HOM resistor power requirement

C.Y. Tan*

Fermi National Accelerator Laboratory,

P.O. Box 500, Batavia, IL 60510-5011, USA.

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Abstract

I want to derive the required power for the HOM resistors for the HOM modes of the cavity.

* cytan@fnal.gov

I. THEORY

The first thing to do is to calculate the spectrum of the beam in Booster. The most obvious thing is that the beam is **not** even in intensity in every bucket and there is a 3 bucket notch as well. So, we cannot expect the spacing of the spectrum be "δ-functions" that are separated by the RF frequency. Instead, I have to derive how the spectrum looks like.

I am going to do this in the steps that I have in mind right now:

A. Single bunch

If there is only one single bunch, that has charge q, in Booster then it is clearly at the wall current monitor, I have

$$\begin{aligned} I(t) &= q \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}}) \\ &= \frac{q}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} \end{aligned} \tag{1}$$

where T_{rev} is the period of revolution and $\omega_{\text{rev}} = 2\pi/T_{\text{rev}}$ is the angular revolution frequency.

The above is, of course, a very well known result that tells us that in Fourier space, the spectrum are δ-functions separated by the revolution frequency.

B. Two bunches

Now instead of one bunch, I have two bunches in Booster. The second bunch is spaced Δt from the first bunch. Also, I will assume that the charges are not equal between the two bunches. Bunch 1 has charge q_1 and bunch 2 has charge q_2. Then clearly, I have the following contributions by each bunch to the wall current monitor:

$$\begin{aligned} I_1(t) &= q_1 \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}}) \\ &= \frac{q_1}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} \end{aligned} \tag{2}$$

$$
\begin{aligned}
I_2(t) &= q_2 \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}} - \Delta t) \\
&= \frac{q_2}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}(t-\Delta t)} \\
&= \frac{q_2}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} e^{-in\omega_{\text{rev}} \Delta t}
\end{aligned}
\tag{3}
$$

And thus the total beam current seen on the wall current monitor is simply the superposition of currents I_1 and I_2, i.e.

$$
I(t) = I_1(t) + I_2(t) \tag{4}
$$

or in terms of sinusoids, I have

$$
I(t) = \frac{q_1}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} + \frac{q_2}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} e^{-in\omega_{\text{rev}} \Delta t} \tag{5}
$$

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1. Check

Now, before I continue, I want to see if the result is consistent for having two bunches of equal intensity that separated equally in Booster. This means that $q_1 = q_2 = q$ and $\Delta t = T_{\text{rev}}/2$. Substituting these values into Eq. 4, I get

$$
\begin{aligned}
I(t) &= q \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}}) + q \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}} - T_{\text{rev}}/2) \\
&= q \left(\sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}}) + \sum_{k=-\infty}^{\infty} \delta\left[t - \left(k + \frac{1}{2}\right) T_{\text{rev}}\right] \right)
\end{aligned}
\tag{6}
$$

which obviously shows that the δ-functions are spaced $T_{\text{rev}}/2$ apart, i.e.

$$
I(t) = q \sum_{k=-\infty}^{\infty} \delta(t - kT_{\text{rev}}/2) \tag{7}
$$

which tells me that the spectrum in Fourier space consists of δ-functions that are spaced $\frac{2\pi}{T_{\text{rev}}/2} = 2\omega_{\text{rev}}$ apart as expected. This can be checked by using Eq. 5,

$$
\begin{aligned}
I(t) &= \frac{q}{T_{\text{rev}}} \left(\sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} + \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}} t} e^{-in\pi} \right) \\
&= \frac{2q}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{i2n\omega_{\text{rev}} t} = \frac{q}{T_{\text{rev}}/2} \sum_{n=-\infty}^{\infty} e^{in(2\omega_{\text{rev}})t}
\end{aligned}
\tag{8}
$$

because the odd terms in n cancel betwen the infinite sums, leaving only the even terms in n to add.

Since everything checks out above, I will continue from Eq. 5,

$$I(t) = \frac{1}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \left(q_1 + q_2 e^{-in\omega_{\text{rev}}\Delta t}\right) \tag{9}$$

which I can generalize! See the next section.

C. Generalization

I can generalize from two bunches to $N = 84$ buckets in Booster. Now, whether there are 84 or 81 bunches (with a 3 bunch notch) in Booster is not relevant in the analysis below. This is because whether the bucket is filled or not comes from the value of the charge q_k in the bucket. In either case, I have, from Eq. 9,

$$\begin{aligned} I(t) &= \frac{1}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \left(q_1 + q_2 e^{-in(1\times 2\pi/N)} + q_3 e^{-in(2\times 2\pi/N)} + \ldots + q_{84} e^{-in(83\times 2\pi/N)}\right) \\ &= \frac{1}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} q_k e^{-in(k-1)2\pi/N} \end{aligned} \tag{10}$$

where I have made $\Delta t = T_{\text{rev}}/N$ which means that $\omega_{\text{rev}}\Delta t = \omega_{\text{rev}} T_{\text{rev}}/N = 2\pi/N$.

Eq. 10 tells me that, in general, the spectrum of $I(t)$ are δ-functions that are spaced ω_{rev} apart and the "strength" of the δ-function at $n\omega_{\text{rev}}$ (note: n can be negative here) is given by $\sum_{k=1}^{84} q_k e^{-in(k-1)2\pi/N}$.

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1. *Check*

My sanity check of Eq. 10 can be done by making every bunch have the same charge, i.e. $q_k = q$. This means that

$$I(t) = \frac{q}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} e^{-in(k-1)2\pi/N} \tag{11}$$

The second sum is a geometric series and has an analytic solution when n is not an integer multiple of N (second line on the rhs below)

$$\sum_{k=1}^{N=84} e^{-in(k-1)2\pi/N} = \begin{cases} N & \text{if } n = pN,\ p \in \mathbb{Z} \\ \dfrac{1-e^{-i2n\pi}}{1-e^{-i2n\pi/N}} = 0 & \text{otherwise} \end{cases} \tag{12}$$

Therefore, $I(t)$ becomes

$$I(t) = \frac{Nq}{T_{\text{rev}}} \sum_{p=-\infty}^{\infty} e^{ipN\omega_{\text{rev}}t} = \frac{q}{T_{\text{RF}}} \sum_{p=-\infty}^{\infty} e^{ip\omega_{\text{RF}}t} \tag{13}$$

where $T_{\text{RF}} = T_{\text{rev}}/N$ is the RF period and $\omega_{\text{RF}} = N\omega_{\text{rev}}$ is the RF angular frequency. Again, in Fourier space, I have δ-functions that are spaced ω_{RF} apart.

D. All buckets filled with nearly the same charge

If all the buckets are filled with nearly the same charge, I have

$$q_k = q_0(1+\epsilon_k) \tag{14}$$

where $q_0 = Q/N$ if all the buckets are equally filled from a total charge Q and ϵ_k is the variation parameter. The requirement on ϵ_k is that

$$\sum_{k=1}^{N} \epsilon_k = 0 \tag{15}$$

so that Eq. 14 sums to the total charge in Booster

$$\sum_{k=1}^{N} q_k = \sum_{k=1}^{N} q_0(1+\epsilon_k) = Nq_0 = Q \tag{16}$$

I will substitute Eq. 14 into Eq. 10 to get

$$\begin{aligned} I(t) &= \frac{q_0}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} (1+\epsilon_k) e^{-in(k-1)2\pi/N} \\ &= \frac{q_0}{T_{\text{rev}}} \left(\sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} e^{-in(k-1)2\pi/N} + \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} \epsilon_k e^{-in(k-1)2\pi/N} \right) \end{aligned} \tag{17}$$

The first sum is exactly Eq. 13 and the second gives me the size of the revolution harmonics that are not also multiples of the RF frequency

$$I(t) = \frac{q_0}{T_{\text{RF}}} \sum_{p=-\infty}^{\infty} e^{ip\omega_{\text{RF}}t} + \frac{q_0}{T_{\text{rev}}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\text{rev}}t} \sum_{k=1}^{N=84} \epsilon_k e^{-in(k-1)2\pi/N} \tag{18}$$

1. *The revolution harmonics term*

Let me concentrate on the revolution harmonics term in Eq. 18

$$I_\epsilon(t) = \frac{q_0}{T_{\rm rev}} \sum_{n=-\infty}^{\infty} e^{in\omega_{\rm rev}t} \sum_{k=1}^{N=84} \epsilon_k e^{-in(k-1)2\pi/N} \tag{19}$$

which in general can only be calculated numerically. Let's see whether I can do more by considering a special case.

2. *Special case*

I will look at the nth revolution harmonic. It has the following strength

$$I_\epsilon(n) = \frac{q_0}{T_{\rm rev}} \sum_{k=1}^{N=84} \epsilon_k e^{-in(k-1)2\pi/N} \tag{20}$$

I am going to take a stab at calculating I_ϵ for the special case when $|\epsilon_k| = \varepsilon$, this means that

$$I_\epsilon(n) = \frac{q_0}{T_{\rm rev}}\varepsilon \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} \tag{21}$$

where I have introduced $\mathcal{P}_k$ that has a 50/50 chance for being either -1 or 1 with the requirement that $\sum_{k=1}^{N} \mathcal{P}_k = 0$ because of Eq. 15.

For $n = pN$, i.e. multiples of the harmonic number,

$$I_\epsilon(pN) = \frac{q_0}{T_{\rm rev}}\varepsilon \sum_{k=1}^{N=84} \mathcal{P}_k e^{-ip(k-1)2\pi} = \frac{q_0}{T_{\rm rev}}\varepsilon \sum_{k=1}^{N=84} \mathcal{P}_k = 0 \tag{22}$$

This means that there are no corrections to the strengths of the δ-functions of the harmonics of the RF and thus power from these RF harmonics.

For the other revolution harmonics, the only way to calculate the value of I_ϵ is numerically. However, since every injection into Booster has a different current distribution, I can find the average power from an infinite number of Booster injections to calculation the power requirement for the HOM resistor.

3. *Special case: power in revolution harmonics*

I will continue to use the special case discussed above for the case when n is not a multiple of the harmonic number N. The time averaged power $\langle P_\epsilon \rangle_m$ for the nth revolution harmonic for injection m is given by

$$\langle P_n \rangle_m = \langle \mathrm{Re}[I_\epsilon(n)] \times \mathrm{Re}[V_\epsilon(n)] \rangle_m \tag{23}$$

since only the real parts of I_ϵ and V_ϵ contribute to any power loss.

From Eq. 21 and including the sinusoidal part, I have

$$I_\epsilon(n;t) = \frac{q_0}{T_{\mathrm{rev}}} e^{in\omega_{\mathrm{rev}}t} \varepsilon \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} \equiv |J_n| e^{i\theta_n} e^{in\omega_{\mathrm{rev}}t} \tag{24}$$

where $J_n = \frac{q_0}{T_{\mathrm{rev}}} \varepsilon \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} = |J_n| e^{i\theta_n}$. The voltage, $V_\epsilon(n)$ is

$$V_\epsilon(n) = I_\epsilon(n) Z(n\omega_{\mathrm{rev}}) = |J_n| e^{i\theta_n} e^{in\omega_{\mathrm{rev}}t} \times |Z(n\omega_{\mathrm{rev}})| e^{i\phi_n} \tag{25}$$

where $Z(\omega)$ is the impedance seen by the beam current.

I can multiply $I_\epsilon(n)$ and $V_\epsilon(n)$ to get

$$I_\epsilon(n) V_\epsilon(n) = |J_n|^2 |Z(n\omega_{\mathrm{rev}})| e^{i[2(n\omega_{\mathrm{rev}}t + \theta_n) + \phi_n]} \tag{26}$$

and the time average of the real part of the above is

$$\begin{aligned} \langle \mathrm{Re}[I_\epsilon(n)] \mathrm{Re}[V_\epsilon(n)] \rangle &= |J_n|^2 |Z(n\omega_{\mathrm{rev}})| \langle \cos(n\omega_{\mathrm{rev}}t + \theta_n) \cos(n\omega_{\mathrm{rev}}t + \theta_n + \phi_n] \rangle \\ &= \frac{|J_n|^2 |Z(n\omega_{\mathrm{rev}})|}{2} \cos\phi_n \\ &= \frac{|J_n|^2 \mathrm{Re}[Z(n\omega_{\mathrm{rev}})]}{2} \end{aligned} \tag{27}$$

which is another way of deriving the rms power. Note: $|Z|\cos\phi = R$

Now, like I said above, I cannot really find a simple formula for $|J_n|$ for each Booster injection. However, for an infinite number of injections, I can calculate the mean $\langle |J_n| \rangle$ because the bunch current variations are random.

4. Special case: $\langle |J_n|^2 \rangle$

For the special case that I am considering, I have found numerically that

$$\langle |J_n|^2 \rangle = \frac{q_0}{T_{\mathrm{rev}}} \varepsilon \left\langle \left| \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} \right|^2 \right\rangle = 84 \left(\frac{q_0}{T_{\mathrm{rev}}} \right)^2 \varepsilon^2 \tag{28}$$

is independent of n. I'll prove the above in the appendix.

TABLE I. HOM modes of the 2nd harmonic cavity at injection

Mode	**Frequency (MHz)**	**Shunt impedance kΩ**	**Q**
1	142.1	3.7	616
2	189.1	6.7	211
4	293.6	6.3	172

5. Numerical example: power from beam RF harmonics

I have to sum the power contributed from each HOM. The HOM frequencies, Q's and shunt impedances at injection are summarized inTable I, i.e. when the fundamental of the 2nd harmonic cavity is at 75.7 MHz. Mode 3 is not listed because the stretched wire does not excite it. These measurments can be found in Table 15 of our writeup. 6 dB attenuators were connected to the HOM cavity to obtain these values.

Since this is just a back of the envelope calculation, I am going to assume that the shunt impedance and Q of each mode remains constant during the ramp. This is a reasonable assumption because for the present operating condition that is constrained to injection only, the modes reach their maximum excursion at 3 ms and then return back to their injection frequencies. See section 10.1 of writeup. Also, since the Q and shunt impedance is also the worst at injection, this will be a worst case calculation. The measured evolution of the modes up to 5 ms are shown in dashed lines in Fig. 1. The red dashed lines show the width of the mode ($\Delta f = f_{\rm res}/Q$) and the situation where the mode frequency stays constant.

The current that comes from the RF harmonic for an infinite number of turns is given by the coefficient of the $e^{\pm i p \omega_{\rm RF} t}$ term from the first sum of Eq. 18, i.e.

$$I_{\rm RF} = \frac{2q_0}{T_{\rm RF}} \tag{29}$$

where the "2" comes from the positive and negative harmonics. Therefore, the rms power for any of the modes that the RF harmonics cross is

$$P_{\rm RF} = I_{\rm RF}^2 R/2 = 2\left(\frac{q_0}{T_{\rm RF}}\right)^2 R \tag{30}$$

Note that $T_{\rm RF}$ clearly depends on where on the RF ramp the harmonic crosses the mode.

Again, since this is the worst case back of the envelope calculation, I will make set $T_{\rm RF}$ is at its minimum. And for Booster, this is when the RF frequency is at 53 MHz. Injection

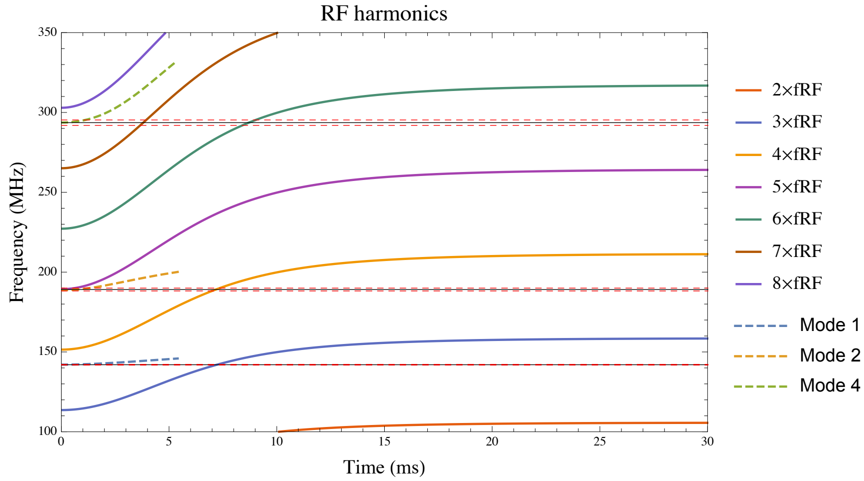

FIG. 1. The measured modes (dashed lines) superimposed onto RF harmonics during the Booster ramp. The red dashed lines show the width of the mode if it is held constant during the ramp.

Q's and shunt impedances are also the worst.

In the worst case, using Table I, and setting the charge per bunch $q_0 = 6 \times 10^{12} \times 1.6 \times 10^{-19}/84$ C $= 1.1 \times 10^{-8}$ C, I can calculate the power dumped into the HOM when the RF harmonic sits indefinitely on the mode

$$\left.\begin{array}{l} \text{mode 1} : P_{\text{mode1}} = 2\left(\dfrac{q_0}{T_{\text{RF}}}\right)^2 R = 2\left(\dfrac{1.1\times 10^{-8}\text{ C}}{1/(53\times 10^6\text{ Hz})}\right)^2 \times (3700\ \Omega) = 2.7\text{ kW} \\ \text{mode 2} : P_{\text{mode2}} = 2\left(\dfrac{q_0}{T_{\text{RF}}}\right)^2 R = 2\left(\dfrac{1.1\times 10^{-8}\text{ C}}{1/(53\times 10^6\text{ Hz})}\right)^2 \times (6700\ \Omega) = 4.9\text{ kW} \\ \text{mode 4} : P_{\text{mode4}} = 2\left(\dfrac{q_0}{T_{\text{RF}}}\right)^2 R = 2\left(\dfrac{1.1\times 10^{-8}\text{ C}}{1/(53\times 10^6\text{ Hz})}\right)^2 \times (6300\ \Omega) = 4.6\text{ kW} \end{array}\right\} \quad (31)$$

This above solution only applies to the beam that sits on the HOMs indefinitely. But in reality, they do not. So, when I zoom into Fig. 1, each mode crosses the RF harmonic at most twice, and each time about 0.5 ms. For simplicity, if I assume 1 ms total crossing time for each mode, then I have the duty factor

$$\eta = (1 \times 10^{-3}\text{ s}) \times (15\text{ Hz})/1\text{ s} = 0.015 \quad (32)$$

Therefore, the power that is deposited in the HOM resistors *in the worst case* for this duty factor is

$$P_{\text{RF}}(1.5\%) = \eta(P_{\text{mode1}} + P_{\text{mode2}} + P_{\text{mode4}}) = 0.015 \times (2.7 + 4.9 + 4.6)\text{ kW} = 183\ \ \text{W} \quad (33)$$

Since we have 4 HOM resistors, the power into each resistor is about 45 W if the mode is 4-fold symmetric. If it is a dipole mode, then the power is dumped into two resistors. The power is thus 90 W. So 150 W resistors are more than sufficient for our purposes.

6. *Numerical example: power from beam revolution harmonics*

In this calculation, I will assume that the bunch variation is ±10%, thus $\eta = 0.1$.

I use these numbers which I substitute into Eq. 27 with the averaged power for each revolution harmonic to get

$$P_{\text{harmonics}} = \sum_n \frac{\langle |J_n|^2 \rangle \text{Re}[Z(n\omega_{\text{rev}})]}{2} = 42 \left(\frac{q_0}{T_{\text{rev}}} \right)^2 \epsilon^2 \sum_n \text{Re}[Z(n\omega_{\text{rev}})] \approx 0.7 \text{ W} \tag{34}$$

where I have summed over 600 revolution harmonics centred around each HOM. See `hom_resistor1.nb` for how $P_{\text{harmonics}}$ came about.

Reminder: The above is for an infinite number of turns averaged over an infinite number of injections.

Therefore, even without taking into account the duty factor and the negative harmonics, the power dumped into the HOM resistors from these revolution harmonics are small when compared to the RF harmonics and can be neglected.

Appendix A: Simplifying $\left\langle \left| \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} \right|^2 \right\rangle$

I want to simplify

$$s = \left\langle \left| \sum_{k=1}^{N=84} \mathcal{P}_k e^{-in(k-1)2\pi/N} \right|^2 \right\rangle \tag{A1}$$

which after expanding is

$$s = \left\langle \left(\sum_{k=1}^{N} \mathcal{P}_k \cos \frac{2\pi n(k-1)}{N} \right)^2 + \left(\sum_{k=1}^{N} \mathcal{P}_k \sin \frac{2\pi n(k-1)}{N} \right)^2 \right\rangle \tag{A2}$$

Since I am going to take an infinite average, the cross terms in the above expansion should vanish because they consist of terms that look like $\mathcal{P}_i\mathcal{P}_j \cos 2\pi n(i-1)/N \cos 2\pi n(j-1)/N$ and $\mathcal{P}_i\mathcal{P}_j \sin 2\pi n(i-1)/N \sin 2\pi n(j-1)/N$ which by construction

$$\sum_{ij} \mathcal{P}_i\mathcal{P}_j = 0 \quad \text{if } i \neq j \tag{A3}$$

Thus

$$
\begin{aligned}
s &= \left\langle \sum_{k=1}^{N} \mathcal{P}_k^2 \cos^2 \frac{2\pi n(k-1)}{N} + \sum_{k=1}^{N} \mathcal{P}_k^2 \sin^2 \frac{2\pi n(k-1)}{N} \right\rangle \\
&= \langle \sum_{k=1}^{N} \mathcal{P}_k^2 \rangle \\
&= N = 84
\end{aligned}
\tag{A4}
$$

The last two lines come from recalling that $\mathcal{P} = \pm 1$. Notice that this result is independent of n.

I have checked that the above result numerically using Mathematica. See `harmonic_size.nb`.