Quadrupole (½ Integer) Errors and Correction

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PSP Taskforce Meeting

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Fermilab

Outline

- Quadrupole error propagation ('exact')
  - quadratic approximation in $q_{i,j}$
- Linear error terms & correction circuits
- Orthogonal quadrupole families
- Parabolic tune shifts & stop-band correction
  - example from MI
- Booster circuits
- Proof-of-principle Booster studies
- Observations & suggestions

Experience

- MI-8 proton line from Booster extraction to MI-10 injection
- MI ½-integer resonant extraction circuits
- NuMI primary beamline
- Tevatron Run II dispersionless Interaction Region (IR) upgrade
- C0 IR design for BTeV
- Delivery Ring lattice modifications for transitionless deceleration
- Replacement of the RR G232A/B magnet pair with separated function magnets for improved aperture
- Muon M1, M2 and M3 beamlines
- LBNF primary proton line

Quadrupole Error Propagation

- Propagation around a ring is described by the transfer matrix:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- Matrix **M** can be expanded in terms of sub-matrices Mji & the N perturbing quadrupole terms:

$$M \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = M_{0,N} \cdot \bar{q}_N \cdot M_{N,N-1} \cdot \bar{q}_{N-1} \cdot \bullet\bullet\bullet \cdot \bar{q}_1 \cdot M_{1,0} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- The quadrupole terms in thin-lens approximation are:

$$\bar{q}_i \equiv \begin{pmatrix} 1 & 0 \\ q_i & 1 \end{pmatrix}$$

- Matrices Mji are functions of the **unperturbed** lattice parameters:

$$M_{j,i} \equiv \begin{pmatrix} \sqrt{\beta_j/\beta_i} \cdot \left(\cos\psi_{ji} + \alpha_i \cdot \sin\psi_{ji}\right) & \sqrt{\beta_j\beta_i} \cdot \sin\psi_{ji} \\ -\dfrac{(1+\alpha_j\alpha_i)\cdot\sin\psi_{ji} + (\alpha_j - \alpha_i)\cdot\cos\psi_{ji}}{\sqrt{\beta_j\beta_i}} & \sqrt{\beta_i/\beta j} \cdot \left(\cos\psi_{ji} - \alpha_j \cdot \sin\psi_{ji}\right) \end{pmatrix}$$

- Switch to normalized co-ordinates $\bar{x} = x/\sqrt{\beta}$, $\bar{x}' = (\beta x' + \alpha x)/\sqrt{\beta}$ via:

$$R \equiv \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix}$$ and its inverse $$R^{-1} \equiv \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}$$

Then $\begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} = R^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}$ and M_{ji} are unitary transformations $M_{ji} = R_j \cdot \bar{M}_{ji} \cdot R_i^{-1}$

$$\bar{M}_{ji} \equiv \begin{pmatrix} \cos\psi_{ji} & \sin\psi_{ji} \\ -\sin\psi_{ji} & \cos\psi_{ji} \end{pmatrix}$$

- Particle transport described in normalized form:

$$\begin{pmatrix} \bar{x}_f \\ \bar{x}'_f \end{pmatrix} = (R_0^{-1} M_{0N} R_N)(R_N^{-1} \bar{q}_N R_N)(R_N^{-1} M_{NN-1} R_{N-1}) \cdot \bullet\bullet\bullet (R_1^{-1} \bar{q}_1 R_1)(R_1^{-1} M_{10} R_0) \cdot \begin{pmatrix} \bar{x}_0 \\ \bar{x}'_0 \end{pmatrix}$$

or

$$\begin{pmatrix} \bar{x}_f \\ \bar{x}'_f \end{pmatrix} = \bar{M}_{0N} \cdot \bar{Q}_N \cdot \bar{M}_{NN-1} \cdot \bar{Q}_{N-1} \cdot \bullet\bullet\bullet \bar{Q}_1 \cdot \bar{M}_{10} \cdot \begin{pmatrix} \bar{x}_0 \\ \bar{x}'_0 \end{pmatrix}$$

where $\bar{Q}_i \equiv R_i^{-1} \cdot \bar{q}_i \cdot R_i = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ q_i & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} = 1 + q_i \cdot \beta_i \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- The effect of the quadrupoles becomes more transparent by projecting them back to the origin. Sequentially, the matrices are rewritten as

$$\bar{M}_{21} \cdot \bar{Q}_1 \cdot \bar{M}_{10} = \bar{M}_{21} \cdot \bar{M}_{10} \cdot \left(\bar{M}_{10}^{-1} \cdot \bar{Q}_1 \cdot \bar{M}_{10}\right) = \bar{M}_{20} \cdot \left(\bar{M}_{10}^{-1} \cdot \bar{Q}_1 \cdot \bar{M}_{10}\right)$$

- Where $\bar{M}_{20}$ is the **unperturbed** transport from the origin to point #2.
- Similarly

$$\bar{M}_{32} \cdot \bar{Q}_2 \cdot \bar{M}_{20} \cdot \left(\bar{M}_{10}^{-1} \cdot \bar{Q}_1 \cdot \bar{M}_{10}\right) = \bar{M}_{30} \cdot \left(\bar{M}_{20}^{-1} \cdot \bar{Q}_2 \cdot \bar{M}_{20}\right)\left(\bar{M}_{10}^{-1} \cdot \bar{Q}_1 \cdot \bar{M}_{10}\right)$$

- etc, etc, etc . . . until, finally:

$$\begin{pmatrix} \bar{x}_f \\ \bar{x}'_f \end{pmatrix} = \bar{M}_0 \cdot \prod_{i=1}^{N} \left(\bar{M}_i^{-1} \cdot \bar{Q}_i \cdot \bar{M}_i\right) \begin{pmatrix} \bar{x}_0 \\ \bar{x}'_0 \end{pmatrix}$$

where $\bar{M}_0$ is the 'ideal' ring transfer matrix for a tune $\mu_0 = \nu 0 / 2\pi$:

$$\bar{M}_0 \equiv \begin{pmatrix} \cos\upsilon_0 & \sin\upsilon_0 \\ -\sin\upsilon_0 & \cos\upsilon_0 \end{pmatrix}$$

- Quad perturbations can be simplified to:

$$\bar{M}_i^{-1} \cdot \bar{Q}_i \cdot \bar{M}_i = 1 + q_i\beta_i \cdot \begin{pmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\psi_i & \sin\psi_i \\ -\sin\psi_i & \cos\psi_i \end{pmatrix}$$

$$\equiv 1 + {q_i\beta_i}/{2} \cdot \begin{pmatrix} -\sin 2\psi_i & -1 + \cos 2\psi_i \\ 1 + \cos 2\psi_i & \sin 2\psi_i \end{pmatrix}$$

- Finally, transport of a particle once around the ring is described **exactly** by† :

$$\boxed{\begin{pmatrix} \bar{x}_f \\ \bar{x}'_f \end{pmatrix} = \bar{M}_0 \cdot \prod_{i=1}^{N} \left(1 + {q_i\beta_i}/{2} \cdot \begin{pmatrix} -\sin 2\psi_i & -1 + \cos 2\psi_i \\ 1 + \cos 2\psi_i & \sin 2\psi_i \end{pmatrix} \right) \cdot \begin{pmatrix} \bar{x}_0 \\ \bar{x}'_0 \end{pmatrix}}$$

† The extension to distributed, rather than point-like, quadrupole sources presents purely a technical complication and does not introduce additional physics issues.

Quadratic Approximation to the Exact Transfer Result

- Assuming quad terms are ‘small’ expand the product as:

$$\prod_{i=1}^{N}\left(1+\frac{q_i\beta_i}{2}\cdot\begin{pmatrix}-\sin 2\psi_i & -1+\cos 2\psi_i\\ 1+\cos 2\psi_i & \sin 2\psi_i\end{pmatrix}\right)\approx \mathbf{1}+\sum_{i=1}^{N}\frac{q_i\beta_i}{2}\cdot\begin{pmatrix}-\sin 2\psi_i & -1+\cos 2\psi_i\\ 1+\cos 2\psi_i & \sin 2\psi_i\end{pmatrix}$$

$$+\sum_{i,j>i}^{N}\frac{q_i\beta_i\cdot q_j\beta_j}{4}\cdot\begin{pmatrix}-(1-\cos 2(\psi_j-\psi_i))+\cos 2\psi_j-\cos 2\psi_i & -\sin 2(\psi_j-\psi_i))+\sin 2\psi_j-\sin 2\psi_i\\ \sin 2(\psi_j-\psi_i))+\sin 2\psi_j-\sin 2\psi_i & -(1-\cos 2(\psi_j-\psi_i))-\cos 2\psi_j+\cos 2\psi_i\end{pmatrix}$$

- The 2nd term above, which depends on quad products $q_i q_j$, introduces a quadratic tune shift. This is relevant for correcting the intrinsic ½ integer stopband of a machine (and also for implementing resonant extraction).

- This term will be returned to.

Linear Error Terms & Correction Circuits

$$1+\sum_{i=1}^{N}\frac{q_i\beta_i}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+\sum_{i=1}^{N}\frac{q_i\beta_i}{2}\begin{pmatrix}-sin\,(2\psi_i) & cos(2\psi_i)\\ cos(2\psi_i) & sin\,(2\psi_i)\end{pmatrix}$$

UNPERTURBED TRANSFER

JUST A TUNE SHIFT – ABSORB IN MACHINE TUNE

THIS IS THE INTERESTING TERM

- The 3rd term doesn't introduce a tune shift†, but does produce a β-wave that advances with twice the phase.

† Homework: It is left as an exercise for the interested reader (assuming, optimistically, that such an entity exists), to show that this statement is true.

- To 1st order the perturbed transfer matrix is:

$$M \approx \begin{pmatrix} \cos\nu_0 + \sum_{i=1}^{N} \frac{q_i\beta_i}{2}\sin(\nu_0 - 2\psi_i) & \sin\nu_0 + \sum_{i=1}^{N} \frac{q_i\beta_i}{2}\cos(\nu_0 - 2\psi_i) \\ -\sin\nu_0 + \sum_{i=1}^{N} \frac{q_i\beta_i}{2}\cos(\nu_0 - 2\psi_i) & \cos\nu_0 - \sum_{i=1}^{N} \frac{q_i\beta_i}{2}\sin(\nu_0 - 2\psi_i) \end{pmatrix}$$

- Assembling the bits & pieces from slides 3 → 6 we find there is a β-wave generated that gets progressively worse as the unperturbed tune approaches ½.

$$\frac{\Delta\beta_\alpha}{\beta_\alpha} \approx \sum_{i=1}^{N} \frac{q_i\beta_i}{2} \cdot \frac{\cos(\nu_0 - 2\Delta\psi_{i\alpha})}{\sin(\nu_0)}$$

- This 1st order β-wave is cancelled by the orthogonal quad correction circuits . . .

Orthogonal Quadrupole Families

- Orthogonality is imposed by requiring that 2 families obey:

$$one\ family: \sum_{i=1}^{N} \cos(2\psi_i) \equiv 0, \quad and\ the\ other: \sum_{j=1}^{N} \sin(2\psi_j) \equiv 0$$

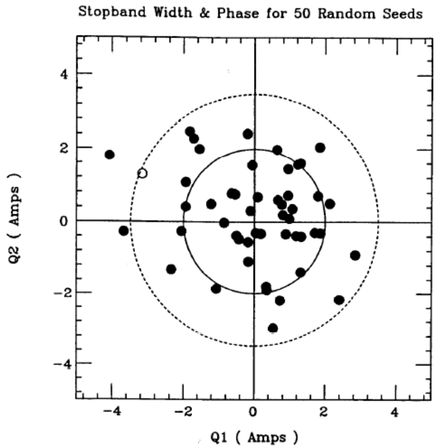
- The families' contributions to the transfer matrix are then:

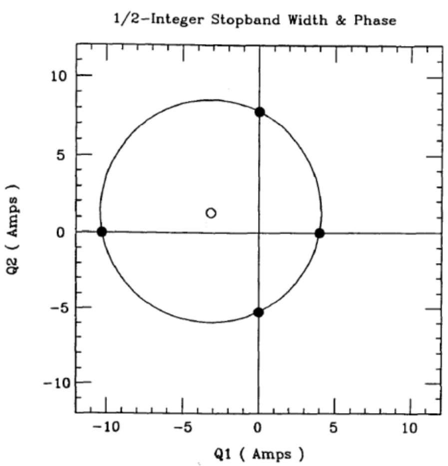
$$1 + \frac{|q_\alpha|\beta_\alpha}{2} \cdot \sum_{i=1}^{N} (-)^i \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{|q_\alpha|\beta_\alpha}{2} \cdot \sum_{i=1}^{N} (-)^i \cdot \begin{pmatrix} 0 & \cos(2\psi_i) \\ \cos(2\psi_i) & 0 \end{pmatrix}$$

$$1 + \frac{|q_\beta|\beta_\beta}{2} \cdot \sum_{j=1}^{N} (-)^j \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{|q_\beta|\beta_\beta}{2} \cdot \sum_{j=1}^{N} (-)^j \cdot \begin{pmatrix} -\sin(2\psi_j) & 0 \\ 0 & \sin(2\psi_j) \end{pmatrix}$$

- The quads in each family have alternating polarity to eliminate the 0th-harmonic (tune-changing) term and add coherently in the phase-dependent term.

- Each individual quad error can be decomposed into projections on the 2 orthogonal families, so the total error can be too.





- The philosophy now is to determine the stopband & inject a quad ‘error’ of the correct strength & out of phase to cancel it.
  - ➢ One way to determine the stopband is to ramp the Q1 & Q2 families to maximize transmission. This is proving to be a little difficult to interpret.
  - ➢ A more accurate technique is to measure the quadratic tune shift & determine ½-integer correction circuit settings that minimize the tune. A complete example from MI will follow.

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Parabolic Tune Shift & Stop-band Correction

- The quadratic transfer term again:

$$+\sum_{i,j>i}^{N}\frac{q_i\beta_i\cdot q_j\beta_j}{4}\cdot\begin{pmatrix}-(1-\cos 2(\psi_j-\psi_i))+\cos 2\psi_j-\cos 2\psi_i & -\sin 2(\psi_j-\psi_i))+\sin 2\psi_j-\sin 2\psi_i\\ \sin 2(\psi_j-\psi_i))+\sin 2\psi_j-\sin 2\psi_i & -(1-\cos 2(\psi_j-\psi_i))-\cos 2\psi_j+\cos 2\psi_i\end{pmatrix}$$

- Creates a tune shift:

$$\boxed{\mu\cong\frac{1}{2\pi}\cos^{-1}\left\{\left[1+2\pi^2Q^2\right]\cdot\cos(2\pi\mu_0)\right\}\Rightarrow\mu\sim\mu_0+{Q^2}/_{2\Delta}}$$

- where $Q\equiv\frac{N\beta}{4\pi}\cdot\sqrt{q_1^2+q_2^2}$; $\Delta\equiv(\mu_0-{}^{13}/_2)$ and:

$$q_1=\frac{1}{4\pi}\cdot\oint\frac{B'\beta}{B_0\rho}\cdot\cos(2\psi)\quad\text{and}\quad q_2=\frac{1}{4\pi}\cdot\oint\frac{B'\beta}{B_0\rho}\cdot\sin(2\psi)$$

Measuring the tune by ramping each family independently determines the strength & phase of the stop-band as corresponding to the minima and, therefore, the quad family offsets required to cancel it.

Warning:

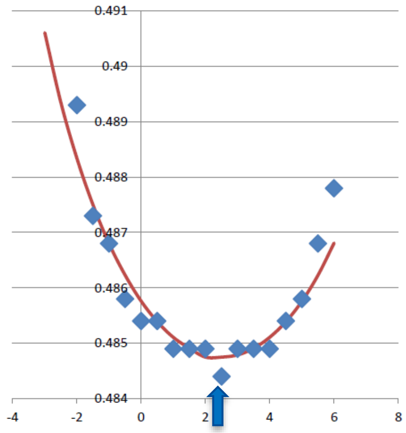
$$\mu \cong \frac{1}{2\pi}\cos^{-1}\left\{\left[1+2\pi^2 Q^2\right]\cdot\cos(2\pi\mu_0)\right\} \Rightarrow \mu \sim \mu_0 + {Q^2}/{2\Delta}$$

Unless *Q* is *really* small this parabolic approximation is truly horrible, so don't use it.

A Correction Example from MI

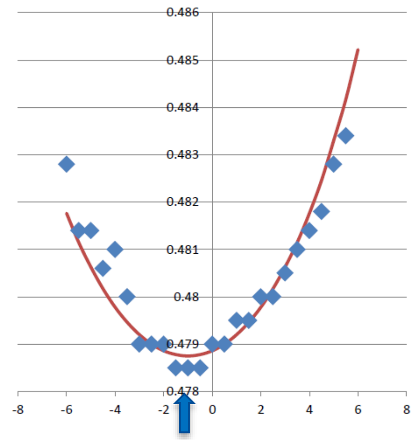
"AS FOUND"

QC206 @ +4.0A



+2.5A

QC328 @ -3.0A

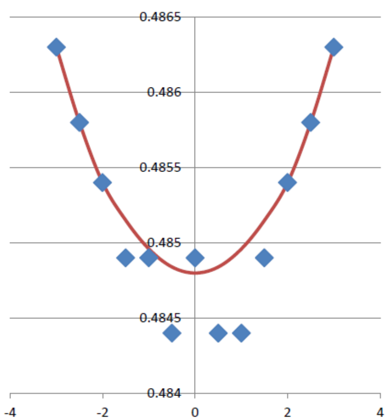


-1.0A

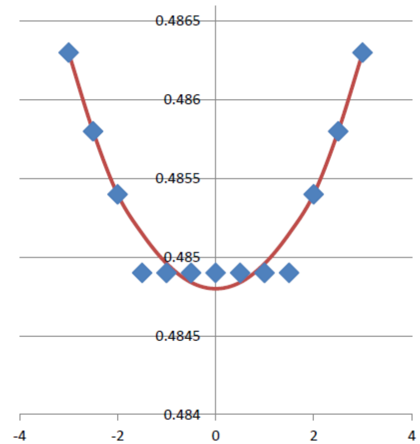
Dots = Measured
Solid = Predicted

CORRECTED

QC206 @ +1.5 A



QC328 @ -2.0 A



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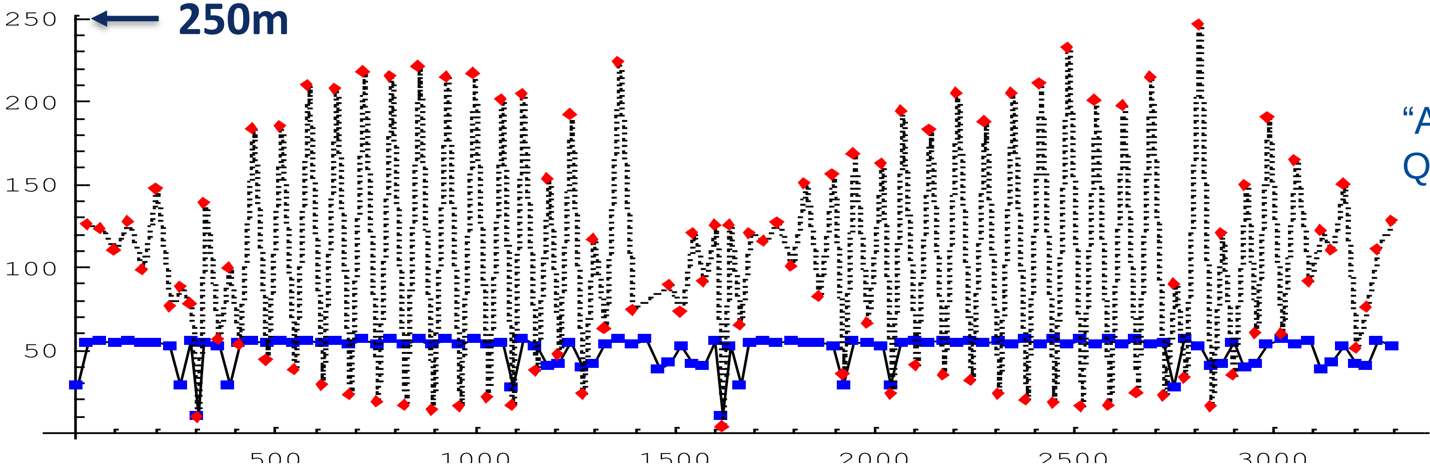
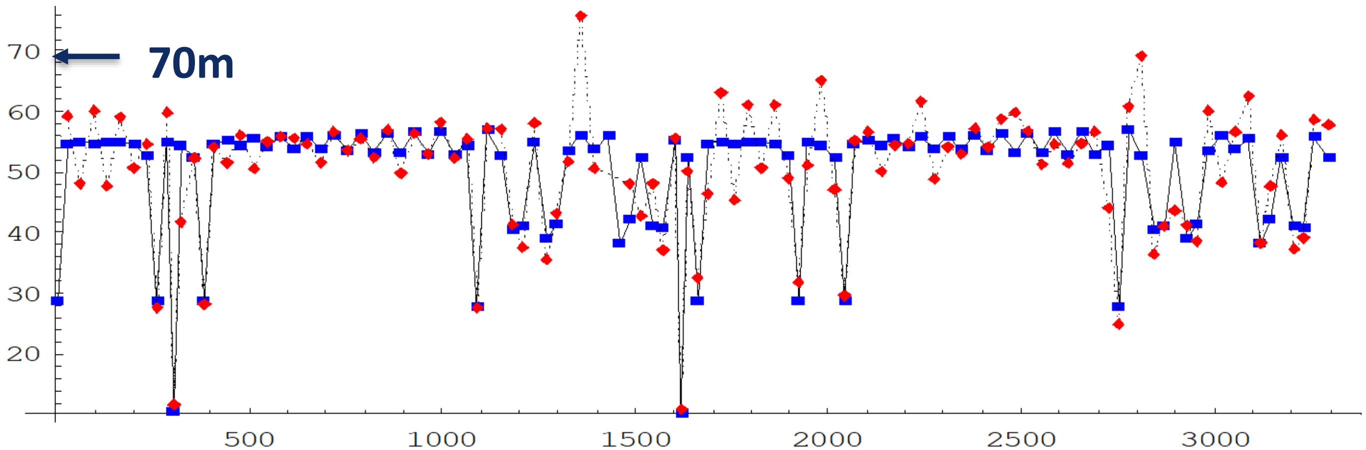
Δβ Effects of Stop-Band Cancellation



“As found” harm. quads, Qx=0.485627



Q206=+1.5A, Q328=-2A, Qx=0.484487

- John’s algorithm works!

Design
Measured

Y. Alexahin

Booster Circuits

- For a machine tune exactly on the half-integer at 6.5 the phase advance per cell in the *design* lattice is 97.5^o.

- To construct 2 orthogonal quad families we want $2\Delta\psi$ to be multiples of 180^o within a family, and $2\Delta\psi$ to be an odd multiple of 90^o between families. These constraints can be satisfied as follows :

  – Quads situated diametrically opposite in the ring are separated in phase by 90^o ($2\Delta\psi = 180^o$), forming one family, and;

  – Quads displaced by 6 cells from the 1st set are 45^o away ($2\Delta\psi = 90^o$), thereby forming the 2nd family.

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- Possible 13th harmonic orthogonal horizontal quad combinations are:

2 Quad Families

Family #1 : QS01 : +qh1 : QS13 : -qh1
Family #2 : QS07 : +qh2 : QS19 : -qh2

4 Quad Families

Family #1 : QS01 : +qh1 : QS02 : -qh1 : QS13 : -qh1 : QS13 : +qh1
Family #2 : QS07 : +qh2 : QS08 : -qh2 : QS19 : -qh2 : QS20 : +qh2

6 Quad Families

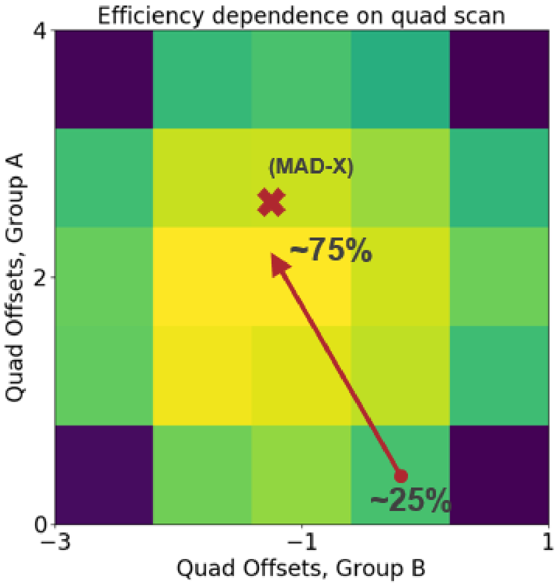
Family #1 : QS24 : -qh1 : QS01 : +qh1 : QS02 : -qh1 : QS12 : +qh1 : QS13 : -qh1 : QS14 : +qh1
Family #2 : QS06 : -qh2 : QS07 : +qh2 : QS08 : -qh2 : QS18 : +qh2 : QS19 : -qh2 : QS20 : +qh2

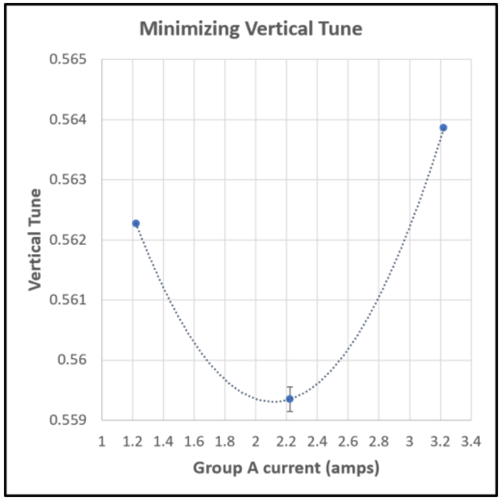
etc, etc . . .

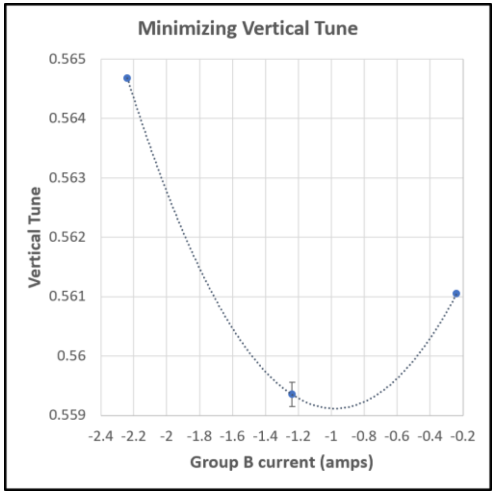
- In the vertical plane the same patterns apply but with the QLxx substituted for QSxx.
- Because of the 24-fold symmetry of the machine these patterns can be rotated to any convenient location around the ring.

- In Booster we have chosen to adopt the 6 quad/family configurations.
- That these are indeed orthogonal families can be easily shown. For convenience choose the QS1 location to correspond to $\psi = 0$ and, with 97.5^{o} of phase advance per cell, compute the sums of $cos(2\psi)$ & $sin(2\psi)$ around the ring. In Family #1 the cosine sum is +5.8637*$|qh_1|$ and the sine sum is exactly 0. In Family #2 it is the reverse -- the cosine sum is 0 and the sine sum is +5.8637*$|qh_2|$.
- The families are orthogonal within their plane, but have collateral impact on the other transverse plane:
  - At QS locations ${}^{\beta_x}/_{\beta_y} \sim 6$ so horizontal corrections will have only a small impact on the vertical optics, but;
  - At QL locations ${}^{\beta_y}/_{\beta_x}$ is only ~ 3 and vertical corrections will bleed about 30% into the horizontal plane. Some iteration between x & y may be necessary.

Proof-of-Principle Booster Studies†





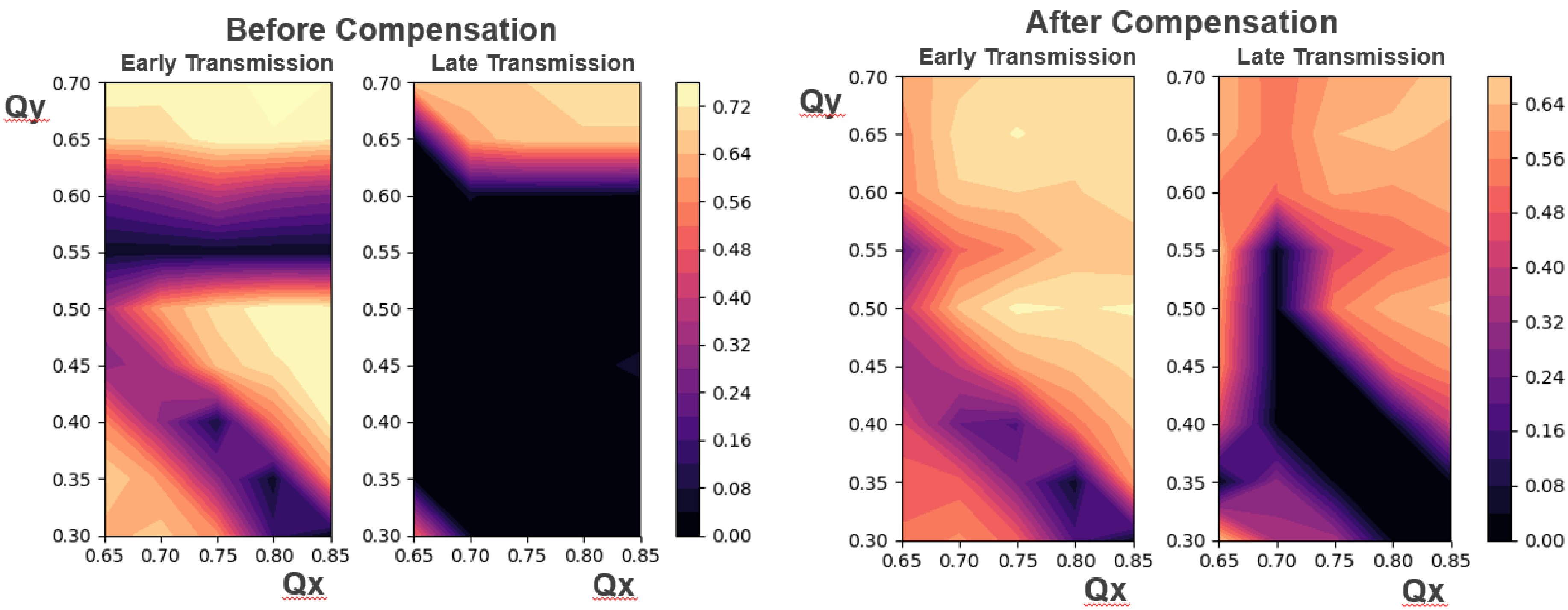


Best efficiency (2.22, -1.24); Tune Minimum (2.11,-0.98)

- There is reasonable agreement between the two approaches, although a finer tune mesh is desirable & many more tune measurements.

† Results are lifted from the following sources:

1) Jeff Eldred, *December 11 Preliminary Half-Integer Study*, PSP/Taskforce meeting on Jan. 2nd , Beams-doc-7905, 2020;
2) Jeff Eldred, *Half-Integer Compensation Studies – Jan. 22nd Results*, PSP/Taskforce meeting Feb. 13th , Beams-doc-8012, 2020;



- ½ integer compensation enlarges the usable tune space.

Observations & Suggestions

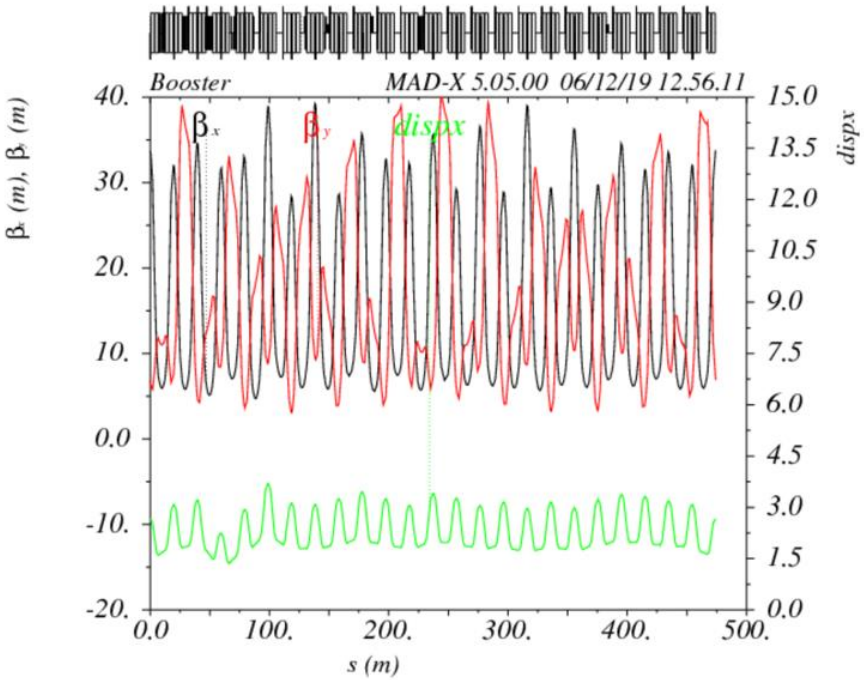
- The orthogonal quad circuits are impacting the ½ integer resonance, as advertised, but considerably more study is needed before quantifying.

- To cancel the intrinsic machine stopband due to the ***main ring magnets*** requires that the **trim quads be *Off*** or they are absorbed in the correction process.

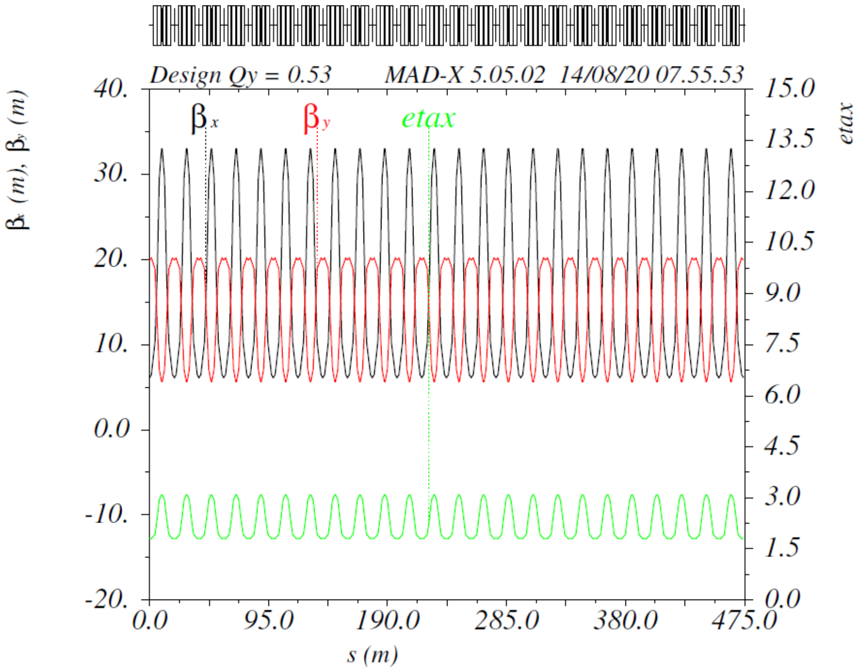
- The philosophy for correcting the half-integer is not unlike that of resonant extraction:
  - Eliminate the intrinsic (unwanted) quad sources driving the ½-integer, then;
  - Inject the desirable quadrupole terms.
  - Application of Jeff's recent works deriving relationships for localized β-bumps† (plus Jeff & Preston Hardcastle's summer work) should prove very useful.

† Jeff Eldred, *Local Beta Bump – Beta Relations for Transfer Matrices*, Beams-doc-8341, 2020.

Operational *vs* Design Injection Optics at Qy = 0.53



That qualifies as a quad 'error' β-beat propagation



- Ideally, correction circuit tuning (with trim quads off) should be applied at multiple energies through the acceleration cycle, as experience with other machines has shown that the intrinsic stopband strength & phase changes with main magnet powering.

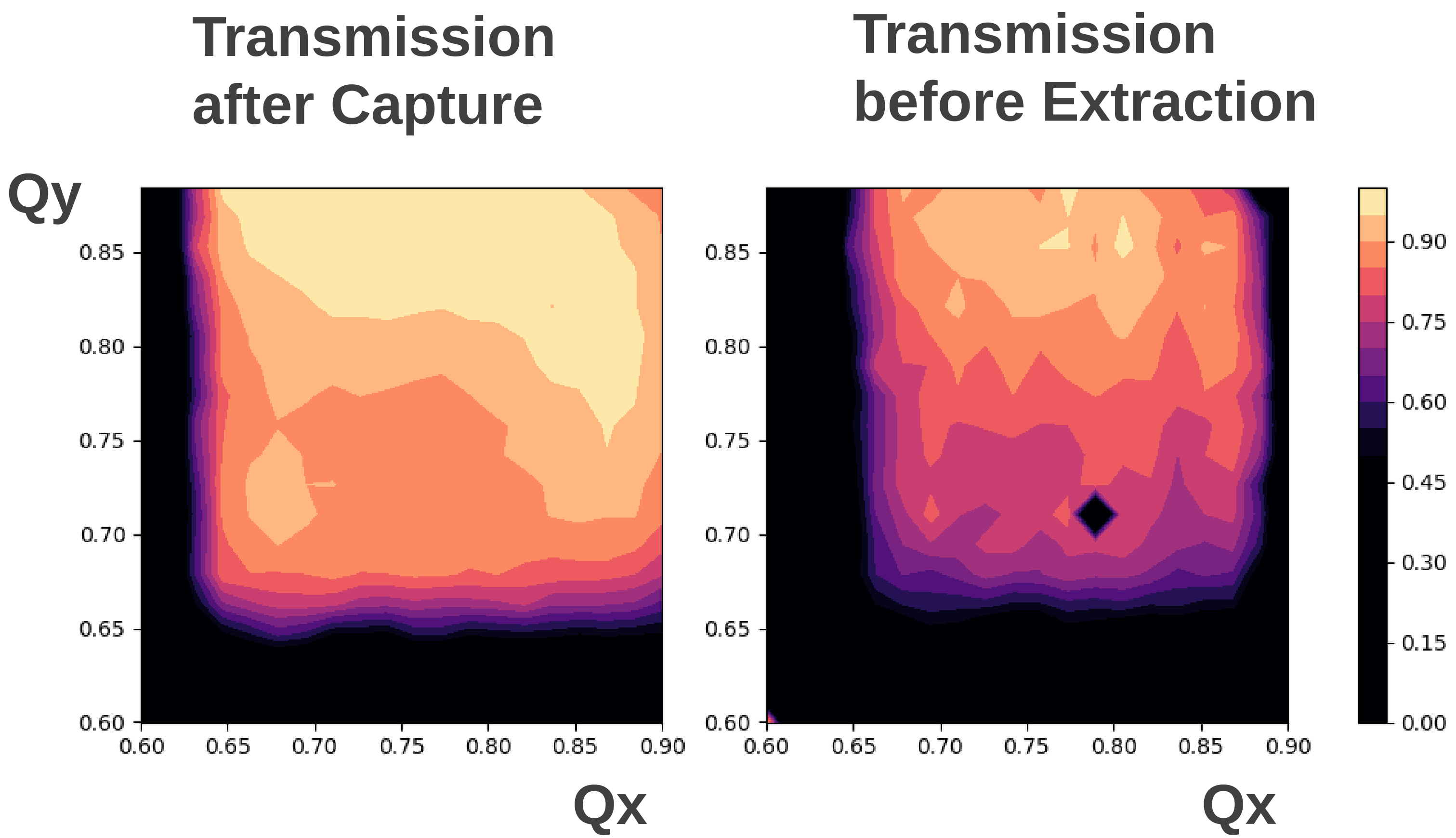
Failing that, at a minimum:

- The optimum circuit values determined at injection need to be scaled with momentum to apply correction equally throughout the ramp.

Ω

BACKOFF
SLIDES

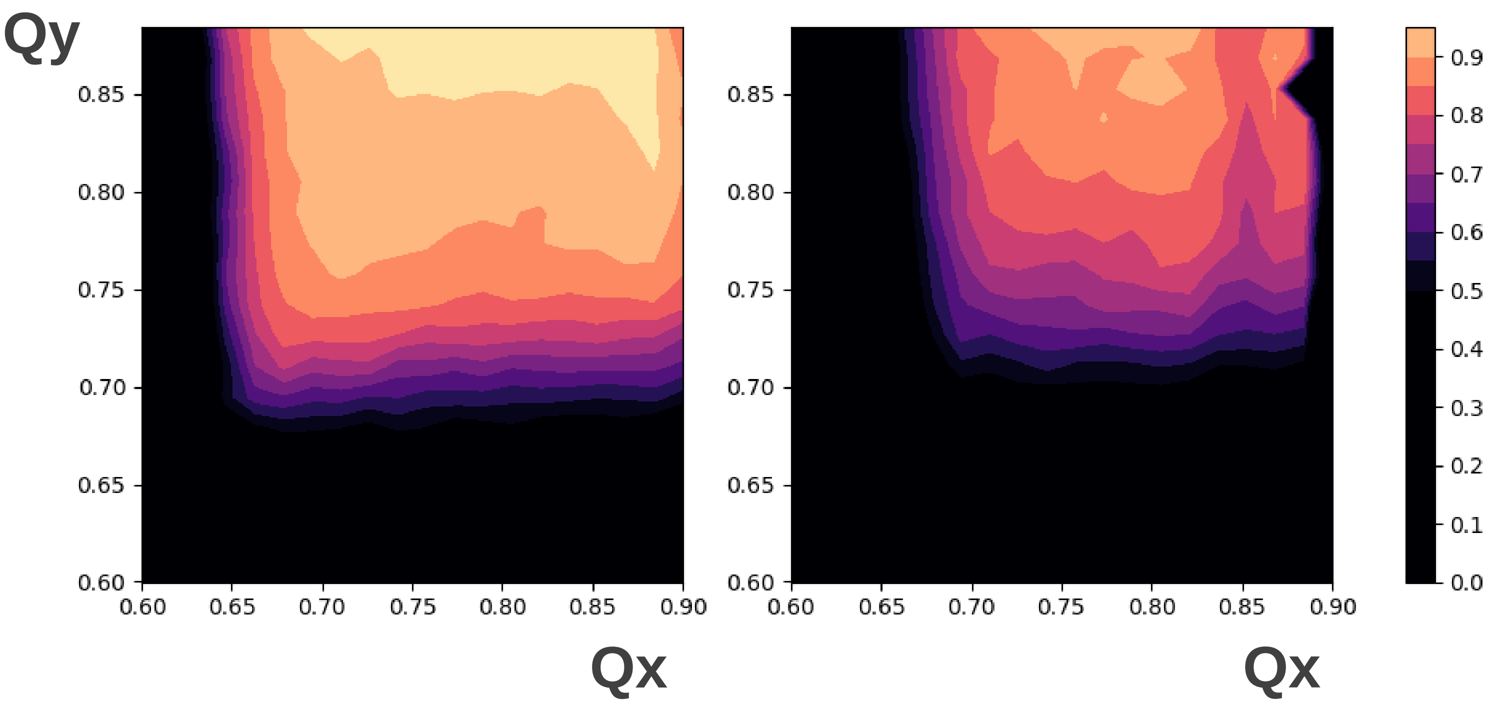
Intensity Reduces Tunespace



1/3 nom. intensity, chromaticity -20 at injection

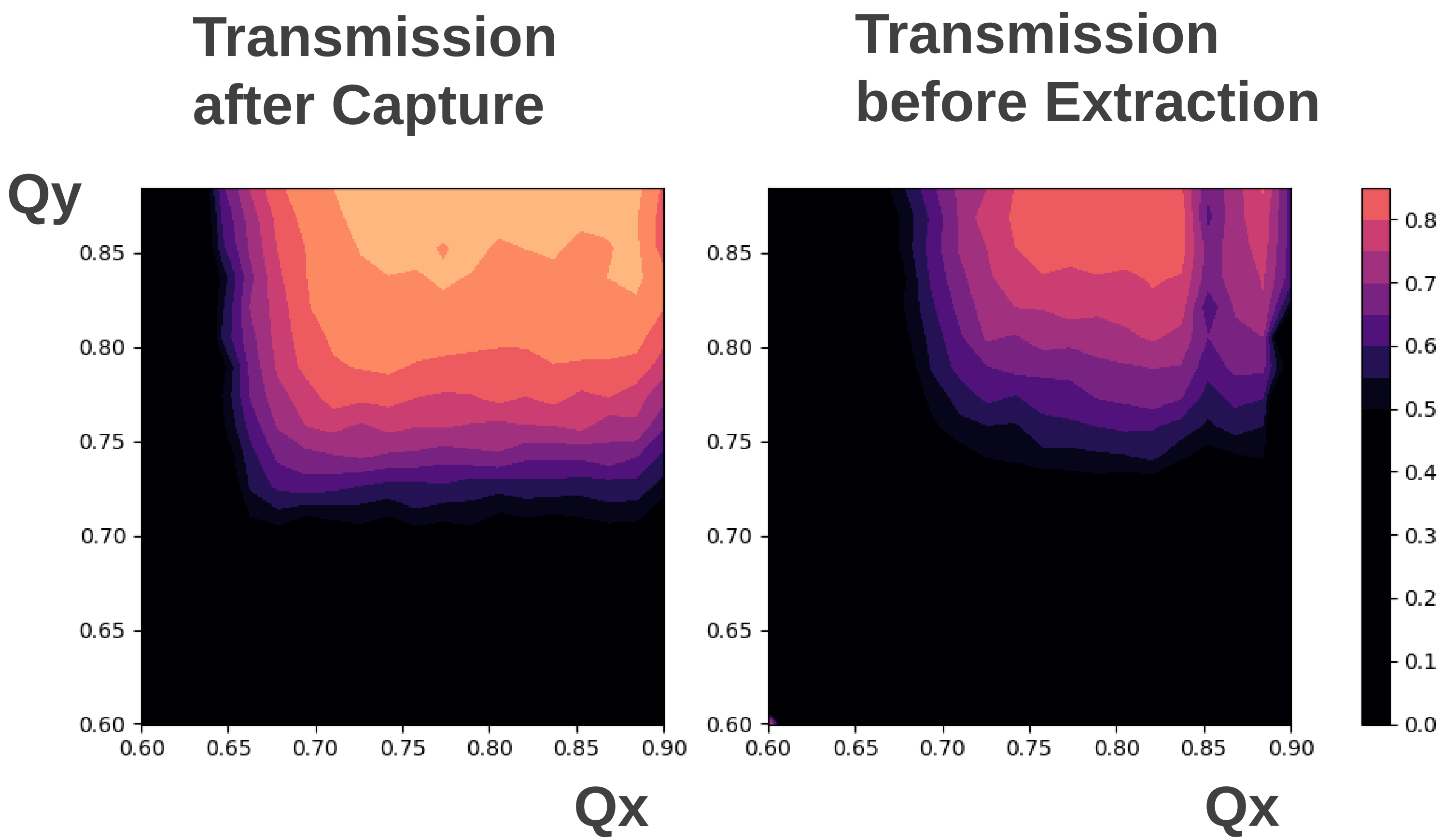
Intensity Reduces Tunespace

Transmission after Capture | **Transmission before Extraction**



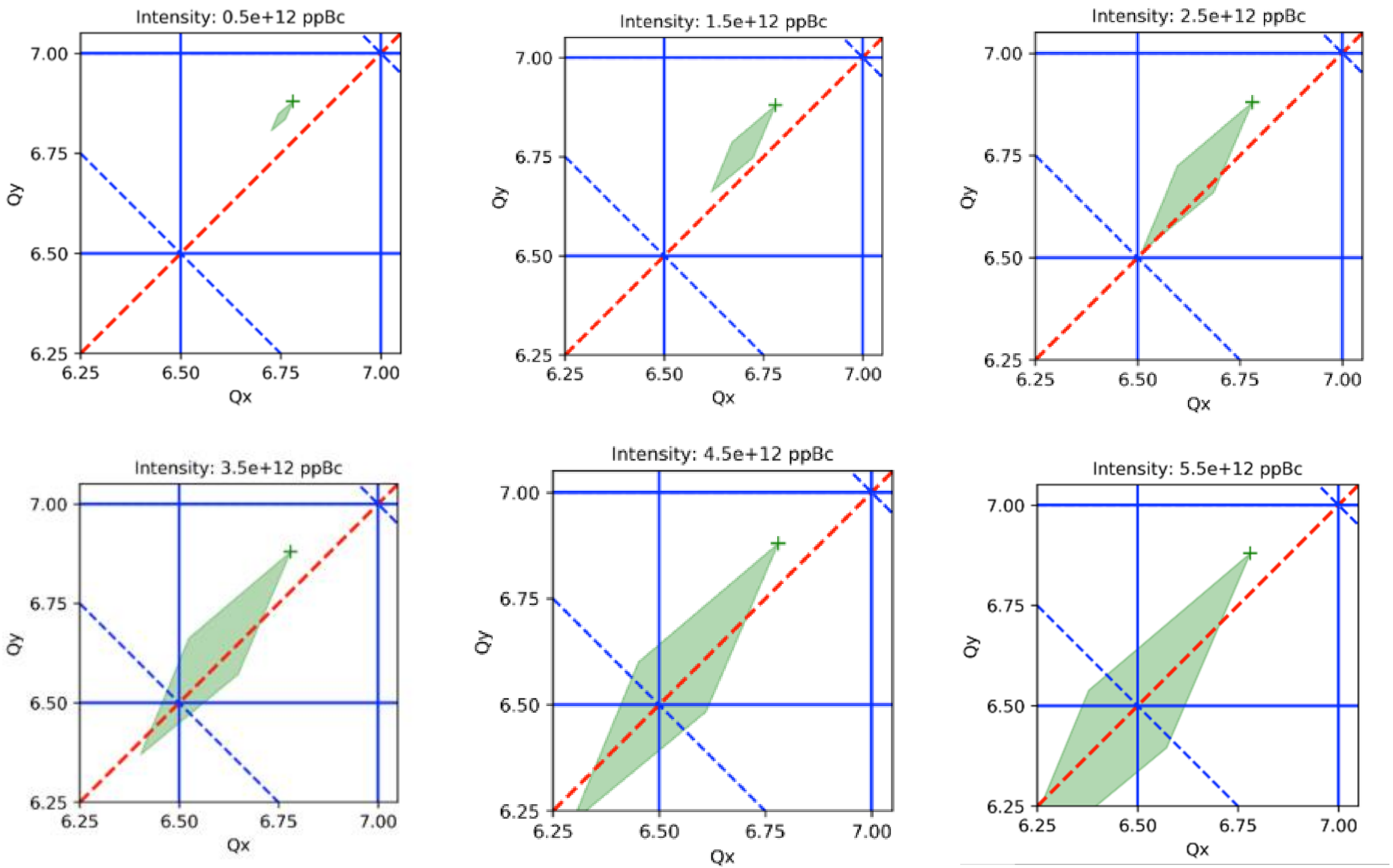
2/3 nom. intensity, chromaticity -20 at injection

Intensity Reduces Tunespace



nominal intensity, chromaticity -20 at injection

Tune Space *vs* Beam Intensity†



† H. Bartosik & a cast of thousands, *Lattice Periodicity & Emittance Growth (part 1)*, Capstone Event, 2019.