Local Beta Bump
and applications for Fermilab Booster

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1 Motivation

In general, changing the focusing at one quadrupole corrector in a ring will have an impact on the beta-function everywhere in a particle accelerator ring. With dipole-correctors, a local effect on the beam orbit can be accomplished by a sequence of three dipole correctors with a generic phase-advance between them. It would be valuable to have something analogous for quadrupole correctors, a sequence of quadrupole correctors with a local-effect on the beta-function (i.e. a nonzero quadrupole correction with no change to beta functions outside of the quadrupole corrector sequence).

The local quad bump could be used to tune the beta functions in one part of the ring without adversely affecting another part of the ring. It can also be used to compensate for a missing quadrupole by using adjacent quadrupoles to provide the same global effect.

2 Three Quad Bump

2.1 Setup

Consider the transfer matrix for three thin quadrupole correctors, separated by arbitrary transfer matrices:

$$\begin{aligned}
M &= Q_3 T_{32} Q_2 T_{21} Q_1 \\
Q_i &= \begin{bmatrix} 1 & 0 \\ q_i & 1 \end{bmatrix}, \\
T_{ji} &= \begin{bmatrix} \sqrt{\frac{\beta_j}{\beta_i}}(\cos\phi_{ji} + \alpha_i \sin\phi_{ji}) & \sqrt{\beta_i\beta_j}\sin\phi_{ji} \\ -\frac{1+\alpha_i\alpha_j}{\sqrt{\beta_i\beta_j}}\sin\phi_{ji} + \frac{\alpha_i-\alpha_j}{\sqrt{\beta_i\beta_j}}\cos\phi_{ji} & \sqrt{\frac{\beta_i}{\beta_j}}(\cos\phi_{ji} - \alpha_j \sin\phi_{ji}) \end{bmatrix}
\end{aligned} \tag{1}$$

To simplify the expression, we should rewrite it in terms of normalized coordinates:

$$\begin{aligned}
&Q_i = B_i \tilde{Q}_i B_i^{-1}, \quad T_{ji} = B_j R_{ji} B_i^{-1}, \\
&B_i = \begin{bmatrix} \sqrt{\beta_i} & 0 \\ \frac{-\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{bmatrix}, \quad B_i^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\beta_i}} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \sqrt{\beta_i} \end{bmatrix}, \\
&\tilde{Q}_i = \begin{bmatrix} 1 & 0 \\ k_i & 1 \end{bmatrix}, \quad R_{ji} = \begin{bmatrix} \cos\phi_{ji} & \sin\phi_{ji} \\ -\sin\phi_{ji} & \cos\phi_{ji} \end{bmatrix}, \\
&\tilde{M} = B_3^{-1} M B_1^{-1} = \tilde{Q}_3 R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1. \\
&k_i = \beta_i q_i
\end{aligned} \tag{2}$$

Now the expression is just thin-kicks separated by rotation matrices. It can be easier to work with the expression if we can separate the thin kick-perturbation from the identity matrix:

$$\begin{aligned}
&\tilde{M} = (I + k_3 E_0)\, R_{32}\, (I + k_2 E_0)\, R_{21}\, (I + k_1 E_0)\,, \\
&I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\end{aligned} \tag{3}$$

2.2 Calculation

Now we need to find the criteria for nontrivial local quad correction consisting of a sequence of three quadrupoles - i.e. non-zero values of k_1, k_2, k_3 such that $\tilde{M}(k_1, k_2, k_3) = R_{32}R_{21} = R_{31}$. We write:

$$
\begin{aligned}
\cancel{R_{31}} &= \cancel{R_{31}} + k_3 E_0 R_{31} + k_2 R_{32} E_0 R_{21} + k_1 R_{31} E_0 \\
&\quad + k_3 k_2 E_0 R_{32} E_0 R_{21} + k_3 k_1 E_0 R_{31} E_0 + k_2 k_1 R_{32} E_0 R_{21} E_0 \\
&\quad + k_3 k_2 k_1 E_0 R_{32} E_0 R_{21} E_0 \\
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= k_3 \begin{bmatrix} 0 & 0 \\ c_{31} & s_{31} \end{bmatrix} + k_2 \begin{bmatrix} s_{32}c_{21} & s_{32}s_{21} \\ c_{32}c_{21} & c_{32}s_{21} \end{bmatrix} + k_1 \begin{bmatrix} s_{31} & 0 \\ c_{31} & 0 \end{bmatrix} \\
&\quad + k_3 k_2 s_{32} \begin{bmatrix} 0 & 0 \\ c_{21} & s_{21} \end{bmatrix} + k_3 k_1 \begin{bmatrix} s_{31} & 0 \\ 0 & s_{31} \end{bmatrix} + k_2 k_1 \begin{bmatrix} s_{32} & 0 \\ c_{32} & 0 \end{bmatrix} s_{21} \\
&\quad + k_3 k_2 k_1 \begin{bmatrix} 0 & 0 \\ s_{32}s_{21} & 0 \end{bmatrix}
\end{aligned} \tag{4}
$$

where for brevity I defined:

$$
\cos \phi_{ji} \equiv c_{ji}, \qquad \sin \phi_{ji} \equiv s_{ji} \tag{5}
$$

Now from Eq. 4 we have obtained four equations for the values of the 2x2 matrix:

$$
0 = k_2 s_{32} c_{21} + k_1 s_{31} + k_3 k_1 s_{31} + k_2 k_1 s_{32} s_{21} \tag{A.1}
$$

$$
0 = k_2 s_{32} s_{21} \tag{A.2}
$$

$$
0 = k_3 c_{31} + k_2 c_{32} c_{21} + k_1 c_{31} + k_3 k_2 s_{32} c_{21} + k_2 k_1 c_{32} s_{21} + k_3 k_2 k_1 s_{32} s_{21} \tag{A.3}
$$

$$
0 = k_3 s_{31} + k_2 c_{32} s_{21} + k_3 k_2 s_{32} s_{21} + k_3 k_1 s_{31} \tag{A.4}
$$

From Eq. A.2, we have s_{32} or s_{31} must be zero (since $k_2 \neq 0$). If $s_{32} = 0$ we find from Eq. A.1 that s_{31} and s_{21} must be zero. And if $s_{21} = 0$ we find from Eq. A.4 that s_{31} and s_{32} must be zero.

Since both s_{31} and s_{32} are zero, that must mean that ϕ_{32} is 0 or π, and ϕ_{21} is 0 or π, which means that $c_{31} = \pm_a 1$, $c_{21} = \pm_b 1$ and $c_{31} = \pm_a \pm_b 1$. Using this into Eq. A.3 we write:

$$
0 = \pm_a \pm_b [k_3 + k_2 + k_1] \tag{6}
$$

This means our final solution is that the quad kicks are related and that phase-advances ϕ_{32} and ϕ_{21} should be integer multiples of π:

$$
\begin{aligned}
k_2 &= -(k_1 + k_3) \\
\phi_{21} &= n_1 \pi \\
\phi_{32} &= n_2 \pi
\end{aligned} \tag{7}
$$

if we let $k_1 = \kappa - \epsilon$ and $k_3 = \kappa + \epsilon$ we see that k_2 sets the overall magnitude and that ϵ is a degree of freedom that changes the nature of the local beta-bump:

$$
\begin{aligned}
k_2 &= -2\kappa \\
k_1 &= \kappa - \epsilon \\
k_3 &= \kappa + \epsilon
\end{aligned}
$$

For $\epsilon = 0$ the bump is symmetric and for $\epsilon = \pm\kappa$ it would be a two quad bump:

2.3 Interpretation

The three-quad bump only works if the phase-advances between the quad kicks are integer multiples of π. In the Booster, when the tune is $\nu_{x,y} = 6.8$ then adjacent Booster cells are separated by a phase advance $(17/30)\pi \approx 0.57\pi$. Which means that two Booster cells are separated by phase advance of $(34/30)\pi = \pi + 2\pi/15 \approx \pi$.

That means we can make a bump at for example QL9 QL11 QL13 and it will be mostly but not completely local. The nearly local bump can be deconstructed as a completely local bump plus a smaller non-local error. For a three quad bump the error in Eq. A.2 and Eq. A.3 should be $k_2 \sin(2\pi/15)^2 \approx 0.17 k_2$ (for small k) and for a two quad bump the error in Eq. A.1 and Eq. A.4 should be $(1/2)k_2 \sin(4\pi/15) \approx 0.37k_2$. Howver the two-quad bump has too much error to use.

If we use three quad-long correctors (or three-quad short correctors), the kick will be stronger in one plane then the other, but not completely decoupled. For the bump to be local in both planes, the conditions should be approximately held in both planes. Fortunately, since $\nu_x \approx \nu_y$ the horizontal and vertical phase-advances are approximately the same cell-to-cell (although not between shorts and longs). And for three quad-longs or three-quad shorts, the beta functions should be (approximately) equal at all three locations, so we don't necessarily need to take into account the ratio of the beta functions. So in this case, we find that relation for k is also the relation for the quadrupole currents, i.e. $I\_QL11 = -(I\_QL09 + I\_QL13)$ and it will local to the $\sim$17% level in both planes.

The current implementation of the three quad bump in the Booster controls has coefficients of 2.74, 1, 2.74, separated by $\approx \pi/2$ phase advances. This analysis would predict that is not a local bump, the quad kicks should mostly add up coherently (i.e. rather than cancelling) which should be similar to adjusting the current of just one quad. MAD-X modeling also confirms this. What this three-quad knob does instead, is make a local dispersion bump (the quad effect on the dispersion is the same as the dipole effect on the closed orbit), which is very useful in the horizontal plane.

3 Five Quad Bump

3.1 Motivation

We should be able to squash the non-local $\sim$17% error in the three-quad bump by using the other correctors located in between (i.e. QL10 and QL12 to supplement QL9, QL10, QL11), so with a little bit more algebra we can consider a five-quad bump instead of a three-quad bump.

For a five-quad bump, we have five quad kicks $(k_1, k_2, k_3, k_4, k_5)$, four phase advances$(\phi_{54}, \phi_{43}, \phi_{32}, \phi_{21})$, four constraining equations (for a 2x2 matrix). That means it may be possible to find a solution in which the phase-advances are unconstrained, the quad-kicks are all defined relative to those phase-advances and an arbitrary k which represents the quad-bump magnitude.

3.2 Calculation

For a local five quad bump we require that:

$$\begin{aligned}
\tilde{M} &= (I + k_5 E_0)\, R_{54}\, (I + k_4 E_0)\, R_{43}\, (I + k_3 E_0)\, R_{32}\, (I + k_2 E_0)\, R_{21}\, (I + k_1 E_0) \\
\cancel{R_{51}} &= \cancel{R_{51}} + k_5 E_0 R_{51} + k_4 R_{54} E_0 R_{41} + k_3 R_{53} E_0 R_{31} \\
&\quad + k_2 R_{52} E_0 R_{21} + k_1 R_{51} E_0 + \mathcal{O}(k^2)
\end{aligned} \tag{8}$$

where we have neglected the k^2 and higher order terms (i.e. considering only small values of k).

From Eq. 8 we can write the four equations for the values of the 2x2 matrix:

$$0 = k_4 s_{54} c_{41} + k_3 s_{53} c_{31} + k_2 s_{52} c_{21} + k_1 s_{51} \tag{B.1}$$
$$0 = k_4 s_{54} s_{41} + k_3 s_{53} s_{31} + k_2 s_{52} s_{21} \tag{B.2}$$
$$0 = k_5 c_{51} + k_4 c_{54} c_{41} + k_3 c_{53} c_{31} + k_2 c_{52} c_{21} + k_1 c_{51} \tag{B.3}$$
$$0 = k_5 s_{51} + k_4 c_{54} s_{41} + k_3 c_{53} s_{31} + k_2 c_{52} s_{21} \tag{B.4}$$

This is a lot to keep track of, so let's assume the phase-advances are symmetric

$$\phi_{54} = \phi_{21} = \phi_A, \qquad \phi_{43} = \phi_{32} = \phi_B \tag{9}$$

and for algebraic convenience let's also define the quad-kicks quasi-symmetrically:

$$k_1 = \kappa_1 - \lambda, \quad k_5 = \kappa_1 + \lambda, \quad k_2 = \kappa_2 - \epsilon, \quad k_4 = \kappa_2 + \epsilon, \quad k_3 = \kappa_3 \tag{10}$$

With these substitutions we rewrite:

$$
\begin{aligned}
0 &= \kappa_1 s_{2A+2B} + \kappa_2(c_A s_{A+2B} + s_A c_{A+2B}) + \kappa_3 s_{A+B} c_{A+B} \\
&\quad - \lambda s_{2A+2B} - \epsilon(c_A s_{A+2B} - s_A c_{A+2B}) && \text{(C.1)} \\
0 &= 2\kappa_2 s_A s_{A+2B} + \kappa_3 s_{A+B}^2 && \text{(C.2)} \\
0 &= 2\kappa_1 c_{2A+2B} + 2\kappa_2 c_A c_{A+2B} + \kappa_3 c_{A+B}^2 && \text{(C.3)} \\
0 &= \kappa_1 s_{2A+2B} + \kappa_2(c_A s_{A+2B} + s_A c_{A+2B}) + \kappa_3 s_{A+B} c_{A+B} \\
&\quad + \lambda s_{2A+2B} + \epsilon(c_A s_{A+2B} - s_A c_{A+2B}) && \text{(C.4)}
\end{aligned}
$$

taking the sum and difference of Eq. C.1 and Eq. C.4 we obtain the equations:

$$
\begin{aligned}
0 &= \kappa_1 s_{2A+2B} + \kappa_2(c_A s_{A+2B} + s_A c_{A+2B}) + \kappa_3 s_{A+B} c_{A+B} && \text{(C.5)} \\
0 &= \lambda s_{2A+2B} + \epsilon(c_A s_{A+2B} - s_A c_{A+2B}) && \text{(C.6)}
\end{aligned}
$$

Using trig. identities we realize that $s_{2A+2B} = c_A s_{A+2B} + s_A c_{A+2B} = 2 s_{A+B} c_{A+B}$ and consequently if $s_{2A+2B} \neq 0$ we find that Eq. C.5 implies that

$$0 = 2\kappa_1 + 2\kappa_2 + \kappa_3. \tag{D.1}$$

Using another trig. identity $(c_A s_{A+2B} - s_A c_{A+2B}) = s_{2B}$ with Eq. C.6 we find:

$$\lambda = -\epsilon s_{2B}/s_{2A+2B} \tag{D.2}$$

Substituting Eq. D.1 into Eq. C.2 we find:

$$0 = 2\kappa_2(s_A s_{A+2B} - s_{A+B}^2) - 2\kappa_1 s_{A+B}^2 \tag{C.7}$$

It can be shown by expanding for trig. functions that $s_A s_{A+2B} - s_{A+B}^2 = -s_B^2$ and consequently we can write:

$$0 = \kappa_1 s_{A+B}^2 + \kappa_2 s_B^2 \tag{D.3}$$

The last condition to resolve is Eq. C.3. Using the trig identity

$$c_{2A+2B} - c_{A+B}^2 = (c_{A+B}^2 - s_{A+B}^2) - c_{A+B}^2 = -s_{A+B}^2$$

and by expanding $c_A c_{A+2B} - c_{A+B}^2 = -s_B^2$, we find that Eq. C.3 is the same as Eq. C.2 (under the symmetric phase-advance constraint).

So now, summarizing the results and solving for $\lambda, \kappa_1, \kappa_2$ we have:

$$
\begin{aligned}
\kappa_1 &= -\frac{1}{2}\kappa_3 (s_{A+B}^2/s_B^2 - 1)^{-1} && \text{(D.1)} \\
\lambda &= -\epsilon s_{2B}/s_{2A+2B} && \text{(D.2)} \\
\kappa_2 &= \frac{1}{2}\kappa_3 (s_B^2/s_{A+B}^2 - 1)^{-1} && \text{(D.3)}
\end{aligned}
$$

We see that there is not constraint on the phase-advances (other than symmetry), they only determine the ratios between the five thin kicks. But as before, κ_3 sets the overall magnitude and ϵ is a degree of freedom that changes the nature of the local beta-bump.

A local four quad bump, unconstrained by phase, is possible for:

$$\epsilon = \pm\frac{1}{2}\kappa_3 \frac{s_{2A+B}}{c_{2B}} \left(1 + \frac{s_{A+B}^2}{s_B^2}\right)^{-1} \tag{11}$$

Another four quad bump, five quads missing the middle quad is possible for $\kappa_3 = 0$, $\epsilon \neq 0$.

Writing Eq. D.1-D.3 in terms of the original kicks:

$$
\begin{aligned}
k_1 &= -\frac{1}{2}k_3 (s_{A+B}^2/s_B^2 - 1)^{-1} + \epsilon s_{2B}/s_{2A+2B} \\
k_5 &= -\frac{1}{2}k_3 (s_{A+B}^2/s_B^2 - 1)^{-1} - \epsilon s_{2B}/s_{2A+2B} \\
k_2 &= \frac{1}{2}k_3 (s_B^2/s_{A+B}^2 - 1)^{-1} - \epsilon \\
k_4 &= \frac{1}{2}k_3 (s_B^2/s_{A+B}^2 - 1)^{-1} + \epsilon
\end{aligned}
\tag{12}
$$

3.3 Interpretation

Using four or five consecutive Boosters cells, we can create a quad bump truly local in both planes. The five quad-long bump QL9 QL10 QL11 QL12 QL13 will primarily impact the local vertical beta functions, but have a 1/3 impact on the horizontal beta function. If this is paired with a quad-short bump which primarily impacts the horizontal plane, then independent control of the horizontal and vertical plane can be achieved.

If the Booster tune is set to $\nu_{x,y} = 6.8$, the appropriate quadrupole current coefficients can be calculated for five adjacent cells:

$$k_1 = -0.60 + 0.55\epsilon, \quad k_2 = +0.10 - \epsilon, \quad k_3 = +1, \quad k_4 = +0.10 + \epsilon, \quad k_5 = -0.60 - 0.55\epsilon$$

for any value of ϵ. If the tune is off by 0.1, it will only result in a nonlocal error of 1.5% so the same coefficients can be kept when the tune is changed.

The appropriate coefficients for the four-bump (with adjacent cells) are given by:

$$k_1 = -1.21, \quad k_2 = -1, \quad k_3 = +1, \quad k_4 = +1.21$$

(It turns out that the four-bump coefficients k_1 and k_2 are the same when $\phi_A = \phi_B$, which is to say that the phase-advances are constant between cells).

It should be acknowledged that a local beta bump which uses four of twenty-four Booster cells may be local in the sense that only impacts those four or five cells, but is a very wide bump.

To compensate for a missing corrector at L11, the QL9 QL10 QL11 QL12 QL13 five-quad-bump would have to be used to avoid impacting either the notch absorber at L13 or the new PIP2-era collimator at L8. To make a bump at L3 while avoiding use of the L3 corrector, a QL1 QL2 QL4 QL5 four-bump can be used. To make a beta bump at L1 injection while avoiding L3 extraction, a QL23 QL24 QL1 QL2 four bump could be used or the five bump centered at L24.

To make a beta-bump that also serves as a local dispersion bump, a convolution of the two bumps can be used:

$$\begin{array}{llll}
k_1 = & +(-0.60 + 0.55\epsilon) \times 2.56 & & \\
k_2 = & +(0.10 - \epsilon) \times 2.56 & +(-0.60 + 0.55\epsilon) \times 1.00 & \\
k_3 = & +(1.00) \times 2.56 & +(0.10 - \epsilon) \times 1.00 & +(-0.60 + 0.55\epsilon) \times 2.56 \\
k_4 = & +(0.10 - \epsilon) * 2.56 & +(1.00) \times 1.00 & +(0.10 - \epsilon) \times 2.56 \\
k_5 = & +(-0.60 + 0.55\epsilon) \times 2.56 & +(0.10 - \epsilon).00 & +(1.00) \times 2.56 \\
k_6 = & & +(-0.60 + 0.55\epsilon) \times 1.00 & +(0.10 - \epsilon) \times 2.56 \\
k_7 = & & & +(-0.60 + 0.55\epsilon) \times 2.56
\end{array}$$