Local Beta Bump -
Beta relations for transfer matricies

July 2, 2020

Consider the transfer matrix for three thin quadrupole correctors, separated by arbitrary transfer matrices:

$$
\begin{aligned}
M &= Q_5 T_{54} Q_4 T_{43} Q_3 T_{32} Q_2 T_{21} Q_1 \\
Q_i &= \begin{bmatrix} 1 & 0 \\ q_i & 1 \end{bmatrix}, \\
T_{ji} &= \begin{bmatrix} \sqrt{\frac{\beta_j}{\beta_i}}(\cos\phi_{ji} + \alpha_i \sin\phi_{ji}) & \sqrt{\beta_i\beta_j}\sin\phi_{ji} \\ -\frac{1+\alpha_i\alpha_j}{\sqrt{\beta_i\beta_j}}\sin\phi_{ji} + \frac{\alpha_i-\alpha_j}{\sqrt{\beta_i\beta_j}}\cos\phi_{ji} & \sqrt{\frac{\beta_i}{\beta_j}}(\cos\phi_{ji} - \alpha_j\sin\phi_{ji}) \end{bmatrix}
\end{aligned} \tag{1}
$$

Next we rewrite it in terms of normalized coordinates:

$$
\begin{aligned}
Q_i &= B_i \tilde{Q}_i B_i^{-1}, \quad T_{ji} = B_j R_{ji} B_i^{-1}, \\
B_i &= \begin{bmatrix} \sqrt{\beta_i} & 0 \\ \frac{-\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{bmatrix}, \quad B_i^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\beta_i}} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \sqrt{\beta_i} \end{bmatrix}, \\
\tilde{Q}_i &= \begin{bmatrix} 1 & 0 \\ k_i & 1 \end{bmatrix}, \quad R_{ji} = \begin{bmatrix} \cos\phi_{ji} & \sin\phi_{ji} \\ -\sin\phi_{ji} & \cos\phi_{ji} \end{bmatrix}, \\
\tilde{M} &= B_5^{-1} M B_1^{-1} = \tilde{Q}_5 R_{54} \tilde{Q}_4 R_{43} \tilde{Q}_3 R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1. \\
k_i &= \beta_i q_i
\end{aligned} \tag{2}
$$

Each quad matrix can be split into two halves by:

$$
\begin{aligned}
\tilde{Q}_i &= \tilde{Q}_i^{1/2} \tilde{Q}_i^{1/2} \\
\tilde{Q}_i^{1/2} &= \begin{bmatrix} 1 & 0 \\ k_i/2 & 1 \end{bmatrix}
\end{aligned}
$$

Now let's define the transfer matrices from quad-midpoint to quad-midpoint.

$$
\begin{aligned}
M(s_2|s_1) &= B_2 \tilde{Q}_2^{1/2} R_{21} \tilde{Q}_1^{1/2} B_1^{-1} \\
M(s_3|s_1) &= B_3 \tilde{Q}_3^{1/2} R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1^{1/2} B_1^{-1} \\
M(s_4|s_1) &= B_4 \tilde{Q}_4^{1/2} R_{43} \tilde{Q}_3 R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1^{1/2} B_1^{-1}
\end{aligned}
$$

Assignment 1: Calculate $M(s_2|s_1)$, $M(s_4|s_1)$, and $M(s_3|s_1)$ for generic values of k_1, k_2, k_3, k_4, and generic phases (ϕ_{43},ϕ_{32},ϕ_{21}).

For a given transfer matrix $M(s_j|s_i)$, the corresponding beta-matrix $M_\beta(s_j|s_i)$ is given by:

$$
\begin{aligned}
\vec{x}_j &= M(s_j|s_i)\vec{x}_i \\
\begin{bmatrix} x_j \\ x'_j \end{bmatrix} &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_i \\ x'_i \end{bmatrix} \\
\vec{\beta}_j &= M_\beta(s_j|s_i)\vec{\beta}_i \\
\begin{bmatrix} \beta_j \\ \alpha_j \\ \gamma_j \end{bmatrix} &= \begin{bmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & (M_{11}M_{22} + M_{12}M_{21}) & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{bmatrix} \begin{bmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{bmatrix}
\end{aligned}
$$

For $\alpha_i = 0$, then $\gamma_i = \beta_i^{-1}$, and then β_j can be calculated to be:

$$\begin{aligned}
\beta_j &= \begin{bmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \end{bmatrix} \begin{bmatrix} \beta_i \\ 0 \\ \beta_i^{-1} \end{bmatrix} \\
\beta_j &= M_{11}^2\beta_i + M_{12}^2\beta_i^{-1}
\end{aligned}$$

where M_{11} is the upper-left value of $M(s_j|s_i)$ and M_{12} is the upper-right value of $M(s_j|s_i)$.

Assignment 2: Calculate the β_2, β_3, β_4 for generic values of k_1, k_2, k_3, k_4, generic phases, and for a given value β_1.

Assignment 3: For β_2, β_3, and β_4, plug in the symmetric phase-advance conditions and the corresponding local bump conditions:

$$\begin{aligned}
\phi_{54} &= \phi_{21} = \phi_A \\
\phi_{43} &= \phi_{32} = \phi_B \\
k_1 &= -\frac{1}{2}k_3(s_{A+B}^2/s_B^2 - 1)^{-1} + \epsilon s_{2B}/s_{2A+2B} \\
k_2 &= \frac{1}{2}k_3(s_B^2/s_{A+B}^2 - 1)^{-1} - \epsilon \\
k_4 &= \frac{1}{2}k_3(s_B^2/s_{A+B}^2 - 1)^{-1} + \epsilon \\
k_5 &= -\frac{1}{2}k_3(s_{A+B}^2/s_B^2 - 1)^{-1} - \epsilon s_{2B}/s_{2A+2B}
\end{aligned}$$

Assignment 4: Verify the expression obtained for assignment 2 and/or assignment 3 for an example local bump calculated by MAD-X.