Thin 1-D sextupole in X

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First, a thin sextupole kick (q1 - strength) and a linear transformation

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & \frac{1}{b} \end{pmatrix} \cdot \begin{bmatrix} x \\ px + q1\cdot\left(x^2 - y^2\right) \\ y \\ py - 2\cdot q1\cdot y\cdot x \end{bmatrix} = \begin{bmatrix} a\cdot x \\ \frac{px}{a} + \frac{q1\cdot\left(x^2 - y^2\right)}{a} \\ b\cdot y \\ \frac{py - 2\cdot q1\cdot x\cdot y}{b} \end{bmatrix}$$

Second thin sextupole (q2) and a linear transformation

$$\begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \cdot \begin{bmatrix} a\cdot x \\ \frac{px}{a} + \frac{q1\cdot\left(x^2 - y^2\right)}{a} + q2\cdot\left[\left[(a\cdot x)^2 - (b\cdot y)^2\right]\right] \\ b\cdot y \\ \frac{py - 2\cdot q1\cdot x\cdot y}{b} - q2\cdot 2\cdot b\cdot y\cdot(a\cdot x) \end{bmatrix}$$

$$\begin{pmatrix} x \\ q2\cdot a^3\cdot x^2 - q2\cdot a\cdot b^2\cdot y^2 + q1\cdot x^2 - q1\cdot y^2 + px \\ y \\ py - 2\cdot a\cdot q2\cdot x\cdot y\cdot b^2 - 2\cdot q1\cdot x\cdot y \end{pmatrix}$$

Now, set x-y coupling to zero:

$$q1 = -q2 \cdot a \cdot b^2$$

$$\begin{pmatrix} x \\ q2 \cdot a^3 \cdot x^2 - q2 \cdot a \cdot b^2 \cdot y^2 + q1 \cdot x^2 - q1 \cdot y^2 + px \\ y \\ py - 2 \cdot a \cdot q2 \cdot x \cdot y \cdot b^2 - 2 \cdot q1 \cdot x \cdot y \end{pmatrix}$$

$$\begin{bmatrix} x \\ a \cdot q2 \cdot \left(a^2 - b^2\right) \cdot x^2 + px \\ y \\ py \end{bmatrix}$$

Notice, that both a and b can be negative or positive

A concept of how to make the required matrix with thin quadrupoles

$$\begin{pmatrix} 1 & 0 \\ k4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k1 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -k4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k1 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$$

$a := -0.55$ $\qquad$ $b := -1.5$

Obviously, L1, L2, L3 must be positive!

$L1 := 1$ -- scaling factor

$$L2 := -L1 \cdot \frac{(a-b)^2}{2 \cdot (a + b - 2 \cdot a \cdot b)} \qquad L2 = 0.122$$

$$L3 := L1 \cdot \frac{a \cdot b \cdot (a + b - 2)}{a + b - 2 \cdot a \cdot b} \qquad L3 = 0.903$$

$$k1 := \frac{1}{L1} \cdot \left[-\frac{1}{2} + \frac{a \cdot (a - 1)}{b - a} - \frac{(a - 1)^3}{a + b - 2} + \frac{a + a^3}{a + b} \right] \qquad k1 = -1.967$$

$$k2 := \frac{1}{L1} \cdot \frac{a + b - 2}{b - a} \qquad k2 = 4.263$$

$$k3 := \frac{1}{L1} \cdot \frac{(a + b - 2 \cdot a \cdot b)^2}{a \cdot b \cdot (a - b) \cdot (a + b - 2)} \qquad k3 = -4.313$$

$$k4 := \frac{1}{L1} \cdot \frac{-7 \cdot a^2 \cdot b - b^3 + a^3 \cdot (4 \cdot b - 1) + a \cdot b \cdot [8 + b \cdot (-7 + 4 \cdot b)]}{2 \cdot a \cdot b \cdot (a^2 - b^2) \cdot (a + b - 2)} \qquad k4 = 2.336$$

$$M1 := \begin{pmatrix} 1 & 0 \\ k4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k1 & 1 \end{pmatrix}$$

$$M2 := \begin{pmatrix} 1 & 0 \\ -k4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -k1 & 1 \end{pmatrix}$$

$$M1 = \begin{pmatrix} -0.55 & 0 \\ 5.329 \times 10^{-15} & -1.818 \end{pmatrix}$$

$$M2 = \begin{pmatrix} -1.5 & 0 \\ -1.776 \times 10^{-15} & -0.667 \end{pmatrix}$$

To create a reverse matrix $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix}$ reverse the order of elements:

$$\begin{pmatrix} 1 & 0 \\ k1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ k4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix}$$