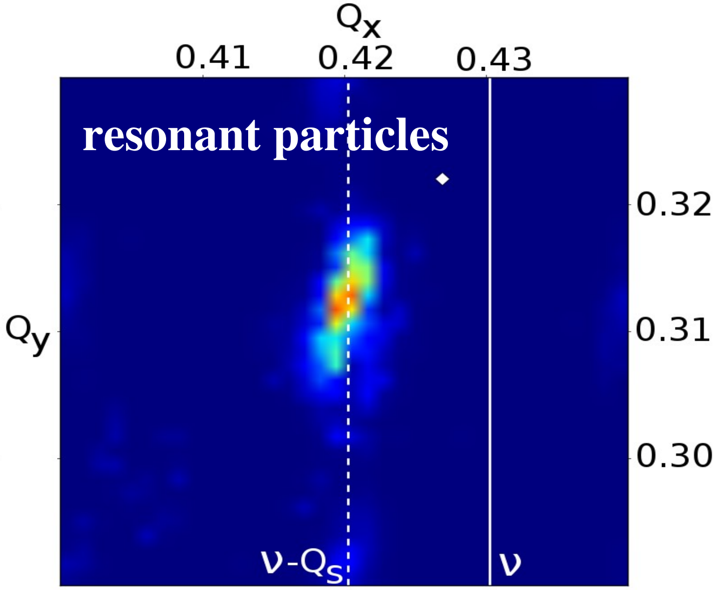
$$\ddot{x}+\omega_0^2(Q_{0x}-\delta Q)^2 x=-2\omega_0^2 Q_{0x}\delta Q\,\bar{x}, \qquad \delta Q(z,J_x,J_y), \qquad \bar{x}(z)\propto e^{i\omega_0(\nu-Q_s)t}$$



bunch tune footprint



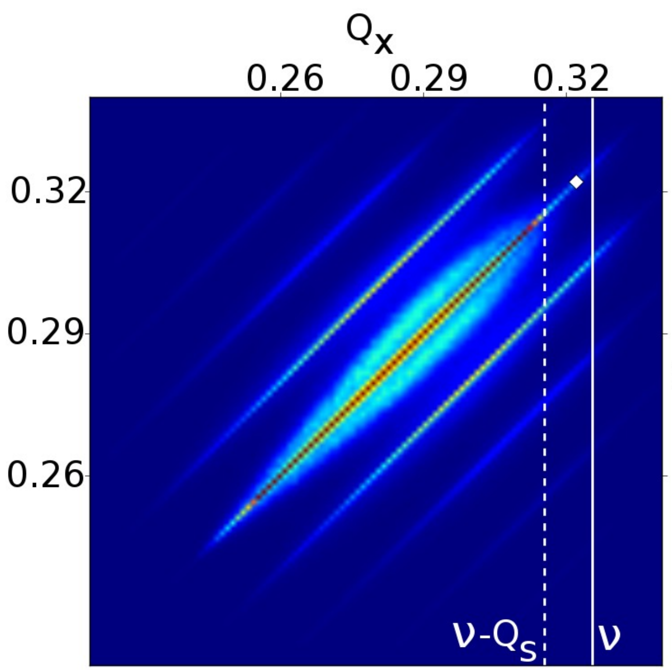
Landau damping (LD) responsible particles increase their energy with time



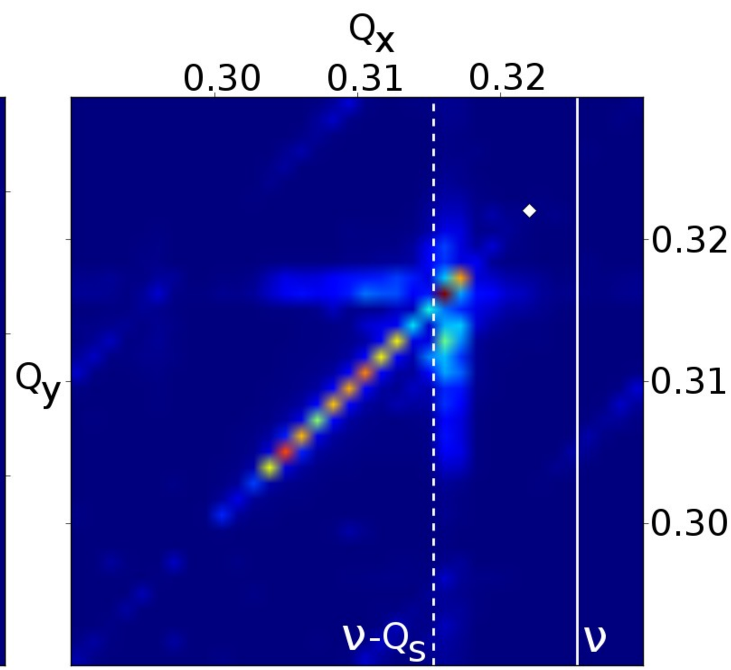
- **Conventional LD mechanism.**
- **The tune of the LD responsible particles is at the coherent tune.**

Landau damping at coupling resonance, $Q_{0x}=Q_{0y}$

LD responsible particles = particles with a large increase in energy



bunch tune footprint



resonant particles

- **The tune of most of the LD responsible particles is not in the vicinity of coherent tune. Why do these particles absorb the mode's energy?**
- **The picture does not fit the conventional LD paradigm.**

Particles dynamics at coupling resonance

- **Particles are trapped around the stable point**
- **Trapped particles properties:**
  - $J_s=J_x+J_y$**, constant of motion**
  - $J_d=J_x-J_y$**, oscillates around the stabled point with frequency** $\omega_0 Q_t$
  - **the trapping frequency** $\omega_0 Q_t$ **is particle dependent**

Exercise:

$$H=\frac{p_x^2}{2}+\frac{1}{2}\omega_x^2 x^2+\frac{p_y^2}{2}+\frac{1}{2}\omega_y^2 y^2+\alpha(x^2+y^2)^2$$

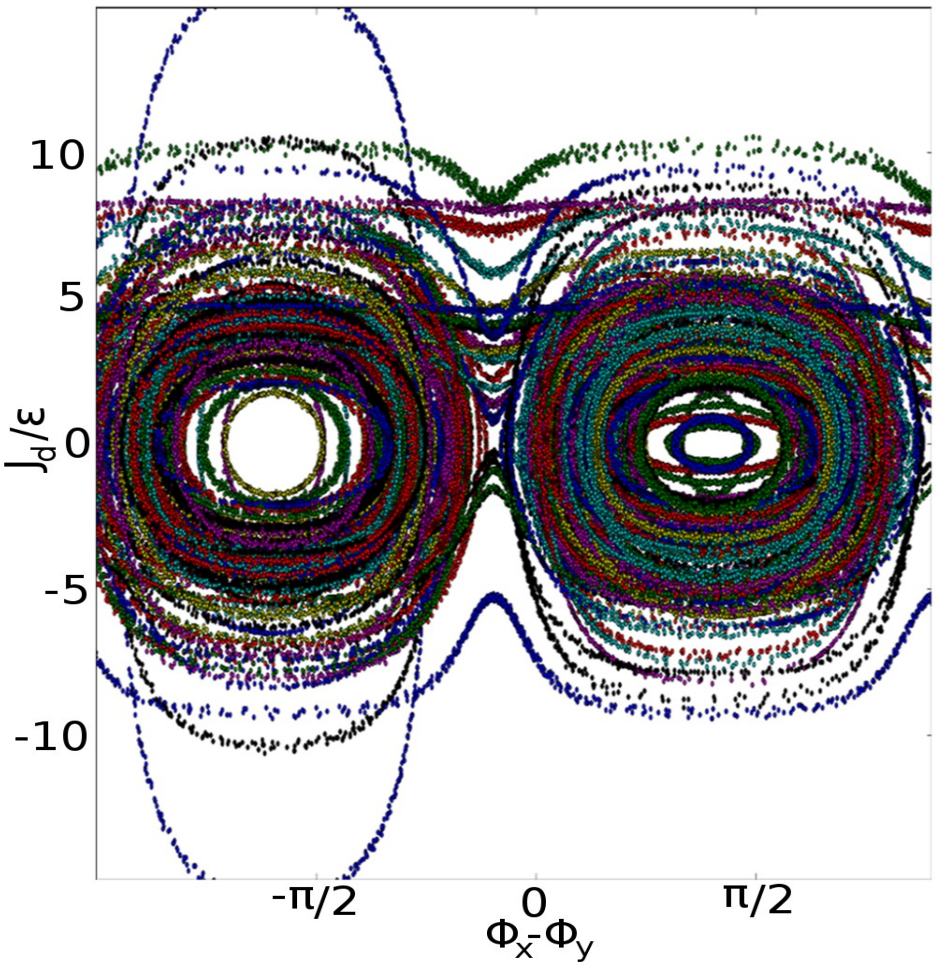
stable point: $\cos 2(\Phi_x^* - \Phi_y^*) = -1, \quad J_d^* = \frac{(\omega_x-\omega_y)(\omega_x+\omega_y)^2}{8\alpha}$

trapping frequency: $Q_t = \frac{8\alpha}{(\omega_x+\omega_y)^2}\sqrt{J_s^2 - J_d^{*\,2}}$

Particles dynamics at coupling resonance

Synergia simulations

Poincaré plots, J_d vs Φ_x-Φ_y

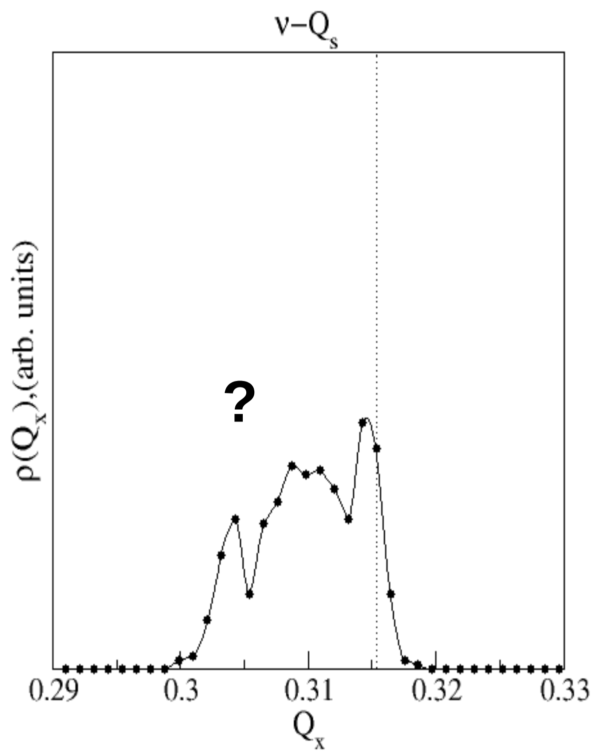


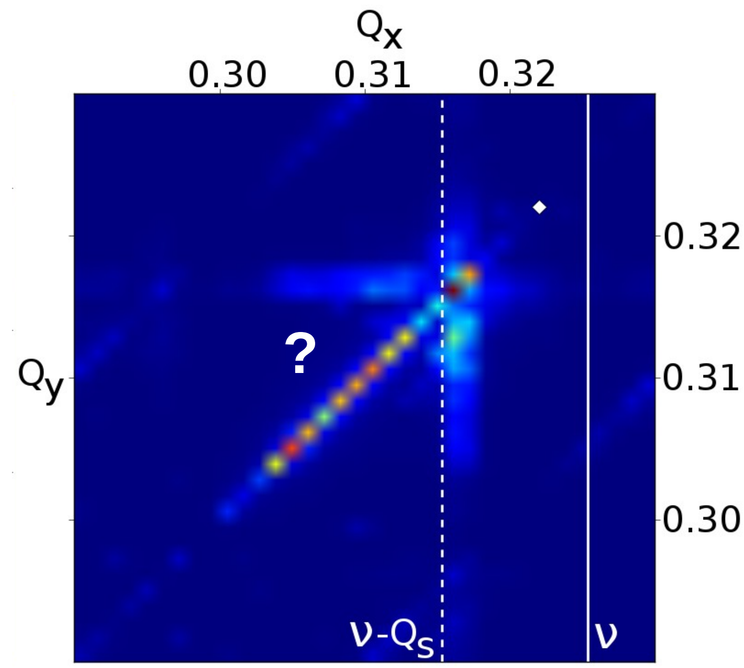
Landau damping responsible particles

- **Most of the LD responsible particles are trapped in resonance islands.**
- **Their actions oscillates with particle dependent frequency.**

LD responsible particles = particles with a large increase in energy

Damping at coupling resonance





resonant particles

$$\ddot{x}_i + \omega_0^2 (Q_{0x} - \delta Q_i)^2 x_i = -2\omega_0^2 Q_{0x} \delta Q_i (z_i, J_{si}, J_{di}) \bar{x}$$

mode-particle coupling

LD responsible particles are trapped around the stable point

$$J_{di} \propto e^{i\omega_0 Q_{ti} t}$$

Parametric Landau damping

$$\ddot{x}_i + \omega_0^2 (Q_{0x} - \delta Q_i)^2 x_i = -2\omega_0^2 Q_{0x} \delta Q_i (z_i, J_{si}, J_{di}) \bar{x}$$

$$\ddot{x}_i + \omega_0^2 Q(z, J_{si}, J_{di})^2 x = -A(z, J_{si}) \bar{x} - B(z, J_{si}) J_{di} \bar{x}, \qquad \bar{x}(z) \propto \mathrm{e}^{i\omega_0(\nu - Q_s)t}, J_{di} \propto \mathrm{e}^{i\omega_0 Q_{ti} t}$$

resonance condition:

- $A\bar{x}$ **coupling** $Q_i = \nu - Q_s$
  - → **conventional LD**

- $BJ_d\bar{x}$ **coupling** $Q_i + Q_{ti} = \nu - Q_s$
  - → **parametric LD**
  - → **mode-particle coupling modulated by** Q_t
  - → **the tune of the LD resonant particles is not at the coherent tune**

Parametric Landau damping

resonance conditions:

- **conventional Landau damping**

$$Q_i = \nu - Q_s$$

The tune density of the LD responsible particles

$$\rho(Q) = \sum_i \delta(Q_i - Q)$$

is peaked at $\nu - Q_s$

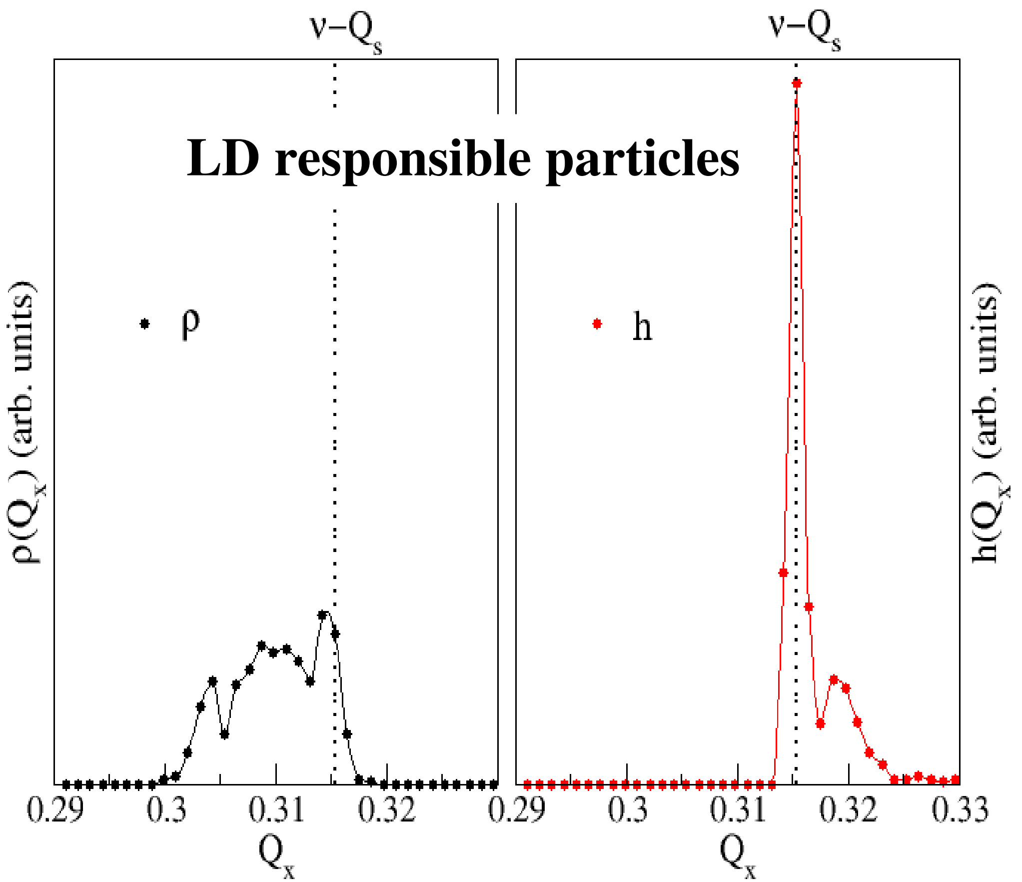
- **parametric Landau damping**

$$Q_i + Q_{ti} = \nu - Q_s$$

The Q_t **shifted tune density of the LD responsible particles**

$$h(Q) = \sum_i \delta(Q_i + Q_{ti} - Q)$$

is peaked at $\nu - Q_s$



conventional LD,
$A\bar{x}$ **coupling**

parametric LD,
$BJ_d\bar{x}$ **coupling**

Conclusions

- Dynamic mode decomposition works well for modal analysis of accelerator beams.
- We employed Synergia and DMD techniques to analyze the transverse space charge modes in bunched beams.
- The SP modes change from the radially degenerate phase space harmonics to momentum independent space charge harmonics with increasing the space charge strength.
- In the proximity of coupling resonance the simulations reveal a novel Landau damping mechanism driven by the modulation of mode-particle interaction.
- The amplitude oscillations of the trapped particles at the coupling resonance enhance the Landau damping rate in bunched beams.

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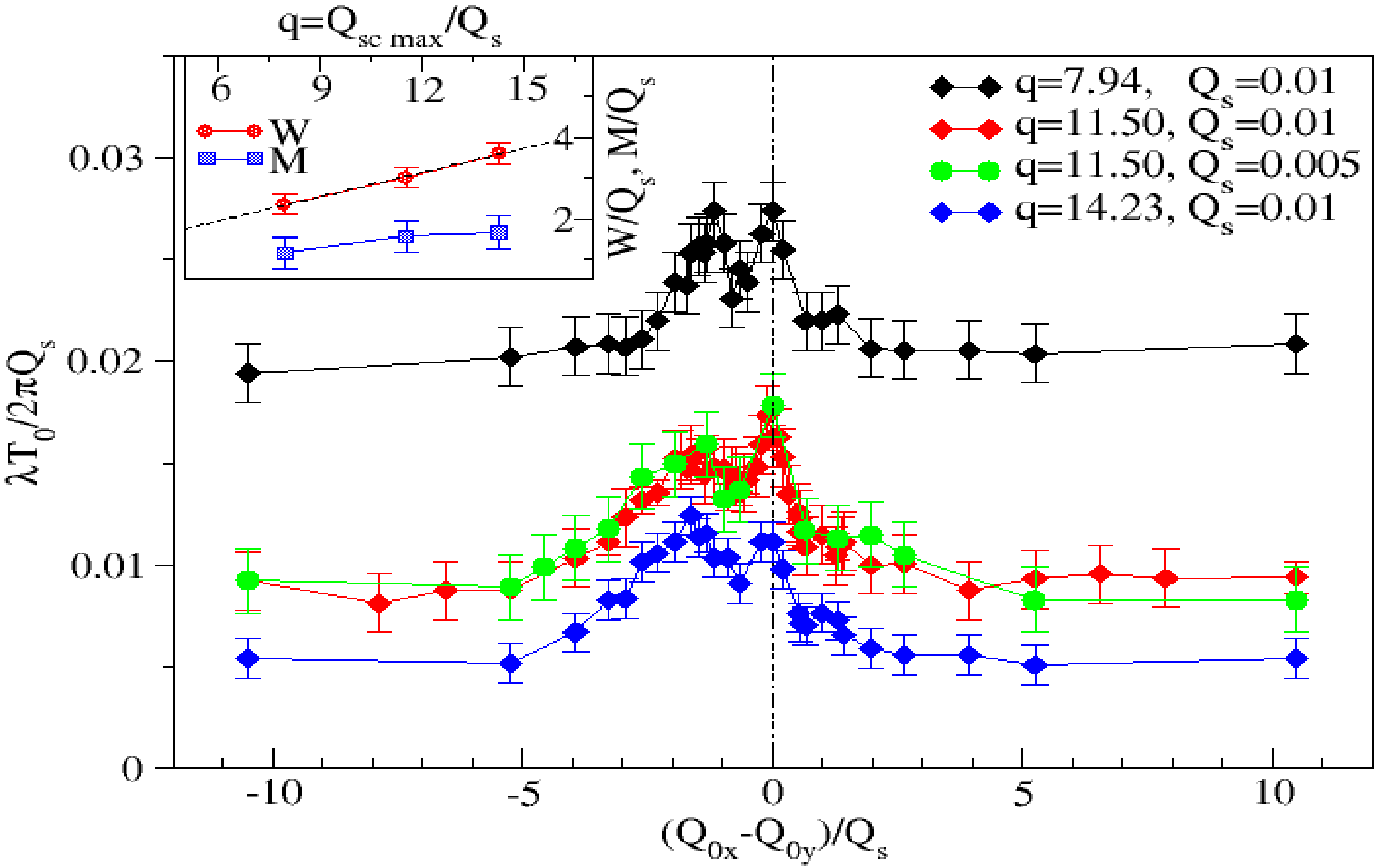
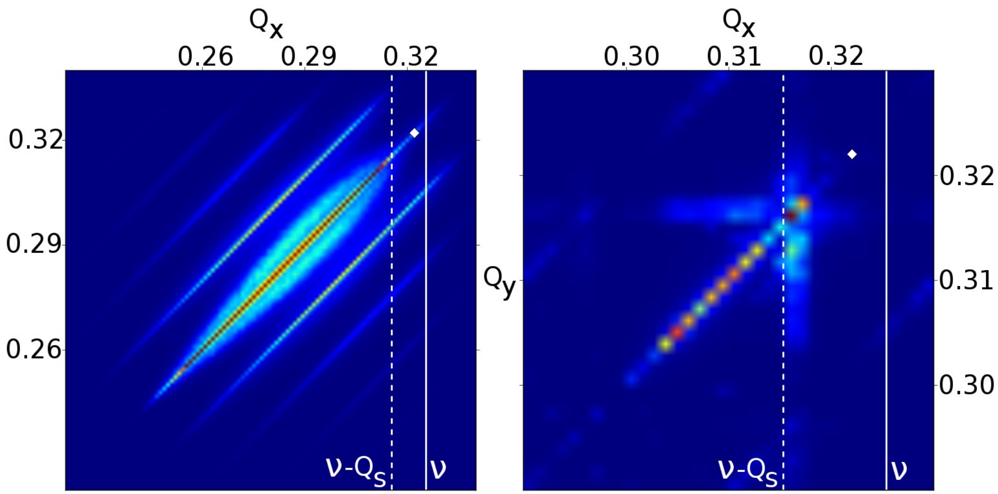


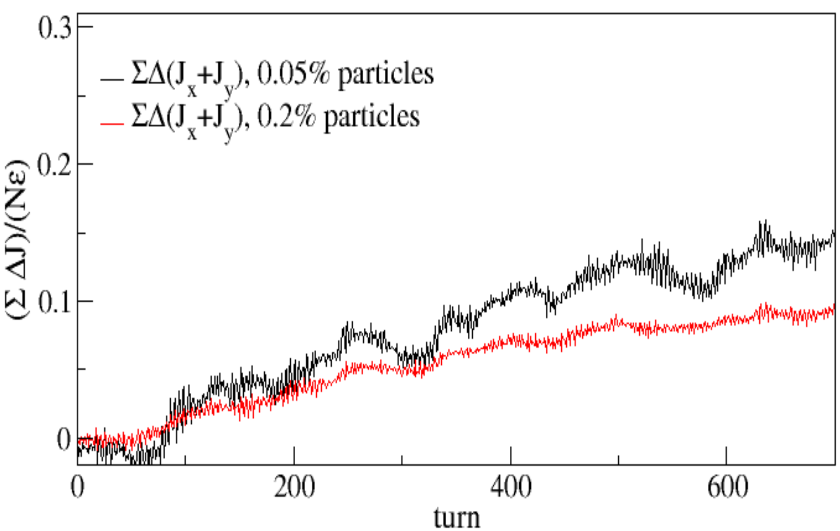
q=Q_sc max/Q_s
6
9
12
15
W
M
4
2
W/Q_s, M/Q_s
q=7.94, Q_s=0.01
q=11.50, Q_s=0.01
q=11.50, Q_s=0.005
q=14.23, Q_s=0.01
λT_0/2πQ_s
0.03
0.02
0.01
0
-10
-5
0
5
10
(Q_0x-Q_0y)/Q_s

Landau damping at coupling resonance

$$Q_{x0} = Q_{y0}$$



bunch tune footprint **resonant particles**



- **Resonant particles are chosen as the ones with the largest energy increase during simulation**