Canonical perturbation theory for symplectic mappings

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0. INTRODUCTION

Basic definitions

Consider a **mapping (map)** $\mathrm{T} : \mathrm{M} \to \mathrm{M}$ defined by a function f

$$\zeta_{n+1} = f(\zeta_n), \qquad\qquad \zeta_i \in \mathrm{M}.$$

Manifold M can be $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$, etc..

The **trajectory** of ζ_0 is the finite set

$$\{\zeta_0, \mathrm{T}(\zeta_0), \mathrm{T}^2(\zeta_0), \ldots, \mathrm{T}^n(\zeta_0)\}$$

The **orbit** of ζ_0, is a set of all points that can be reached

$$\{\ldots, \mathrm{T}^{-2}(\zeta_0), \mathrm{T}^{-1}(\zeta_0), \zeta_0, \mathrm{T}(\zeta_0), \mathrm{T}^2(\zeta_0), \ldots\}$$

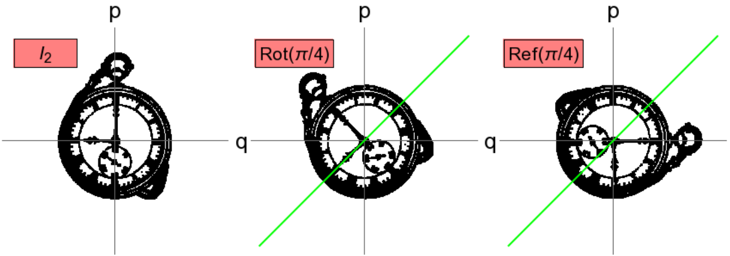
The *n*-**cycle** (or **periodic orbit** of period n) is a solution of

$$\mathrm{T}^n(\zeta_0) = \zeta_0$$

Symplectic mappings of the plane

We will consider area-preserving mappings of the plane

$$q' = q'(q,p), \qquad p' = p'(q,p), \qquad \det \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix} = 1.$$



Identity, Id | Rotation, Rot | Reflection*,**, Ref

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Integrable systems

A map T in the plane is called **integrable**, if there exists a non-constant real valued continuous functions $\mathcal{K}(q,p)$, called **integral**, which is invariant under T:

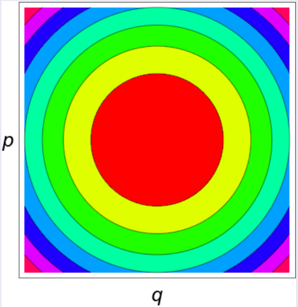
$$\forall\,(q,p): \qquad \mathcal{K}(q,p) = \mathcal{K}(q',p')$$

where primes denote the application of the map, $(q',p') = \mathrm{T}(q,p)$.

Example: Rotation transformation

$$\mathrm{Rot}(\theta): \quad \begin{aligned} q' &= q\,\cos\theta - p\,\sin\theta \\ p' &= q\,\sin\theta + p\,\cos\theta \end{aligned}$$

has the integral $\mathcal{K}(q,p) = q^2 + p^2$.



McMillan form of the map

McMillan considered a special form of the map

$$\boxed{\begin{aligned} \mathrm{M}: \quad q' &= p, \\ p' &= -q + f(p), \end{aligned}}$$

where $f(p)$ is called **force function** (or simply **force**).

a. Fixed point

$$p = q \;\cap\; p = \frac{1}{2}\,f(q).$$

b. 2-cycles

$$q = \frac{1}{2}\,f(p) \;\cap\; p = \frac{1}{2}\,f(q).$$

1D accelerator lattice with thin nonlinear lens, $\mathrm{T} = \mathrm{F} \circ \mathrm{M}$

$$\mathrm{M}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha\,\sin\Phi & \beta\,\sin\Phi \\ -\gamma\,\sin\Phi & \cos\Phi - \alpha\,\sin\Phi \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

$$\mathrm{F}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ F(y) \end{bmatrix},$$

where α, β and γ are Courant-Snyder parameters at the thin lens location, and, Φ is the betatron phase advance of one period.

Mapping in McMillan form after CT to (q,p), $\mathrm{T} = \widetilde{\mathrm{F}} \circ \mathrm{Rot}(-\pi/2)$

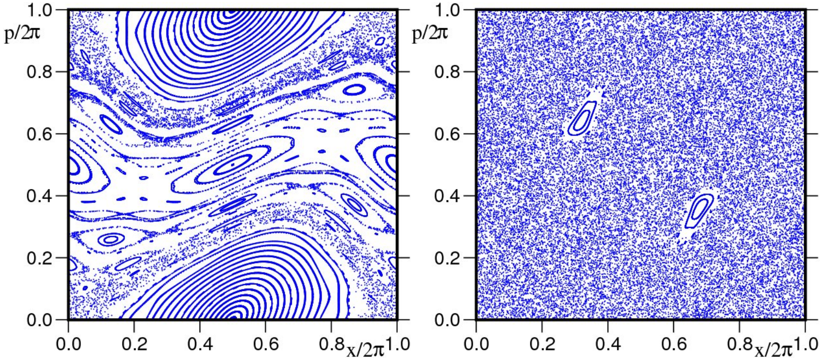
$$q = y,$$
$$p = y\,(\cos\Phi + \alpha\,\sin\Phi) + \dot{y}\,\beta\,\sin\Phi,$$

$$\boxed{\widetilde{\mathrm{F}}(q) = 2\,q\,\cos\Phi + \beta\,F(q)\,\sin\Phi}.$$

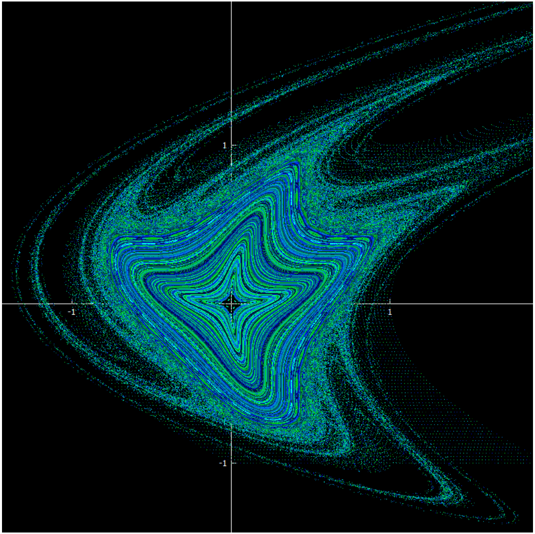
Example 1: Standard map/Chirikov-Taylor map/Chirikov standard map ($f = \cos p$)

$$\Delta E_{n+1} = \Delta E_n + e\,V\,(\sin\phi_n - \sin\phi_s)$$

$$\phi_{n+1} = \phi_n + \frac{2\,\pi\,h\,\eta}{\beta^2\,E}\Delta E_{n+1}$$



Example 2: Hénon quadratic map ($f = p^2$)



Turaev theorem

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Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps

Dmitry Turaev

Recommended by C Liverani

Abstract

It is shown that every symplectic diffeomorphism of R^{2n} can be approximated, in the C^∞-topology, on any compact set, by some iteration of some map of the form $(x, y) \mapsto (y + \eta, -x + \nabla V(y))$ where $x \in R^n$, $y \in R^n$, and V is a polynomial $R^n \to R$ and $\eta \in R^n$ is a constant vector. For the case of area-preserving maps (i.e. $n = 1$), it is shown how this result can be applied to prove that C^r-universal maps (a map is universal if its iterations approximate dynamics of all C^r-smooth area-preserving maps altogether) are dense in the C^r-topology in the Newhouse regions.

1. PERTURBATION THEORY

Consider a map in **McMillan form**:

$$\mathrm{T}: \qquad q' = p, \qquad p' = -q + f(p),$$

where function $f(p)$ is of the class C^∞ and will be referred to as a **force function**, or simply **force**.

In order to construct a perturbation theory, we shall introduce a small positive parameter ϵ characterizing the amplitude of oscillations. It can be done using a change of variables $(q,p) \to \epsilon\,(q,p)$:

$$\mathrm{T}: \qquad q' = p$$
$$p' = -q + \tfrac{1}{\epsilon}\, f(\epsilon\, p) = -q + a\, p + \epsilon\, \tfrac{b}{2!}\, p^2 + \epsilon^2\, \tfrac{c}{3!}\, p^3 + \ldots.$$

where we expanded the force function in a power series of $(\epsilon\, p)$ and

$$a \equiv \partial_p f(0), \qquad b \equiv \partial_p^2 f(0), \qquad c \equiv \partial_p^3 f(0), \qquad \ldots.$$

Linearization of map

$$\mathrm{T}: \qquad q' = p$$
$$p' = -q + a\,p + \epsilon\,\tfrac{b}{2!}\,p^2 + \epsilon^2\,\tfrac{c}{3!}\,p^3 + \ldots.$$

Jacobian of transformation

$$\mathrm{J}_\mathrm{T} = \begin{bmatrix} \frac{\partial\, q'}{\partial q} & \frac{\partial\, q'}{\partial p} \\ \frac{\partial\, p'}{\partial q} & \frac{\partial\, p'}{\partial p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$$

Courant-Snyder invariant

$$\mathrm{C.S.} = p^2 - a\,p\,q + q^2$$

Betatron frequency

$$\mu = \frac{1}{2\,\pi}\,\arccos\,\frac{a}{2}$$

$\mathcal{K}^{(n)}(p', q') - \mathcal{K}^{(n)}(p, q) = \mathcal{O}(\epsilon^{n+1})$

We seek for an invariant of motion expanded in powers of a small parameter:

$$\mathcal{K}^{(n)} = \mathcal{K}_0 + \epsilon\,\mathcal{K}_1 + \epsilon^2\,\mathcal{K}_2 + \ldots + \epsilon^n\,\mathcal{K}_n$$

such that $\mathcal{K}_m$ are degree $(m+2)$ polynomials

$$\begin{aligned}
\mathcal{K}_0 &= C_{2,0}\,p^2 + C_{1,1}\,p\,q\; + C_{0,2}\,q^2, \\
\mathcal{K}_1 &= C_{3,0}\,p^3 + C_{2,1}\,p^2 q + C_{1,2}\,p\,q^2 + C_{0,3}\,q^3, \\
\mathcal{K}_2 &= C_{4,0}\,p^4 + C_{3,1}\,p^3 q + C_{2,2}\,p^2 q^2 + C_{1,3}\,p\,q^3 + C_{0,4}\,q^4, \\
&\cdots .
\end{aligned}$$

Due to the first symmetry, $\mathcal{K}(q,p) = \mathcal{K}(p,q)$, it is convenient to introduce the following notations:

$$\Sigma = p + q \qquad \Pi = p\,q \qquad \mathrm{C.S.} = \Sigma^2 - (2+a)\,\Pi = p^2 - a\,p\,q + q^2$$

Then we perform the expansion for even and odd orders of PT as

$$\begin{aligned}
\mathcal{K}_0 &= \mathrm{C.S.} \\
\mathcal{K}_1 &= A_1^{(1)}\,\Pi\Sigma \\
\mathcal{K}_2 &= A_1^{(2)}\,\Pi^2 + \boxed{C^{(2)}\,\mathrm{C.S.}^2} \\
\mathcal{K}_3 &= A_1^{(3)}\,\Pi^2\Sigma + A_2^{(3)}\,\Pi\Sigma\mathrm{C.S.} \\
\mathcal{K}_4 &= A_1^{(4)}\,\Pi^3 + A_2^{(4)}\,\Pi^2\mathrm{C.S.} + \boxed{C^{(4)}\,\mathrm{C.S.}^3} \\
\mathcal{K}_5 &= A_1^{(5)}\,\Pi^3\Sigma + A_2^{(5)}\,\Pi^2\Sigma\mathrm{C.S.} + A_3^{(5)}\,\Pi\Sigma\mathrm{C.S.}^2 \\
&\ldots
\end{aligned}$$

Averaging

1. Canonical change of variables to Floquet coordinates

$$q = \left(1 - a^2/4\right)^{1/4}\ \sqrt{2\,J}\cos(\varphi) + \tfrac{a}{2}\ \left(1 - a^2/4\right)^{-1/4}\sqrt{2\,J}\sin(\varphi),$$
$$p = \left(1 - a^2/4\right)^{-1/4}\sqrt{2\,J}\sin(\varphi),$$

2. Rewriting the residual in terms of (J, φ)

It is periodic function of φ, so its average over a full period vanishes:

$$\int_0^{2\pi}\left[\mathcal{K}^{(2)}(q', p') - \mathcal{K}^{(2)}(q, p)\right]\,\mathrm{d}\varphi = 0.$$

3. Minimization of the average of the squared residual

$$I_1 = \int_0^{2\pi}\left[\mathcal{K}^{(2)}(q', p') - \mathcal{K}^{(2)}(q, p)\right]^2\,\mathrm{d}\varphi$$

and solve for C_1 from $\frac{\mathrm{d}}{\mathrm{d}C_1}\,I_1 = 0$

Approximated invariant for Hénon map

$$\mathcal{K}_{\text{sex}}^{(3)} = \text{C.S.} - \frac{b}{r_3}\,\Sigma\,\Pi\,\epsilon^1 + \left(\frac{b^2}{r_3 r_4}\,\Pi^2 + C_1\,\text{C.S.}^2\right)\epsilon^2 - \\ - \frac{b}{r_3}\left(\frac{b^2}{r_4 r_5}\,\Sigma\,\Pi^2 - \left[\frac{b^2}{r_3 r_4 r_5} - 2\,C_1\right]\Sigma\,\Pi\,\text{C.S.}\right)\epsilon^3$$

$$\langle\mathcal{K}_{\text{sex}}^{(0)}\rangle = r_1 r_2\,\text{C.S.}$$

$$\langle\mathcal{K}_{\text{sex}}^{(1)}\rangle = r_1 r_2 r_3\,\text{C.S.} - r_1 r_2\,\Sigma\,\Pi\,\epsilon\,b$$

$$\langle\mathcal{K}_{\text{sex}}^{(2)}\rangle = r_1 r_2 r_3 r_4\,\text{C.S.} - r_1 r_2 r_4\,\Sigma\,\Pi\,\epsilon\,b + \left(r_1 r_2\,\Pi^2 + \tfrac{5}{4}\,\text{C.S.}^2\right)\epsilon^2 b^2$$

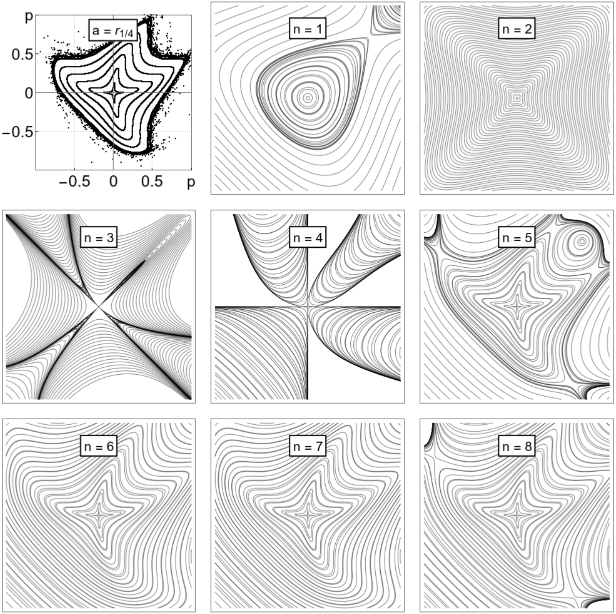
$$\langle\mathcal{K}_{\text{sex}}^{(3)}\rangle = r_1 r_2 r_3 r_4 r_5\,\text{C.S.} - r_1 r_2 r_4 r_5\,\Sigma\,\Pi\,\epsilon\,b + \\ + r_1\left(r_2 r_5\,\Pi^2 + \frac{P_1}{P_0}\,\text{C.S.}^2\right)\epsilon^2 b^2 - r_1\left(r_2\,\Sigma\,\Pi^2 + 7\,\frac{P_2}{P_0}\,\Sigma\,\Pi\,\text{C.S.}\right)\epsilon^3 b^3$$

where

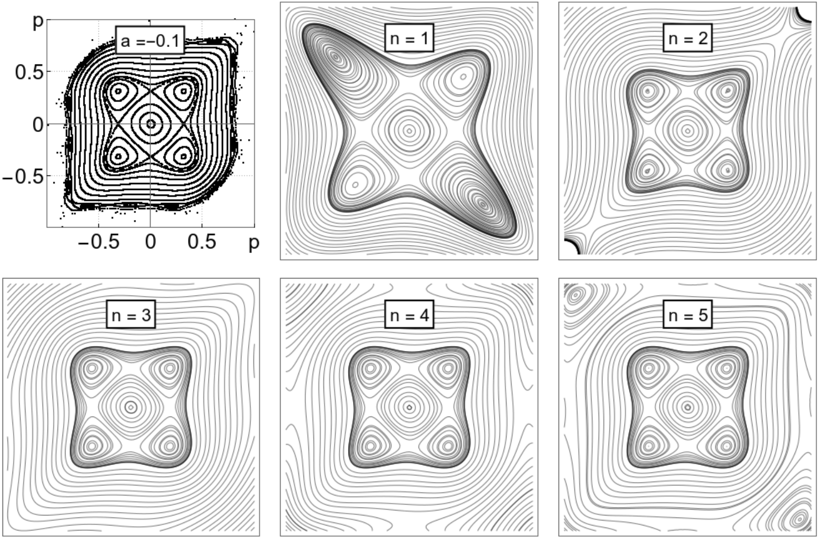
$$r_1 = a - 2 \qquad r_2 = a + 2 \qquad r_3 = a + 1$$

$$r_4 = a \qquad r_5 = \left(a + \tfrac{1+\sqrt{5}}{2}\right)\left(a + \tfrac{1-\sqrt{5}}{2}\right)$$

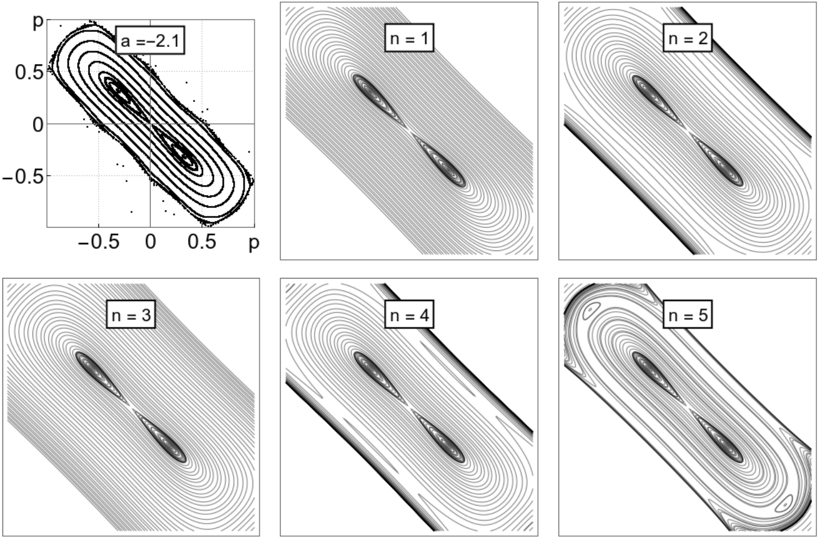
a. Resonance cases (Sextupole on a 1/4 resonance)



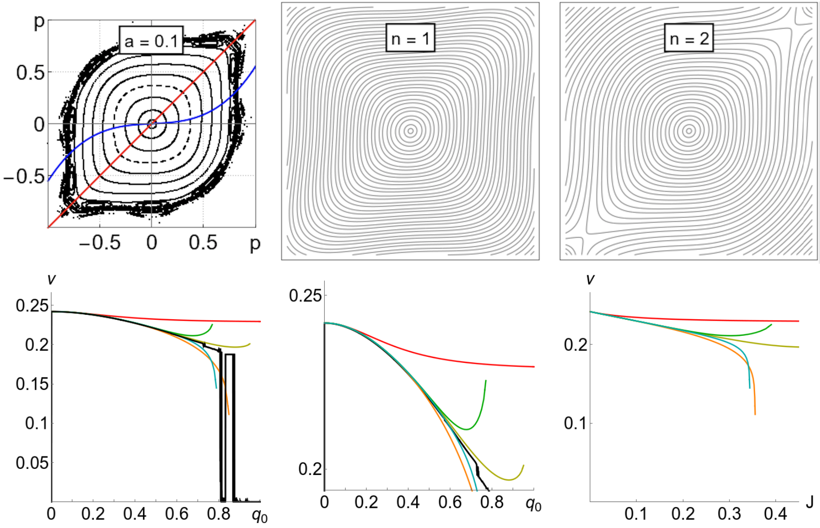
b. Islands (Octupole below 1/4 resonance)



c. Unstable fixed point (Octupole below 1/2 resonance)



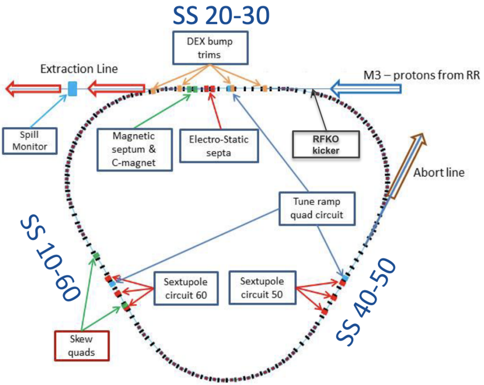
d. Frequency as a function of amplitude (Octupole above 1/4 resonance)



2. DELIVERY RING EXTRACTION FOR **Mu2e**

Implementation of Resonant Extraction in the Delivery Ring for Mu2e

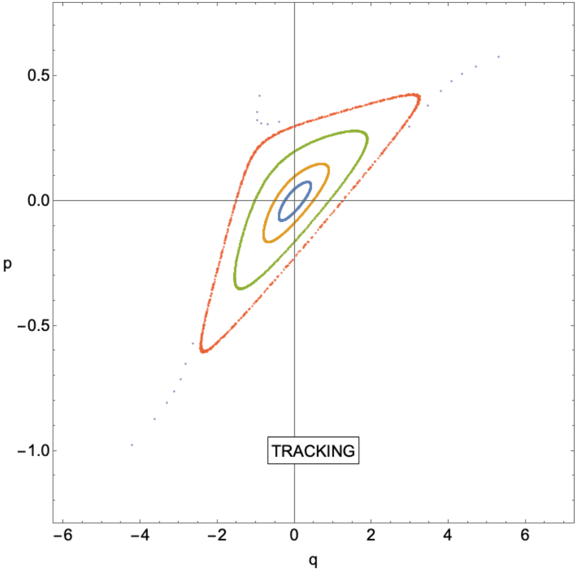
- New injection point
- Extraction in SS 20-30
- Electro-static septa
- 2 families of harmonic Sextupoles
- A family of tune Quadrupoles
- Extraction Lambertson
- Dynamic orbit control
- Abort line
- RFKO system
- Spill monitoring
- Spill regulation



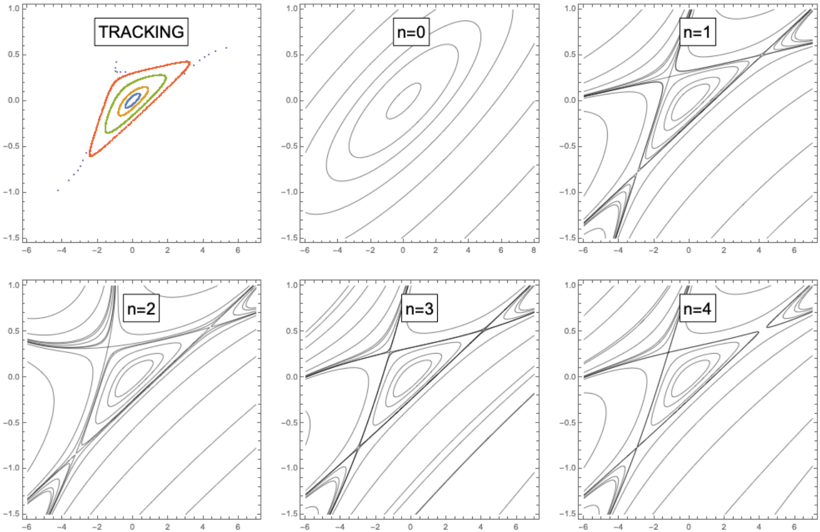
- Horizontal 3rd Integer resonance
- Q_x / Q_y=9.650 / 9.735

Vladimir Nagaslaev

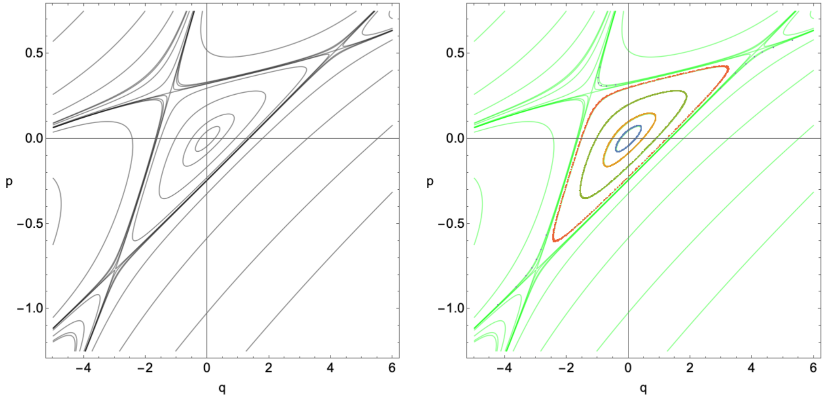
Tracking with 6 sextupoles



0-th — 4-th order approximated invariants, $\mathcal{K}^{(n)}(p, q)$



4-th order vs. tracking



3. NONLINEAR OPTICAL FUNCTIONS

AND

GENERALIZED COURANT – SNYDER INVARIANT

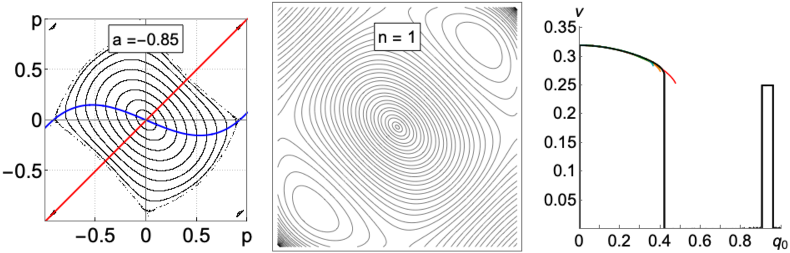
Nonlinear optical functions

$$\mathbf{inv}(s) = \underbrace{\alpha(s)\,p^2 + \beta(s)\,p\,q + \gamma(s)\,q^2}_{\mathrm{C.S.}} + \underbrace{\delta(s)\,p^2\,q + \epsilon(s)\,p\,q^2}_{\textit{sextupoles}} +$$

$$+ \underbrace{\zeta(s)\,p^2\,q^2}_{\textit{octupoles}} + \underbrace{\eta(s)\,\mathrm{C.S.}^2}_{\textit{2nd order correction}}$$

- Sextupole and octupole terms are in the form of McMillan integrable mappings
- Estimate of dynamical aperture near 1st, 2nd, 3rd and 4th order resonances (critical points of the invariant)
- Distortion of the ellipse trajectories on larger amplitudes ($\triangle$, $\square$, C- or S-shapes)
- Amplitude dependent betatron frequency $\mu(q_0, p_0)$

Example for Hénon octupole map



Summary

- We developed a very powerful tool for studying discrete transformations
- Relative mathematical simplicity allows higer order analysis
- Fast estimate of dynamic aperture and frequency spread without exact tracking (minimization of losses, brightness increment etc.)
- Optimization of accelerator design or improvement procedure
- Analytical and semi-analytical models are helping us to understand and verify our numerical simulations
- Introduction of nonlinear optical functions

LAST SLIDE

Thank you for your attention!

Questions?