Mechanical Safety Subcommittee
Guideline for Design of Thin Windows
Regarding Roark’s Edge Condition Coefficient

C. Ader, E. Voirin, M. McGee, and L. Nobrega
An error was found in an edge stress coefficient used to calculate stresses in thin windows. This error is present in “Roark’s Formulas for Stress and Strain” 7th and 8th Edition. The 6th Edition is correct. This guideline specially discusses a major difference in regards to a coefficient used in calculating the edge stress in “Roark’s Formulas for Stress and Strain” 6th Edition compared to the 7th and 8th Editions. In Chapter 10: Flat Plates under “Circular plates under distributed load producing large deflections,” Case 3, which is “Fixed and held. Uniform pressure q over entire plate.” The coefficient for a fixed edge condition in the 6th Edition1 $K_4 = 0.476$ while in the 7th and 8th Edition2, the coefficient is 1.73 which is significant difference.


Two independent analyses were done which show that the coefficient in the 6th Edition is correct. The first analysis, “Thin Circular Windows Under Uniform Pressure: Equation Analysis” by Erik Voirin (attached) uses the same case as in APM-56-12. Voirin’s analysis compares hand calculations and FEA and concludes that $K_4$ for the edge should be 0.476, not the published value of 1.73 as seen in Roark and Young’s 7th and 8th Edition.

The second analysis “Proof of Bending of Circular Plates with Large Deflection” by Michael McGee (attached) uses Way’s book. The edge coefficient is derived and found to be $K_4 = 0.476$. The third author of the 8th Edition of “Roark’s Formulas for Stress and Strain”, Dr. Ali M. Sadegh7, has agreed in an email dated Feb. 21, 2017 that in the next edition he will revise the value $K_4$ back to equal 0.476 and was not sure why it had been changed.

References
7. A. Sadegh, email, 2/21/17
Thin Circular Windows under Uniform Pressure: Equation Analysis

Erik Voirin - March 21, 2017 - evoirin@fnal.gov - 630-840-5168

**Dimensions:** Same as case in APM-56-12: Bending of Circular Plates with Large Deflection

\[ \frac{D_{\text{plate}}}{2} = 4.5 \text{ in} \quad E_{\text{young}} = 3 \times 10^7 \text{ psi} = 3 \times 10^4 \text{ ksi} \quad \nu = 0.3 \quad \text{Pressure} = 10 \text{ psi} \]

\[ t_{\text{plate}} = 0.032 \text{ in} \quad r = \frac{D_{\text{plate}}}{2} = 2.25 \text{ in} \quad u_0 = \frac{r}{t_{\text{plate}}} = 70.313 \]

**Equations in TM1380 and Roark and Young 6th Ed. for fixed edge.**

\[ K_1 = \frac{5.33}{1 - \nu} = 5.857 \quad K_2 = \frac{2.6}{1 - \nu} = 2.857 \quad K_3 = \frac{2}{1 - \nu} = 2.857 \quad K_4 = 0.976 \]

\[ K_{2\text{edge}} = \frac{4}{1 - \nu} = 4.396 \quad K_{4\text{edge}} = 0.476 \]

**Deflection at Center**

\[ \delta = \text{Find}(\delta) = 0.03071 \text{ in} \]

**TM1380: Equation 5.1b**

**Stress at Center (Membrane + Bending)**

\[ \sigma_{\text{center}} = \frac{E_{\text{young}}}{t_{\text{plate}}} \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \left[ K_3 \left( \frac{\delta}{t_{\text{plate}}} \right) + K_4 \left( \frac{\delta}{t_{\text{plate}}} \right) \right] = 22.092 \text{ ksi} \]

\[ \sigma_{\text{FEA Center}} = 19.308 \text{ ksi} \]

**Stress at Edge (Membrane + Bending)**

\[ \sigma_{\text{edge}} = \frac{E_{\text{young}}}{r^2} \left( \frac{t_{\text{plate}}}{r} \right)^2 \left[ K_{3\text{edge}} \delta + K_{4\text{edge}} \left( \frac{\delta}{t_{\text{plate}}} \right) \right] = 28.261 \text{ ksi} \]

\[ \sigma_{\text{FEA Edge}} = 24.261 \text{ ksi} \]

**FEA Results for thin circular window with non-linear deformation:**

![Graph showing deformation vs. radius](image)

**FEA Gives:**

\[ \delta_0 = 0.02992 \text{ in} \quad \frac{\delta_0}{t_{\text{plate}}} = 0.935 \]
Membrane and Bending Stress at Center of Plate

\[
\begin{array}{c|c|c}
\text{Membrane [psi]} & \text{Bending [psi]} & \text{Membrane+Bending [psi]} \\
5084.9 & 14223 & 19308 \\
\end{array}
\]

\[\sigma_{\text{FEA Membrane Center}} = 5084.9 \text{ psi} \quad \sigma_{\text{FEA Bending Center}} = 14223 \text{ psi} \quad \sigma_{\text{FEA Bend Plus Mem Center}} = 19308 \text{ psi}\]

Membrane and Bending Stress at Edge of Plate

\[
\begin{array}{c|c|c}
\text{Membrane [psi]} & \text{Bending [psi]} & \text{Membrane+Bending [psi]} \\
2449.4 & 26626 & 29005 \\
\end{array}
\]

\[\sigma_{\text{FEA Membrane Edge}} = 2449.4 \text{ psi} \quad \sigma_{\text{FEA Bending Edge}} = 26626 \text{ psi} \quad \sigma_{\text{FEA Bend Plus Mem Edge}} = 29005 \text{ psi}\]
**FEA Von Mises Stress**

At: Static Structural  
Equivalent Stress  
Type: Equivalent (von-Mises) Stress  
Unit: psi

Equations from APM-56-12 Bending of Circular Plates with Large Deflection

**Membrane Stresses**

\[
\sigma_{\text{MemCenter}} := 0.976 \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \frac{E_{\text{young}}}{r} \left( \frac{t_{\text{plate}}}{r} \right)^2 = 5455 \text{ psi} \\
\sigma_{\text{EMembraneEdge}} := 0.476 \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \frac{E_{\text{young}}}{r} \left( \frac{t_{\text{plate}}}{r} \right)^2 = 2661 \text{ psi} \\
\]

*FEA Gives:*  
\[\sigma_{\text{FEA Membrane Center}} = 5085 \text{ psi} \]
\[\sigma_{\text{FEA Membrane Edge}} = 2449 \text{ psi} \]

Good agreement with \( K_4 = 0.476 \), which states the value is conservative.

**Other form of equation in Roark & Young 7th and 8th editions:** (Wrong \( K_4 \) value of 1.73 for edge)

\[
\sigma_{\text{MembraneEdge Wrong}} := 1.73 \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \frac{E_{\text{young}}}{r} \left( \frac{t_{\text{plate}}}{r} \right)^2 = 9670 \text{ psi} \]

*No agreement with anything.*

\[
\sigma_{\text{EdgeWrong}} := \frac{E_{\text{young}}}{r} \left[ K_3 \text{edge} \left( \frac{\delta}{t_{\text{plate}}} \right) + 1.73 \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \right] = 35269 \text{ ksi} \\
\sigma_{\text{EdgeCorrect}} := \frac{E_{\text{young}}}{r} \left[ K_3 \text{edge} \left( \frac{\delta}{t_{\text{plate}}} \right) + 0.476 \left( \frac{\delta}{t_{\text{plate}}} \right)^2 \right] = 28.26 \text{ ksi} \\
\sigma_{\text{FEBend Plus Mem Edge}} = 29.005 \text{ ksi} \\
\]

*We have concluded that \( K_4 \) for the edge should be 0.476, not the published value of 1.73 as seen in R&Y editions 7 and 8.*
Proof of Bending of Circular Plates With Large Deflection

M. McGee

February 2017

Ref. [1]: Stewart Way, "Bending of Circular Plates With Large Deflection"

Nomenclature

\( a \) = radius
\( h \) = thickness
\( p \) = load intensity, assumed uniform
\( \sigma_r, \sigma_\theta, \sigma_r^\prime = \) radial stresses
\( \sigma_\theta, \sigma_\phi, \sigma_\phi^\prime = \) circumferential stresses
\( z \) = distance from middle surface, downward direction positive
\( r \) = distance from the axis of symmetry to a point in the plate before deflection
\( \omega \) = vertical displacement of points of the middle surface relative to the center of the middle surface
\( E \) = Young's modulus
\( \mu \) = Poisson's ratio

Consider methods by G.B. Galerkin: Displacement \( \omega \) perpendicular to plate and displacement \( p \) (radial direction)

Symbolically

\[ E(\omega) = 0 \quad \text{and} \quad E(p) = 0 \]

Integrate over the entire circular plate

\[ \int E(\omega) \delta \omega \, d\Lambda = 0 \quad \text{and} \quad \int E(p) \delta p \, d\Lambda = 0 \]

Substitute into arbitrary functions that will satisfy boundary conditions and find the constants. Considering the membrane stresses at the circular plate center (of maximum stress).

The condition of vertical equilibrium of a disk of the plate of radius \( r \):

\[ D \left[ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \cdot \varphi \right) \right] \right] = p \left( \frac{r}{2} \right) + h \left( \sigma_r p + \ldots \right) \quad \text{eqn (8) ref [1]}.

Multiply eqn (8) by \( r \), differentiating and dividing by \( r \):

\[ p = D \left[ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} \omega \right) \right] \right] - h \left[ \frac{d}{dr} \left[ r \sigma_r \left( \frac{d}{dr} \omega \right) \right] \right] \quad \text{eqn (42a)}

Differentiate and use relation (eqn 9) between \( \sigma_r p \) and \( \sigma_\theta p \)

\[ p = D \left[ \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} \omega \right) \right] \right] + h \left( \frac{\sigma_r p}{R_r} + \frac{\sigma_\theta p}{R_t} \right) \quad \text{eqn (42b)}

where, \( R_r \) & \( R_t \) are radial and tangential radii of curvature of the middle surface.
If no membrane stresses exist, the third term would go to 0 and external work is defined as

$$\pi p \int_0^a \omega \cdot r \, dr$$

and equivalent strain energy of bending is

$$\pi D \int_0^a \frac{1}{r} \left[ \frac{d}{dr} \left( \frac{1}{r} \left( \frac{d}{dr} \omega \right) \right) \right] \cdot r \cdot \omega \cdot r \, dr$$

where load is proportional to deflection.

If membrane stresses are considered, and we neglect all displacements except those normal to the middle surface, \( e/\omega = 1/R \), and the strain energy of stretching is expressed by multiplying the third term above by \( 1/2 \omega \) and integrating over the entire plate. The total external work is no longer

$$\pi P \int_0^a \omega \cdot r \, dr$$

However, since the strain energy of stretching is a small part of the total, roughly 20% when the deflection at the center equals the plate thickness, we assume that the straight line relation approximation still holds. Giving,

$$\pi D \int_0^a \frac{1}{r} \left[ \frac{d}{dr} \left( \frac{1}{r} \left( \frac{d}{dr} \omega \right) \right) \right] \cdot r \cdot \omega \, dr = \pi D \int_0^a \frac{1}{r} \left[ \frac{d}{dr} \left( \frac{1}{r} \left( \frac{d}{dr} \sigma_{tp} \right) \right) \right] \cdot r \cdot \omega \, dr$$

Eqn 43.

Then we may assume a value for \( \omega \) in terms of \( r \) and a constant, complete the operations and solve for the constant and its relation to \( p \). Assume

$$\omega = \omega_0 \left( 1 - \frac{r^2}{a^2} \right)^2$$

as this is the expression for \( \omega \) when the membrane stresses are neglected. Letting \( r = a S \), and from the form of Eqn (10):

$$S \frac{d}{dS} (\sigma_{tp} + \sigma_{ip}) = -8 \cdot \frac{E \omega_0}{a^2} \left( S^6 - 2 \cdot S^4 - S^2 \right)$$

Integrate both sides in terms of \( S \),

$$\sigma_{tp} + \sigma_{ip} = -8 \cdot \frac{E \omega_0}{a^2} \left( \left( \frac{1}{6} \cdot S^6 - \frac{1}{2} \cdot S^4 + \frac{1}{2} \cdot S^2 + A \right) \right)$$

From (eqn 9),

$$\frac{d}{dr} \left[ \frac{1}{2} \sigma_{tp} \right] - \sigma_{tp} = \sigma_{tp} \quad \text{(Eqn 9)}$$

$$\sigma_{tp} + \frac{d}{dS} (S \sigma_{tp})$$

and

$$\sigma_{tp} + \frac{d}{dS} (S \sigma_{tp}) = \frac{1}{S} \frac{d}{dS} \left( S \cdot \sigma_{tp} \right)$$
Find \( \sigma_r \) and \( \sigma_l \):

\[
(\sigma_{rp}) = -E\cdot\frac{\omega_0}{a^2}\left(\frac{1}{2}A + \frac{B}{S^2} - \frac{2}{3}S^4 + S^2\right)
\]

\[
(\sigma_{rt}) = -E\cdot\frac{\omega_0}{a^2}\left(\frac{5}{2}A - \frac{B}{S^2} + \frac{7}{6}S^6 - \frac{10}{3}S^4 + 3S^2\right)
\]

To determine the constants of integration, we have \( \sigma_{rp} \) and \( \sigma_{tp} \) finite when \( S = 0 \), and \( \sigma_{tp} = \mu \sigma_{rp} \) when \( S = 1 \). Therefore,

\[
(\sigma_{rp}) = E\cdot\frac{\omega_0}{6a^2}\left(\frac{5}{2} - \frac{3\mu}{1 - \mu} - \frac{3}{2}S^6 + 4S^4 - 6S^2\right)
\]

\[
(\sigma_{rt}) = E\cdot\frac{\omega_0}{6a^2}\left(\frac{5}{2} - \frac{3\mu}{1 - \mu} - 7S^2 + 20S^4 - 18S^2\right)
\]

Return to the energy equation (eqn 43)

\[
\pi\cdot D\int_0^a \frac{1}{r}\left[\frac{d}{dr}\left(\frac{d}{dr}\left(\frac{d}{dr}\left(\frac{d}{dr}\left(\omega - r\cdot h\cdot \int_0^a \frac{1}{r}\left[\frac{d}{dr}\left(r\cdot\sigma_{rp}\frac{d}{dr}\left(h\cdot \omega\right)\right]\right)\cdot r\cdot h\cdot \omega\cdot dr\cdot r\cdot r\right)\right)\right)\right)\right] \cdot r\cdot h\cdot \omega\cdot dr\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r\cdot r}\]

Coefficients divined above (by S. Way) used in 3. Fixed and Held Plate condition as $K_3$ and $K_4$.

- \( K_3 := 0.976 \) (at center)
- \( K_4 := 0.476 \) (at edge)