Measurement of Elliptical Beam Emittance in the Transverse Plane

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ABSTRACT

Two methods of transverse emittance measurement are derived and discussed in this paper, both using beam profile monitors. Both methods rely on the assumption that the transverse phase space distribution is elliptical; beam that is not in a ring or has been subjected to significant nonlinear forces may not fit this assumption. The first method requiring a single quadrupole and profile monitor, developed by Ross *et. al.*,[1] is referred to as the "Quadrupole Scan Method". The second method requires three profile monitors, and is thus known as the "Three Profile Method". Note that both methods are ignoring chromatic effects due to non-zero particle momentum spread, and some correction must be made to the measured beam width to compensate.

Quadrupole Scan Method

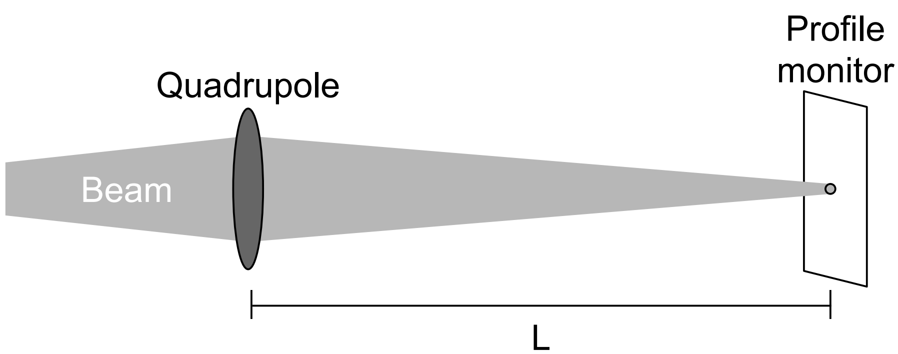

Figure 1. Magnet and instrumentation configuration for the Quadrupole Scan Method.

The single-profile method of measuring the transverse beam emittance involves changing the current in a quadrupole magnet and measuring the change in RMS beam size on a downstream profile monitor, as pictured in *Fig. 1*. The resulting plot of RMS beam size squared as a function of quadrupole strength can be fit to a parabola, and the fit parameters used to calculate the beam emittance and Courant-Snyder parameters at the location of the quadrupole.[1]

To derive the quadratic relationship between quadrupole magnet strength and beam width on the profile monitor, we make use of the beam Σ matrix, defined in *Eq. 1*. This matrix contains the second-order statistical moments of the particle distribution, which can be directly related to the Courant-Snyder beam parameters.[2]

$$\Sigma = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} <x^2> - <x>^2 & <xx'> - <x><x'> \\ <xx'> - <x><x'> & <x'^2> - <x'>^2 \end{pmatrix} \tag{1}$$

In *Eq. 1*, $<x>$ refers to the average transverse position over all particles, $<x'>$ the average angle $\frac{dx}{ds}$, $<x^2>$ the squared RMS position, $<x^2>$ the squared RMS angle, and $<xx'>$ the average correlation between position and angle in the beam. Note that Equation 1 represents the Courant-Snyder parameters and emittance for a single plane (i.e. horizontal or vertical). For the following derivation, plane notation will be suppressed, with the understanding that the derivation proceeds identically for both horizontal and vertical planes.

If we are only concerned with the particle motion about the closed orbit, we may re-define the coordinate system such that the average position is zero, i.e. $<x>= 0$, so the sigma matrix simplifies to:

$$\Sigma = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} <x^2> & <xx'> \\ <xx'> & <x'^2> \end{pmatrix}. \tag{2}$$

Elliptical beam distributions generate Gaussian profiles, so it is often convenient to refer to the second moment of the position $< x^2 >$ in terms of the standard deviation σ_x of the Gaussian fit to the profile data. With the assumptions taken for *Eq. 2*, i.e. motion about the closed orbit, this relationship is simply that the standard deviation is the RMS ("root mean square") of the position distribution:[2]

$$\sigma = \sqrt{< x^2 >}. \tag{3}$$

In the linear paraxial approximation, the beam Σ matrix propagates through transformation matrix "R" as:[2]

$$\Sigma_{mon} = R\Sigma_{quad}R^T. \tag{4}$$

In the case of the setup in Figure **??**, the R matrix is simply that of a quadrupole and a drift:

$$R = SQ = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}. \tag{5}$$

where L is the drift length in $[m]$ between the quadrupole and profile monitor, and $\kappa = \frac{B'L}{B\rho}$ is the magnetic field strength in $[\frac{T}{m}]$. The single variable "$B\rho$" is known as the "magnetic rigidity" of the particle beam, and is equal to the particle momentum divided by the charge $\frac{p}{q}$. A short-hand method of calculating the rigidity is $B\rho[\frac{T}{m}] = \frac{10}{3}p[MeV/c]$.

Substituting R from Equation 5 into the propagation relationship of Equation 4, solving for the (1,1) element of the Σ matrix before the quadrupole, and collecting like terms in powers of κ, we get the following relationship between the measured RMS beam size on the profile monitor, the strength of the quadrupole magnet, and the Courant-Snyder beam parameters *just before* the quadrupole:

$$\Sigma^{11}_{mon} = \sigma^2 = L^2\beta\varepsilon\kappa^2 + \kappa\left(-2L^2\alpha\varepsilon + 2L\beta\varepsilon\right) + \frac{L^2\varepsilon}{\beta} - 2L\alpha\varepsilon + \beta\varepsilon + \frac{L^2\varepsilon}{\beta}\alpha^2. \tag{6}$$

More simply, we rewrite *Eq. 6* by collecting the coefficients of the κ terms:

$$\Sigma^{11}_{mon} = \sigma^2_{mon} = A\kappa^2 + B\kappa + C, \tag{7}$$

where

$$A = L^2\beta\varepsilon \qquad B = -2L^2\alpha\varepsilon + 2L\beta\varepsilon \qquad C = \frac{L^2\varepsilon}{\beta} - 2L\alpha\varepsilon + \beta\varepsilon + \frac{L^2\varepsilon}{\beta}\alpha^2 \tag{8}$$

Solving the system of equations in Equation 8 for ε, β, and α, we get:

$$\varepsilon = \pm\frac{1}{2L}\sqrt{4AC - B^2} \qquad \beta = \pm\frac{2A}{\sqrt{4AC - B^2}} \qquad \alpha = \pm\frac{2A - BL}{L\sqrt{4AC - B^2}}. \tag{9}$$

Since both ε and β must be positive to have physical meaning, it is trivial to decide which of the two possible solutions are viable for these parameters. However, since α may be positive or negative depending on whether the quadrupole is focusing or defocusing in that particular plane, we cannot *a priori* determine which solution to choose for α. This is the limitation of the method described above: more information is needed to determine the sign of α for a given plane, though we can find its magnitude $|\alpha|$ from the parabolic fit.

Therefore, to summarize the information at the quadrupole calculated from the parabolic fit,

$$\varepsilon_{quad} = \frac{1}{2L}\sqrt{4AC - B^2} \qquad \beta_{quad} = \frac{2A}{\sqrt{4AC - B^2}} \qquad |\alpha_{quad}| = \frac{2A - BL}{L\sqrt{4AC - B^2}} \tag{10}$$

1 Three-Profile Method

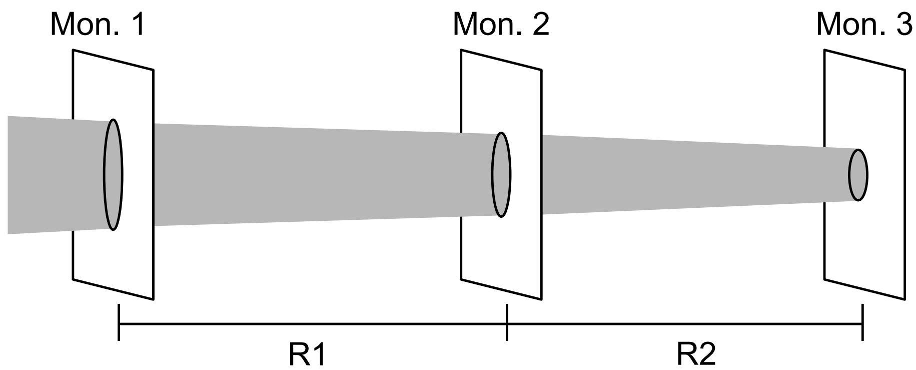

Figure 2. Instrumentation configuration for the Three-Profile Method.

The Three-Profile Method requires three beam profile monitors with a known linear transformation matrix "*R*" between them, as pictured in *Fig. 2*. This method determines the beam sigma matrix just before profile monitor 3 by using measured beam RMS widths at all three monitors. The beam sigma matrices just before each profile monitor are referred to as Σ_1, Σ_2, and Σ_3 respectively. The RMS beam size measured at monitor 2 σ_2 is expressed as a function of Σ_1 as follows, where we have used "[1,1]" to indicate the upper-left element of a matrix:

$$\Sigma_2[1,1] = \sigma_2^2 = (R_1 \Sigma_1 R_1^T)[1,1]. \tag{11}$$

The rest of this derivation proceeds assuming drifts of length L_1 and L_2 between profile monitors, such that

$$R_1 = \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

and

$$R_2 = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

Therefore *Eq. 11* simplifies to:

$$\sigma_2^2 = \sigma_1^2 + 2L_1 < x_1 x_1' >_1 + L_1^2 < x_1'^2 > . \tag{14}$$

Similarly, we express the square of the measured RMS beam size on monitor 3 as a function of the initial beam sigma matrix:

$$\Sigma_3[1,1] = \sigma_3^2 = ((R_2 R_1)\Sigma_1 (R_2 R_1)^T)[1,1], \tag{15}$$

which simplifies to

$$\sigma_3^2 = \sigma_1^2 + 2(L_1 + L_2) < x_1 x_1' > + (L_1 + L_2)^2 < x_1'^2 > . \tag{16}$$

We now have a system of two equations, *Eq. 14* and *Eq. 19* with two unknowns, $< {x_1}^{‘2} >$ and $< x_1 x_1‘ >$. This system has the following solutions for the angle-dependent unknowns:

$$< x_1 x_1' > = \frac{L_1^2(\sigma_2^2 - \sigma_3^2) + (L_2^2 + 2L_1 L_2)(\sigma_2^2 - \sigma_1^2)}{2L_1 L_2 (L_1 + L_2)} \tag{17}$$

and

$$< x_1'^2 > = \frac{L_1(\sigma_3^2 - \sigma_2^2) + L_2(\sigma_1^2 - \sigma_2^2)}{L_1 L_2 (L_1 + L_2)} \tag{18}$$

After solving for the two unknowns, we can compute the emittance as[3]

$$\varepsilon = \sqrt{< x_1^2 >< x_1'^2 > - < x_1 x_1' >^2}. \tag{19}$$

2 Accounting for Momentum

For a single particle with fractional momentum deviation from the reference momentum $\delta = \frac{dp}{p}$, its transverse displacement will differ from the ideal particle position x_0 by:[4]

$$x = x_0 + D\delta \tag{20}$$

If we now consider the RMS beam width of the group of particles, we get

$$\sigma^2 =< x^2 >=< (x_0 + D\delta)^2 >=< x_0^2 + 2x_0 D\delta + D^2\delta^2 > . \tag{21}$$

Distributing the average through the summation, we get

$$\sigma^2 =< x_0^2 > +2D < x_0 >< \delta > + < \delta^2 > \tag{22}$$

We again make the argument that $< x_0 >= 0$, because we have chosen a coordinate system that looks at particle motion about the reference orbit. Also, we know that $< x_0^2 >= \sigma_0^2$, i.e. the beam size if all particles had the reference momentum. Furthermore, $< \delta^2 >= \sigma_p^2$, the RMS width of the momentum distribution. So simplifying *Equation 22* and solving for the achromatic RMS beam width σ_0^2, we get

$$\sigma_0 = \sqrt{\sigma_{measured}^2 - D^2\sigma_p^2}, \tag{23}$$

where $\sigma_{measured}$ is the RMS beam width as measured by the profile monitor. Therefore, *Equation 23* prescribes how to correct for dispersion the measured RMS beam widths before using the aforementioned emittance measurement methods.

References

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