Fermi National Accelerator Laboratory

Mechanical Safety Subcommittee Guideline for Design of Thin Windows Regarding Roark's Edge Condition Coefficient

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An error was found in an edge stress coefficient used to calculate stresses in thin windows. This error is present in "Roark's Formulas for Stress and Strain" 7th and 8th Edition. The 6th Edition is correct. This guideline specially discusses a major difference in regards to a coefficient used in calculating the edge stress in "Roark's Formulas for Stress and Strain" 6th Edition compared to the 7th and 8th Editions. In Chapter 10: Flat Plates under "Circular plates under distributed load producing large deflections," Case 3, which is "Fixed and held. Uniform pressure *q* over entire plate." The coefficient for a fixed edge condition in the 6th Edition[1] K_4 = 0.476 while in the 7th and 8th Edition[2], the coefficient is 1.73 which is significant difference.

Fermilab-TM-1380: "Mechanical Safety Subcommittee Guideline for Design of Thin Windows for Vacuum Vessels"[3] references Roark's 6th edition. TM-1380 is referenced in FESHM FESHM Chapter 5033.1 - Vacuum Window Safety[4].

Case 3 in all three Roark's editions references two documents: APM-56-12: "Bending of Circular Plates with Large Deflection" by Stewart Way[5] and "Vibration Problems in Engineering" by Stephen Timoshenko[6].

Two independent analyses were done which show that the coefficient in the 6th Edition is correct. The first analysis, "Thin Circular Windows Under Uniform Pressure: Equation Analysis" by Erik Voirin (attached) uses the same case as in APM-56-12. Voirin's analysis compares hand calculations and FEA and concludes that K_4 for the edge should be 0.476, not the published value of 1.73 as seen in Roark and Young's 7th and 8th Edition.

The second analysis "Proof of Bending of Circular Plates with Large Deflection" by Michael McGee (attached) uses Way's book. The edge coefficient is derived and found to be K_4 = 0.476. The third author of the 8th Edition of "Roark's Formulas for Stress and Strain", Dr. Ali M. Sadegh[7], has agreed in an email dated Feb. 21, 2017 that in the next edition he will revise the value K_4 back to equal 0.476 and was not sure why it had been changed.

References

1. W. Young, "Roark's Formulas for Stress & Strain," 6th Edition, p. 478, 1989.
2. W. Young; R. Budynas; and A. Sadegh; "Roark's Formulas for Stress & Strain, 8th Edition, p. 464, 2012.
3. J. L. Western, "Mechanical Safety Subcommittee Guideline for Design of Thin Windows for Vacuum Vessels," Fermilab – TM – 1380, Revised November 2014.
4. Fermilab, Fermilab ES&H Manual, Chapter 5033.1, "Vacuum Window Safety."
5. S. Way, Bending of Circular Plates with Large Deflection, Trans. ASME, vol. 56, no. 8, 1934.
6. S. Timoshenko, Vibration Problems in Engineering, p. 319, D. Van Nostrand Company, 1928.
7. A. Sadegh, email, 2/21/17

Thin Circular Windows under Uniform Pressure: Equation Analysis

Erik Voirin - March 21, 2017 - evoirin@fnal.gov - 630-840-5168

Dimensions: Same as case in APM-56-12: Bending of Circular Plates with Large Deflection

$D_{plate} := 4.5in$ $E_{young} := 3\cdot10^{7}psi = 3\times10^{4}\cdot ksi$ $\nu := 0.3$ $Pressure := 10psi$

$t_{plate} := 0.032in$ $r := \frac{D_{plate}}{2} = 2.25\cdot in$ $u_0 := \frac{r}{t_{plate}} = 70.313$

Equations in TM1380 and Roark and Young 6th Ed. for fixed edge.

$K_1 := \frac{5.33}{1-\nu^2} = 5.857$ $K_2 := \frac{2.6}{1-\nu^2} = 2.857$ $K_3 := \frac{2}{1-\nu} = 2.857$ $K_4 := 0.976$

$K_{3edge} := \frac{4}{1-\nu^2} = 4.396$ $K_{4edge} := 0.476$

Deflection at Center

$$\frac{Pressure\cdot r^4}{E_{young}\cdot t_{plate}^{\,4}} = K_1\cdot\frac{\delta}{t_{plate}} + K_2\cdot\left(\frac{\delta}{t_{plate}}\right)^3$$

$\delta := Find(\delta) = 0.03071\cdot in$

TM1380: Equation 5.1b

Stress at Center (Membrane + Bending) *TM1380: Equation 5.1a*

$$\sigma_{center} := E_{young}\cdot\left(\frac{t_{plate}}{r}\right)^2\cdot\left[K_3\cdot\left(\frac{\delta}{t_{plate}}\right) + K_4\cdot\left(\frac{\delta}{t_{plate}}\right)^2\right] = 22.095\cdot ksi$$

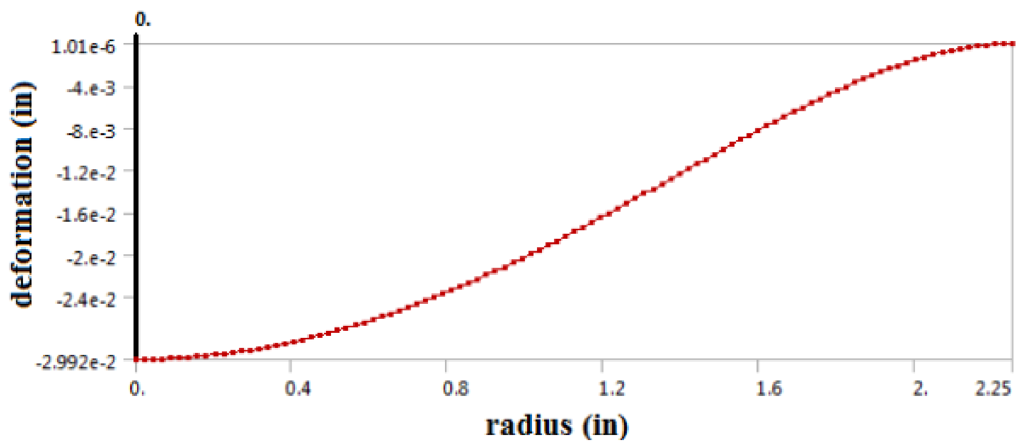
$\sigma_{FEACenter} := 19.308ksi$

Stress at Edge (Membrane + Bending)

$$\sigma_{edge} := E_{young}\cdot\frac{t_{plate}^{\,2}}{r^2}\cdot\left[K_{3edge}\cdot\frac{\delta}{t_{plate}} + K_{4edge}\cdot\left(\frac{\delta}{t_{plate}}\right)^2\right] = 28.26\cdot ksi$$

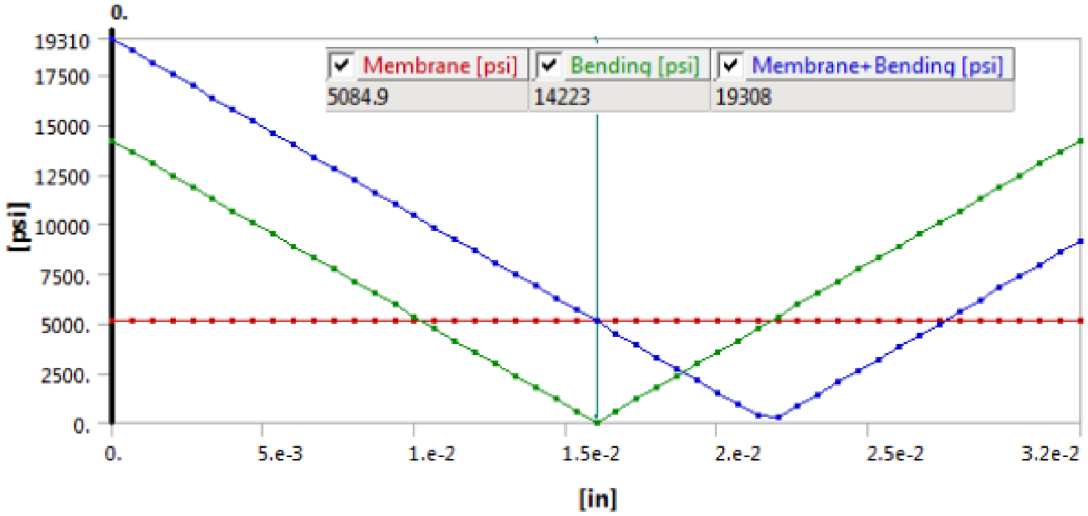
$\sigma_{FEAEdge} := 24.261ksi$

FEA Results for thin circular window with non-linear deformation:



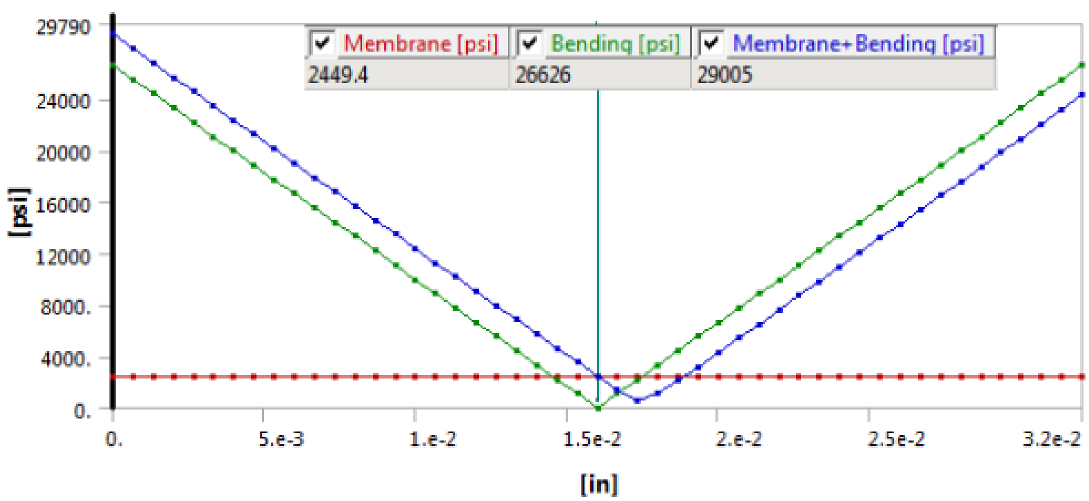
FEA Gives: $\delta_0 := 0.02992in$ $\frac{\delta_0}{t_{plate}} = 0.935$

Membrane and Bending Stress at Center of Plate



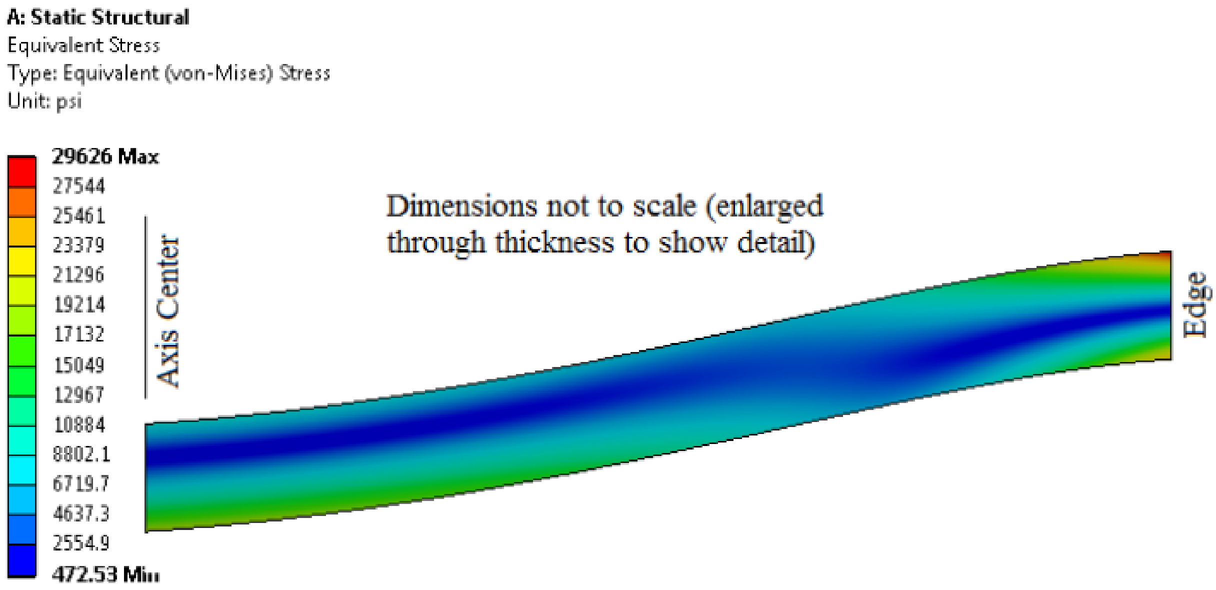
$\sigma_{FEAMembraneCenter} := 5084.9psi$ $\sigma_{FEABendingCenter} := 14223psi$ $\sigma_{FEABendPlusMemCenter} := 19308psi$

Membrane and Bending Stress at Edge of Plate



$\sigma_{FEAMembraneEdge} := 2449.4psi$ $\sigma_{FEABendingEdge} := 26626psi$ $\sigma_{FEABendPlusMemEdge} := 29005psi$

FEA Von Mises Stress



Equations from APM-56-12 Bending of Circular Plates with Large Deflection

Membrane Stresses

$$\sigma_{MemCenter} := 0.976 \cdot \left(\frac{\delta}{t_{plate}}\right)^2 \cdot E_{young} \cdot \left(\frac{t_{plate}}{r}\right)^2 = 5455 \cdot psi$$ *FEA Gives:* $\sigma_{FEAMembraneCenter} = 5085 \cdot psi$

$$\sigma_{EMemEge} := 0.476 \cdot \left(\frac{\delta}{t_{plate}}\right)^2 \cdot E_{young} \cdot \left(\frac{t_{plate}}{r}\right)^2 = 2661 \cdot psi$$ *FEA Gives:* $\sigma_{FEAMembraneEdge} = 2449.4 \cdot psi$

Good agreement with $K_4 = 0.476$, which states the value is conservative.

Other form of equation in Roark & Young 7th and 8th editions: (Wrong K_4 value of 1.73 for edge)

$$\sigma_{MembraneEdgeWRONG} := 1.73 \cdot \left(\frac{\delta}{t_{plate}}\right)^2 \cdot E_{young} \cdot \left(\frac{t_{plate}}{r}\right)^2 = 9670 \cdot psi$$ *No agreement with anything.*

$$\sigma_{edgeWRONG} := E_{young} \cdot \frac{t_{plate}^2}{r^2} \cdot \left[K_{3edge} \cdot \frac{\delta}{t_{plate}} + 1.73 \cdot \left(\frac{\delta}{t_{plate}}\right)^2\right] = 35.269 \cdot ksi$$

$$\sigma_{edgeCorrect} := E_{young} \cdot \frac{t_{plate}^2}{r^2} \cdot \left[K_{3edge} \cdot \frac{\delta}{t_{plate}} + 0.476 \cdot \left(\frac{\delta}{t_{plate}}\right)^2\right] = 28.26 \cdot ksi$$ $\sigma_{FEABendPlusMemEdge} = 29.005\,ksi$

We have concluded that K4 for the edge should be 0.476, not the published value of 1.73 as seen in R&Y editions 7 and 8.

Proof of Bending of Circular Plates With Large Deflection

M. McGee February 2017

Ref. [1]: Stewart Way, "Bending of Circular Plates With Large Deflection"

Nomenclature

a = radius
h = thickness
p = load intensity, assumed uniform
σr, σr', σr" = radial stresses
σt, σt', σt" = circumferential stresses
z = distance from middle surface, downward direction positive
r = distance from the axis of symmetry to a point in the plate before deflection
ω = vertical displacement of points of the middle surface relative to the center of the middle surface
E = Young's modulus
μ = Poisson's ratio

Consider methods by G.B. Galerkin: Displacement ω perpendular to plate and displacement ρ (radial direction)

Symbolically $E(\omega) \equiv 0$ $E(\rho) \equiv 0$

Integrate over the entire circular plate

$$\int E(\omega)\cdot\delta\omega\, dA \equiv 0 \qquad \int E(\rho)\cdot\delta\rho\, dA \equiv 0$$

Substitute into arbitrary functions that will satisfy boundary conditions and find the constants. Considering the membrane stresses at the circular plate center (of maximum stress).

The condition of vertical equilibrium of a disk of the plate of radius r:

$$D\cdot\left[\frac{d}{dr}\left[\frac{1}{r}\cdot\left[\frac{d}{dr}(r\cdot\varphi\right]\right]\right] \equiv p\left(\frac{r}{2}\right) + h\cdot\left(\sigma_{rp}\right) + \ldots \qquad \text{eqn (8) ref [1].}$$

Multiply eqn (8) by r, differentiating and dividing by r:

$$p \equiv \frac{D}{r}\cdot\left[r\cdot\frac{d}{dr}\left[\frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\left(\frac{d}{dr}\omega\right)\right]\right]\right]\right] - \frac{h}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\sigma_{rp}\cdot\left(\frac{d}{dr}\omega\right)\right]\right] \qquad \text{eqn (42a)}$$

Differentiate and use relation (eqn 9) between σrp and σtp)

$$p \equiv \frac{D}{r}\cdot\left[r\cdot\frac{d}{dr}\left[\frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\left(\frac{d}{dr}\omega\right)\right]\right]\right]\right] + h\cdot\left(\frac{\sigma_{rp}}{R_r} + \frac{\sigma_{tp}}{R_t}\right) \qquad \text{eqn (42b)}$$

where, Rr & Rt are radial and tangential radii of curvature of the middle surface.

1

If no membrane stresses exist, the third term would go to 0 and external work is defined as

$$\pi \cdot p \cdot \int_0^a \omega \cdot r \, dr$$

and equivalent strain energy of bending is

$$\pi \cdot D \cdot \int_0^a \frac{1}{r} \cdot \left[\frac{d}{dr}\left[\frac{1}{r} \cdot \left[\frac{d}{dr}\left[r \cdot \left(\frac{d}{dr}\omega\right)\right]\right]\right]\right] \cdot \omega \cdot r \, dr$$

where load is portional to deflection

If membrane stresses are considered, and we neglect all displacements except those normal to the middle surface, e/ω = 1/R, and the strain energy of stretching is expressed by multiplying the third term above by 1/2 ω and integrating over the entire plate. The total external work is no longer

$$\pi \cdot p \cdot \int_0^a \omega \cdot r \, dr$$

However, since the strain energy of stretching is a small part of the total, roughly 20% when the deflection at the center equals the plate thickness, we assume that the straight line relation approximation still holds. Giving,

$$\pi \cdot D \cdot \int_0^a \frac{1}{r} \cdot \left[\frac{d}{dr}\left[\frac{1}{r} \cdot \left[\frac{d}{dr}\left[r \cdot \left(\frac{d}{dr}\omega\right)\right]\right]\right]\right] \cdot \omega \cdot r \, dr - \pi \cdot h \cdot \int_0^a \frac{1}{r} \cdot \left[\frac{d}{dr}\left[r \cdot \sigma_{rp} \cdot \left(\frac{d}{dr}\omega\right)\right]\right] \cdot r \cdot \omega \, dr \qquad \text{Eqn 43.}$$

Then we may assume a value for ω in terms of r and a constant, complete the operations and solve for the constant and its relation to p. Assume

$$\omega \equiv \omega_o \cdot \left(1 - \frac{r^2}{a^2}\right)^2$$

as this is the expression for ω when the membrane stresses are neglected. Letting r = aS, and from the form of Eqn (10):

$$S \cdot \frac{d}{dS}\left(\sigma_{rp} + \sigma_{tp}\right) \equiv -8 \cdot \frac{E\omega_o^{\,2}}{a^2} \cdot \left(S^6 - 2 \cdot S^4 + S^2\right)$$

Integrate both sides in terms of S,

$$\sigma_{rp} + \sigma_{tp} \equiv -8 \cdot \frac{E\omega_o^{\,2}}{a^2} \cdot \left(\left(\frac{1}{6} \cdot S^6 - \frac{1}{2} \cdot S^4 + \frac{1}{2} \cdot S^2 + A\right)\right)$$

From (eqn 9),

$$\frac{d}{dr}\left[\left(\left(r \cdot \sigma_{rp}\right)\right)\right] - \sigma_{tp} = \blacksquare \cdot 0 \qquad \text{(Eqn 9)}$$

$$\sigma_{rp} + \sigma_{tp} \equiv \sigma_{rp} + \frac{d}{dS}\left(S \cdot \sigma_{rp}\right) \qquad \text{and} \qquad \sigma_{rp} + \frac{d}{dS}\left(S \cdot \sigma_{rp}\right) \equiv \frac{1}{S} \cdot \left[\frac{d}{dS}\left(S^2 \cdot \sigma_{rp}\right)\right]$$

2

Find σr'and σt':

$$(\sigma rp) \equiv -E\cdot\frac{\omega_o^{\,2}}{a^2}\cdot\left(\frac{1}{2}\cdot A+\frac{B}{S^2}-\frac{2}{3}\cdot S^4+S^2\right)$$

$$\sigma_{rt} \equiv -E\cdot\frac{\omega_o^{\,2}}{a^2}\cdot\left(\frac{1}{2}\cdot A-\frac{B}{S^2}+\frac{7}{6}\cdot S^6-\frac{10}{3}\cdot S^4+3\cdot S^2\right)$$

To determine the constants of integration, we have σrp and σtp finite when S = 0, and σtp = μσrp when S = 1. Therefore,

$$\sigma_{rp} \equiv E\cdot\frac{\omega_o^{\,2}}{6\cdot a^2}\cdot\left(\frac{5-3\cdot\mu}{1-\mu}-\frac{2}{3}\cdot S^6+4\cdot S^4-6\cdot S^2\right)$$

$$\sigma_{rt} \equiv E\cdot\frac{\omega_o^{\,2}}{6\cdot a^2}\cdot\left(\frac{5-3\cdot\mu}{1-\mu}-7\cdot S^2+20\cdot S^4-18\cdot S^2\right)$$

Return to the energy equation (eqn 43)

$$\pi\cdot D\cdot\int_0^a \frac{1}{r}\cdot\left[\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}\left[r\cdot\left(\frac{d}{dr}\omega\right)\right]\right]\right]\cdot\omega\cdot r\,dr-\pi\cdot h\cdot\int_0^a \frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\sigma_{rp}\cdot\left(\frac{d}{dr}\omega\right)\right]\right]\cdot r\cdot\omega\,dr \qquad \text{Eqn 43.}$$

and substituting the values of ω and σrp, the first gives;

$$\frac{\pi\cdot p\cdot a^2\cdot b}{6} \quad \text{and} \quad \frac{32\cdot\pi\cdot b^2\cdot D}{3\cdot a^2} \quad \text{and} \quad \frac{-2\cdot\pi\cdot b^4\cdot h\cdot E}{3\cdot a^2}\cdot\left[\frac{23-9\cdot\mu}{42\cdot(1-\mu)}\right]$$

Reduce these to the same form of eqn 33 and eqn 35 and μ = 0.25

$$\frac{\omega_o}{h}+0.463\cdot\left(\frac{\omega_o}{h}\right)^2 \equiv \frac{3}{16}\cdot\frac{p}{E}\cdot\left(\frac{a}{h}\right)^4\cdot\left(1-\mu^2\right)...$$

Furthermore, when S is given the values 0 and 1 in eqn 44 and 45, and μ = 0.3

Therefore, membrane stress at the center

$$\frac{\sigma\cdot\mu_o^{\,2}}{E} \equiv 0.976\cdot\left(\frac{w_o}{h}\right)^2$$

at the edge

$$\frac{\sigma_p\cdot r_o\cdot\mu_o^{\,2}}{E} \equiv 0.476\cdot\left(\frac{w_o}{h}\right)^2$$

3

Ref [2]: Roark & Young, Edition 6, p. 477 & 478, under 3. Fixed and held condition.

Coefficients dirvired above (by S. Way) used in 3. Fixed and Held Plate condition as K3 and K4.

(at center) $K_3 := 0.976$

(at edge) $K_4 := 0.476$

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